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On Hadamard and Fejér-Hadamard inequalities for Caputo k-fractional derivatives

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Abstract

In this paper we will prove certain Hadamard and Fejér-Hadamard inequalities for the functions whose n^{th} derivatives are convex by using Caputo k-fractional derivatives. These results have some relationship with inequalities for Caputo fractional derivatives.

Keywords: Hadamard inequality; convex functions; Fejér-Hadamard inequality; Caputo fractional derivatives.
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1. Introduction

Fractional calculus is the generalization of classical calculus which is mainly concerned with operations of integration and differentiation of non–integer (fractional) order. Since 19th century, the theory of fractional calculus developed rapidly, mostly as a foundation for number of applied disciplines which include fractional geometry, fractional differential equations and fractional dynamics. The applications of fractional calculus are very wide nowadays. Almost no discipline of modern engineering and science remains untouched by the tools and techniques of fractional calculus. Fractional calculus has wide applications in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc. (see [7] for details).

The history of fractional calculus is as old as the history of differential calculus. Fractional calculus is a natural extension of standard mathematics. Fractional calculus also has a lot of applications in the

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fields of science counting rheology, fluid flow, diffusive transport, electrical networks, electromagnetic theory and probability (see [3]). Fourier, Abel, Lacroix, Leibniz, Letnikov and Grunwald contributed a lot in this subject (see [6, 8, 9] and references there in). We give some preliminaries that we will use for our results. For this we will define convex functions, Hadamard inequality for convex functions, Fejér–Hadamard inequality for convex functions, Caputo fractional derivatives and finally Caputo k–fractional derivatives.

Definition 1.1. A function $f : I \to \mathbb{R}$ is convex if the following inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If reverse of the above inequality holds, then f is said to be concave function.

Theorem 1.2. Let $f : I \to \mathbb{R}$ be a convex function defined on interval I of real numbers with $a, b \in I$ and $a < b$. Then the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

It is well known in the literature as the Hadamard inequality [10]. In [5], Fejér established the following weighted generalization of the Hadamard inequality.

Definition 1.3. Let $f : I \to \mathbb{R}$ be a convex function defined on interval I of real numbers with $a, b \in I$ and $a < b$. Then the following inequality holds

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.3}$$

where $g : I \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$.

It is well known in the literature as the Fejér–Hadamard inequality.

Definition 1.4. Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous. The left–sided and right–sided Caputo fractional derivatives of order α are defined as follows:

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt \quad (x > a) \tag{1.4}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt \quad (x < b). \tag{1.5}$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^{n}_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^{n}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^{0}_{a+} f)(x) = ({}^C D^{0}_{b-} f)(x) = f(x), \tag{1.6}$$

where $n = 1$ and $\alpha = 0$.

For further details see [6].

Fractional integral inequalities play a very important role in establishing the uniqueness of solutions for certain fractional partial differential equations. They provide bounds for the solution of fractional boundary value problems. Due to these considerations many researchers explore certain extensions and generalizations of several kinds of inequalities by involving fractional calculus operators (see, [1, 2, 3, 4, 6] and references therein).

In this paper, in Section 2 we define Caputo k-fractional derivatives and utilize them to give the Hadamard inequality for functions whose nth derivatives are convex. We also find the bound of a difference of this inequality. In Section 3 we derive the Fejér–Hadamard inequality via Caputo k–fractional derivatives and find bounds of a difference of this inequality. We also deduce some related results.

2. Hermite Hadamard inequalities for Caputo k-fractional derivatives

First we give definition of the left sided and the right sided Caputo k-fractional derivatives.

Definition 2.1. Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The right–sided and left–sided Caputo k–fractional derivatives of order α are defined as follows:

$$({}^C D^{\alpha,k}_{a+} f)(x) = \frac{1}{k\Gamma_k(n - \frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x - t)^{\frac{\alpha}{k} - n + 1}} dt \quad (x > a) \tag{2.1}$$

and

$$({}^C D^{\alpha,k}_{b-} f)(x) = \frac{(-1)^n}{k\Gamma_k(n - \frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t - x)^{\frac{\alpha}{k} - n + 1}} dt \quad (x < b), \tag{2.2}$$

where $\Gamma_k(\alpha)$ is the k–Gamma function defined as:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha - 1} e^{\frac{-t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo k–fractional derivative $({}^C D^{n,1}_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^{n,1}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In particular we have

$$({}^C D^{0,1}_{a+} f)(x) = ({}^C D^{0,1}_{b-} f)(x) = f(x), \tag{2.3}$$

where $n, k = 1$ and $\alpha = 0$. For $k = 1$, Caputo k–fractional derivatives give the definition of Caputo fractional derivatives. In the following we give the Hadamard inequality for functions whose n^{th} derivatives are convex via Caputo k–fractional derivatives.

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in AC^n[a, b]$, $a < b$. If $f^{(n)}$ is convex on $[a, b]$, then the following inequality for Caputo k–fractional derivatives holds*

$$f^{(n)}\left(\frac{a + b}{2}\right) \leq \frac{k\Gamma_k(n - \frac{\alpha}{k} + k)}{2(b - a)^{n - \frac{\alpha}{k}}} \left[({}^C D^{\alpha,k}_{a+} f)(b) + (-1)^n ({}^C D^{\alpha,k}_{b-} f)(a)\right] \leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}. \tag{2.4}$$

Proof . Since $f^{(n)}$ is convex, so

$$f^{(n)}\left(\frac{x+y}{2}\right) \leq \frac{f^{(n)}(x)+f^{(n)}(y)}{2}. \tag{2.5}$$

Let $x, y \in [a,b]$, such that $x = ta+(1-t)b, y = (1-t)a+tb$ where $t \in [0,1]$. Then from (2.5) we have

$$2f^{(n)}\left(\frac{a+b}{2}\right) \leq f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb). \tag{2.6}$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we get

$$2f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt \leq \int_0^1 \frac{f^{(n)}(ta+(1-t)b)}{t^{\frac{\alpha}{k}-n+1}}dt + \int_0^1 \frac{f^{(n)}((1-t)a+tb)}{t^{\frac{\alpha}{k}-n+1}}dt.$$

By change of variables, we have

$$f^{(n)}\left(\frac{a+b}{2}\right) \leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{a+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b-}f)(a)\right]. \tag{2.7}$$

Also convexity of $f^{(n)}$ gives

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb) \leq f^{(n)}(a)+f^{(n)}(b). \tag{2.8}$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we get

$$\int_0^1 \frac{f^{(n)}(ta+(1-t)b)}{t^{\frac{\alpha}{k}-n+1}}dt + \int_0^1 \frac{f^{(n)}((1-t)a+tb)}{t^{\frac{\alpha}{k}-n+1}}dt \leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}dt.$$

Now by change of variables we get

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{a+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b-}f)(a)\right] \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \tag{2.9}$$

From inequalities obtained in (2.7) and (2.9) we get inequality in (2.4). □

Corollary 2.3. If we take $k=1$, we get the following inequality for Caputo fractional derivatives [4]

$$f^{(n)}\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{a+}f)(b)+(-1)^n({}^C D^{\alpha}_{b-}f)(a)\right] \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.$$

For next result we need the following lemma.

Lemma 2.4. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f \in AC^{n+1}[a,b]$, $a<b$. If $f^{(n+1)}$ is convex on $[a,b]$, then the following equality for Caputo k–fractional derivatives holds*

$$\begin{aligned}&\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{a+}f)(b)+(-1)^n({}^C D^{\alpha,k}_{b-}f)(a)\right]\\ &= \frac{b-a}{2}\int_0^1\left((1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right)f^{(n+1)}(ta+(1-t)b)dt.\end{aligned} \tag{2.10}$$

Proof . Consider the right hand side

$$\frac{b-a}{2}\int_0^1 \left((1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right) f^{(n+1)}(ta+(1-t)b)dt$$
$$= \frac{b-a}{2}\int_0^1 (1-t)^{n-\frac{\alpha}{k}} f^{(n+1)}(ta+(1-t)b)dt - \frac{b-a}{2}\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(ta+(1-t)b)dt.$$

Now we compute the first and the second terms of last expression as follows respectively

$$\frac{b-a}{2}\int_0^1 (1-t)^{n-\frac{\alpha}{k}} f^{(n+1)}(ta+(1-t)b)dt$$
$$= \frac{b-a}{2}\left[(1-t)^{n-\frac{\alpha}{k}} \frac{f^{(n)}(ta+(1-t)b)}{a-b}|_0^1 + \frac{(n-\frac{\alpha}{k})}{a-b}\int_0^1 \frac{f^{(n)}(ta+(1-t)b)}{(1-t)^{\frac{\alpha}{k}-n+1}}dt\right]$$
$$= \frac{b-a}{2}\left[\frac{f^{(n)}(b)}{b-a} - \frac{n-\frac{\alpha}{k}}{b-a}\int_a^b \left(\frac{b-a}{x-a}\right)^{\frac{\alpha}{k}-n+1} \frac{f^{(n)}(x)}{b-a}dx\right]$$
$$= \frac{f^n(b)}{2} - \frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{2(b-a)^{n-\frac{\alpha}{k}}}(-1)^n({}^C D_{b-}^{\alpha,k} f)(a)$$

and

$$-\frac{b-a}{2}\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(ta+(1-t)b)dt$$
$$= \frac{b-a}{2}\left[\frac{f^{(n)}(a)}{b-a} - \frac{n-\frac{\alpha}{k}}{b-a}\int_a^b \left(\frac{b-a}{b-x}\right)^{\frac{\alpha}{k}-n+1} \frac{f^{(n)}(x)}{b-a}dx\right]$$
$$= \frac{f^{(n)}(a)}{2} - \frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{2(b-a)^{n-\frac{\alpha}{k}}}({}^C D_{a+}^{\alpha,k} f)(b).$$

Hence the required equality can be established. □

Using the above lemma we establish the bounds of a difference of (2.4).

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in AC^{n+1}[a,b]$, $a<b$. If $|f^{(n+1)}|$ is convex on $[a,b]$, then the following inequality for Caputo k–fractional derivatives holds*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^C D_{a+}^{\alpha,k} f)(b) + (-1)^n({}^C D_{b-}^{\alpha,k} f)(a)\right]\right|$$
$$\leq \frac{b-a}{2(n-\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right]. \tag{2.11}$$

Proof . Using convexity of $|f^{(n+1)}|$ and Lemma 2.4, we have

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^C D_{a+}^{\alpha,k} f)(b) + (-1)^n({}^C D_{b-}^{\alpha,k} f)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\int_0^1 \left|(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right| |f^{(n+1)}(ta+(1-t)b)|dt \tag{2.12}$$
$$\leq \frac{b-a}{2}\int_0^1 \left(\left|(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right|\right)[t|f^{(n+1)}(a)|+(1-t)|f^{(n+1)}(b)|]dt$$

$$= \frac{b-a}{2}\left[\int_0^{\frac{1}{2}}\left[(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right][t|f^{(n+1)}(a)| + (1-t)|f^{(n+1)}(b)|]dt\right.$$
$$\left. + \int_{\frac{1}{2}}^1 \left[t^{n-\frac{\alpha}{k}} - (1-t)^{n-\frac{\alpha}{k}}\right][t|f^{(n+1)}(a)| + (1-t)|f^{(n+1)}(b)|]dt\right].$$

Now we have

$$\int_0^{\frac{1}{2}}\left[(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right][t|f^{(n+1)}(a)| + (1-t)|f^{(n+1)}(b)|]dt$$
$$= |f^{(n+1)}(a)|\left[\int_0^{\frac{1}{2}} t(1-t)^{n-\frac{\alpha}{k}}dt - \int_0^{\frac{1}{2}} t^{n-\frac{\alpha}{k}+1}dt\right] + |f^{(n+1)}(b)|\left[\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}+1}dt - \int_0^{\frac{1}{2}}\frac{1-t}{t^{\frac{\alpha}{k}-n}}dt\right]$$
$$= |f^{(n+1)}(a)|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right] + |f^{(n+1)}(b)|\left[\frac{1}{n-\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right].$$

Similarly

$$\int_{\frac{1}{2}}^1\left[t^{n-\frac{\alpha}{k}} - (1-t)^{n-\frac{\alpha}{k}}\right][t|f^{(n+1)}(a)| + (1-t)|f^{(n+1)}(b)|]dt$$
$$= |f^{(n+1)}(a)|\left[\frac{1}{n-\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right] + |f^{(n+1)}(b)|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right].$$

Therefore (2.12) becomes

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{a+}f)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}f)(a)\right]\right|$$
$$\leq \frac{b-a}{2}\left(|f^{(n+1)}(a)|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right] + |f^{(n+1)}(b)|\left[\frac{1}{n-\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\right.$$
$$\left. + |f^{(n+1)}(a)|\left[\frac{1}{n-\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right] + |f^{(n+1)}(b)|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\right).$$

From which after a little computation we get the required result. □

Corollary 2.6. If we take $k = 1$, we get the following inequality for Caputo fractional derivatives [4]

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{a+}f)(b) + (-1)^n({}^C D^{\alpha}_{b-}f)(a)\right]\right|$$
$$\leq \frac{b-a}{2(n-\alpha+1)}\left(1 - \frac{1}{2^{n-\alpha}}\right)\left[|f^{(n+1)}(a)| + |f^{(n+1)}(b)|\right].$$

3. Fejér–Hadamard inequalities for Caputo *k*–fractional derivatives

In this section Fejér–Hadamard and Fejér Hadamard–type inequalities for Caputo k–fractional derivatives are given.

Lemma 3.1. For $0 < \lambda \leq 1$ and $0 \leq a < b$, we have

$$|a^\lambda - b^\lambda| \leq (b-a)^\lambda.$$

In this section we use $\|g^{(n)}\|_\infty = \sup_{x\in[a,b]} |g^{(n)}(x)|$ and the convolution $f * g$ of functions f and g for Caputo k–fractional derivatives as follows

$$({}^C D^{\alpha,k}_{a+} f * g)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t) g^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt \quad (x > a) \tag{3.1}$$

and

$$({}^C D^{\alpha,k}_{b-} f * g)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t) g^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt \quad (x < b). \tag{3.2}$$

Here first we prove the following lemma.

Lemma 3.2. *Let $g : [a,b] \to \mathbb{R}$ be a function such that $g \in AC^n[a,b]$, $a < b$. If $g^{(n)}$ is symmetric to $\frac{a+b}{2}$, then*

$$({}^C D^{\alpha,k}_{a+} g)(b) = (-1)^n ({}^C D^{\alpha,k}_{b-} g)(a) = \frac{1}{2}[({}^C D^{\alpha,k}_{a+} g)(b) + (-1)^n ({}^C D^{\alpha,k}_{b-} g)(a)].$$

Proof . By definition we have

$$({}^C D^{\alpha,k}_{a+} g)(b) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})} \int_a^b \frac{g^{(n)}(x)dx}{(b-x)^{\frac{\alpha}{k}-n+1}}.$$

Substituting x by $a+b-x$ in the above integral we have

$$({}^C D^{\alpha,k}_{a+} g)(b) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})} \int_a^b \frac{g^{(n)}(a+b-x)dx}{(x-a)^{\frac{\alpha}{k}-n+1}}.$$

By symmetricity of $g^{(n)}$ we have $g^{(n)}(a+b-x) = g^{(n)}(x)$, therefore

$$({}^C D^{\alpha,k}_{a+} g)(b) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})} \int_a^b \frac{g^{(n)}(x)dx}{(x-a)^{\frac{\alpha}{k}-n+1}}$$

and we have

$$({}^C D^{\alpha,k}_{a+} g)(b) = (-1)^n ({}^C D^{\alpha,k}_{b-} g)(a).$$

Hence the required equality can be obtained. □

Using above lemma we prove the following results.

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in AC^n[a,b]$, $a < b$. Also let $f^{(n)}$ be convex function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be such that $g \in AC^n[a,b]$. If $g^{(n)}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the following inequality for Caputo k–fractional derivatives holds*

$$\begin{aligned} & f^{(n)}\left(\frac{a+b}{2}\right) \left[({}^C D^{\alpha,k}_{a+} g)(b) + (-1)^n ({}^C D^{\alpha,k}_{b-} g)(a)\right] \\ & \leq ({}^C D^{\alpha,k}_{a+} f * g)(b) + (-1)^n ({}^C D^{\alpha,k}_{b-} f * g)(a) \\ & \leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2} \left[({}^C D^{\alpha,k}_{a+} g)(b) + (-1)^n ({}^C D^{\alpha,k}_{b-} g)(a)\right]. \end{aligned} \tag{3.3}$$

Proof . Since $f^{(n)}$ is convex multiplying both sides of inequality (2.6) with $\frac{2g^{(n)}(tb+(1-t)a)}{t^{\frac{\alpha}{k}-n+1}}$ and integrating the resulting inequality over $[0,1]$, we have

$$2f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 \frac{g^{(n)}(tb+(1-t)a)dt}{t^{\frac{\alpha}{k}-n+1}}$$
$$\leq \int_0^1 \frac{f^{(n)}(ta+(1-t)b)g^{(n)}(tb+(1-t)a)dt}{t^{\frac{\alpha}{k}-n+1}} + \int_0^1 \frac{f^{(n)}(tb+(1-t)a)g^{(n)}(tb+(1-t)a)dt}{t^{\frac{\alpha}{k}-n+1}}.$$

Putting $x = tb + (1-t)a$, we get

$$\frac{2}{(b-a)^{n-\frac{\alpha}{k}}} f^{(n)}\left(\frac{a+b}{2}\right)\int_a^b \frac{g^{(n)}(x)}{(x-a)^{\frac{\alpha}{k}-n+1}}dx$$
$$\leq \frac{1}{(b-a)^{n-\frac{\alpha}{k}}}\left[\int_a^b \frac{f^{(n)}(a+b-x)g^{(n)}(x)}{(x-a)^{\frac{\alpha}{k}-n+1}}dx + \int_a^b \frac{f^{(n)}(x)g^{(n)}(x)}{(x-a)^{\frac{\alpha}{k}-n+1}}dx\right]$$
$$= \frac{1}{(b-a)^{n-\frac{\alpha}{k}}}\left[\int_a^b \frac{f^{(n)}(x)g^{(n)}(a+b-x)}{(b-x)^{\frac{\alpha}{k}-n+1}}dx + \int_a^b \frac{f^{(n)}(x)g^{(n)}(x)}{(x-a)^{\frac{\alpha}{k}-n+1}}dx\right]$$
$$= \frac{1}{(b-a)^{n-\frac{\alpha}{k}}}\left[\int_a^b \frac{f^{(n)}(x)g^{(n)}(x)}{(b-x)^{\frac{\alpha}{k}-n+1}}dx + \int_a^b \frac{f^{(n)}(x)g^{(n)}(x)}{(x-a)^{\frac{\alpha}{k}-n+1}}dx\right].$$

By using Lemma 3.2 we get the first inequality of (3.3). For the second inequality of (3.3) multiplying both sides of inequality (2.8) with $\frac{g^{(n)}(tb+(1-t)a)}{t^{\frac{\alpha}{k}-n+1}}$ and integrating the resulting inequality over $[0,1]$ we get

$$\int_0^1 \frac{f^{(n)}(ta+(1-t)b)g^{(n)}(tb+(1-t)a)}{t^{\frac{\alpha}{k}-n+1}}dt + \int_0^1 \frac{f^{(n)}(tb+(1-t)a)g^{(n)}(tb+(1-t)a)}{t^{\frac{\alpha}{k}-n+1}}dt$$
$$\leq \left(f^{(n)}(a)+f^{(n)}(b)\right)\int_0^1 \frac{g^{(n)}(tb+(1-t)a)}{t^{\frac{\alpha}{k}-n+1}}dt$$

from which after using change of variables and a little computation we get the required result. □

Corollary 3.4. If we take $k = 1$ in Theorem 3.3, we get the following result for Caputo fractional derivatives [4]

$$f^{(n)}\left(\frac{a+b}{2}\right)\left[({}^CD^{\alpha}_{a+}g)(b) + (-1)^n({}^CD^{\alpha}_{b-}g)(a)\right]$$
$$\leq ({}^CD^{\alpha}_{a+}(f*g))(b) + (-1)^n({}^CD^{\alpha}_{b-}(f*g))(a)$$
$$\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^CD^{\alpha}_{a+}g)(b) + (-1)^n({}^CD^{\alpha}_{b-}g)(a)\right].$$

Next we need the following lemma.

Lemma 3.5. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in AC^{n+1}[a,b]$, $a < b$. Also let $f^{(n+1)}$ be convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be a function such that $g \in AC^n[a,b]$. Then the following equality for Caputo k–fractional derivatives holds*

$$\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^CD^{\alpha,k}_{a+}g)(b) + (-1)^n({}^CD^{\alpha,k}_{b-}g)(a)\right] - \left[({}^CD^{\alpha,k}_{a+}f*g)(b) + (-1)^n({}^CD^{\alpha,k}_{b-}f*g)(a)\right] \tag{3.4}$$
$$= \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^b\left[\int_a^t \frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds - \int_t^b \frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds\right] f^{(n+1)}(t)dt.$$

Proof . We note that

$$\frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^b\left[\int_a^t\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds-\int_t^b\frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds\right]f^{(n+1)}(t)dt$$
$$=\frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\int_a^b\left(\int_a^t\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right)f^{(n+1)}(t)dt+\int_a^b\left(-\int_t^b\frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds\right)f^{(n+1)}(t)dt\right].\tag{3.5}$$

By simple calculation we get

$$\int_a^b\left(\int_a^t\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right)f^{(n+1)}(t)dt$$
$$=\left[\left(\int_a^b\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right)f^{(n)}(b)-\int_a^b\frac{f^{(n)}(t)g^{(n)}(t)}{(b-t)^{\frac{\alpha}{k}-n+1}}dt\right]$$
$$=k\Gamma_k(n-\frac{\alpha}{k})\left[f^{(n)}(b)({}^CD_{a+}^{\alpha,k}g)(b)-({}^CD_{a+}^{\alpha,k}(f*g))(b)\right]$$
$$=k\Gamma_k(n-\frac{\alpha}{k})\left[\frac{f^{(n)}(b)}{2}[({}^CD_{a+}^{\alpha,k}g)(b)+(-1)^n({}^CD_{b-}^{\alpha,k}g)(a)]-({}^CD_{a+}^{\alpha,k}f*g)(b)\right]$$

and

$$\int_a^b\left(-\int_t^b\frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds\right)f^{(n+1)}(t)dt$$
$$=\left(\int_a^b\frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds\right)f^{(n)}(a)-\int_a^b\frac{f^{(n)}(t)g^{(n)}(t)}{(t-a)^{\frac{\alpha}{k}-n+1}}dt$$
$$=k\Gamma_k(n-\frac{\alpha}{k})\left[\frac{f^{(n)}(a)}{2}[({}^CD_{a+}^{\alpha,k}g)(b)+(-1)^n({}^CD_{b-}^{\alpha,k}g)(a)]-(-1)^n({}^CD_{b-}^{\alpha,k}f*g)(a)\right].$$

Hence by addition (3.4) is established. □

Theorem 3.6. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in AC^{n+1}[a,b]$, $a<b$. If $|f^{(n+1)}|$ is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is such that $g\in AC^n[a,b]$. If $g^{(n)}$ is symmetric to $\frac{a+b}{2}$, then the following inequality for Caputo k-fractional derivatives holds*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^CD_{a+}^{\alpha,k}g)(b)+(-1)^n({}^CD_{b-}^{\alpha,k}g)(a)\right]-\left[({}^CD_{a+}^{\alpha,k}f*g)(b)+(-1)^n({}^CD_{b-}^{\alpha,k}f*g)(a)\right]\right|$$
$$\leq\frac{(b-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_\infty}{(n-\frac{\alpha}{k}+1)\Gamma_k(n-\frac{\alpha}{k}+k)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right].$$

Proof . Since $|f^{(n+1)}|$ is convex, so we have

$$|f^{(n+1)}(t)|\leq\frac{b-t}{b-a}|f^{(n+1)}(a)|+\frac{t-a}{b-a}|f^{(n+1)}(b)|,\tag{3.6}$$

where $t\in[a,b]$. From symmetricity of $g^{(n)}$, we have

$$\int_t^b\frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds=\int_a^{a+b-t}\frac{g^{(n)}(a+b-s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds=\int_a^{a+b-t}\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds.$$

This gives

$$\left|\int_a^t \frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds - \int_t^b \frac{g^{(n)}(s)}{(s-a)^{\frac{\alpha}{k}-n+1}}ds\right| = \left|\int_t^{a+b-t} \frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right|$$
$$\leq \begin{cases} \int_t^{a+b-t} \left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right| ds, & t \in [a, \frac{a+b}{2}] \\ \int_{a+b-t}^{t} \left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right| ds, & t \in [\frac{a+b}{2}, b]. \end{cases} \tag{3.7}$$

By virtue of Lemma 3.5 and inequalities (3.6), (3.7) we have

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^C D_{a+}^{\alpha,k} g)(b) + (-1)^n ({}^C D_{b-}^{\alpha,k} g)(a)\right] - \left[({}^C D_{a+}^{\alpha,k} f*g)(b) + (-1)^n ({}^C D_{b-}^{\alpha,k} f*g)(a)\right]\right| \tag{3.8}$$
$$\leq \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right| ds\right)\left(\frac{b-t}{b-a}|f^{(n+1)}(a)| + \frac{t-a}{b-a}|f^{(n+1)}(b)|\right) dt\right.$$
$$\left. + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}} ds\right|\right)\left(\frac{b-t}{b-a}|f^{(n+1)}(a)| + \frac{t-a}{b-a}|f^{(n+1)}(b)|\right) dt\right]$$
$$\leq \frac{\|g^{(n)}\|_\infty}{k\Gamma_k(n-\frac{\alpha}{k}+k)(b-a)}\left[\int_a^{\frac{a+b}{2}} \left(\frac{1}{(b-t)^{\frac{\alpha}{k}-n}} - \frac{1}{(t-a)^{\frac{\alpha}{k}-n}}\right)\left((b-t)|f^{(n+1)}(a)| + (t-a)|f^{(n+1)}(b)|\right) dt\right.$$
$$\left. + \int_{\frac{a+b}{2}}^b \left(\frac{1}{(t-a)^{\frac{\alpha}{k}-n}} - \frac{1}{(b-t)^{\frac{\alpha}{k}-n}}\right)\left((b-t)|f^{(n+1)}(a)| + (t-a)|f^{(n+1)}(b)|\right) dt\right].$$

We have

$$\int_a^{\frac{a+b}{2}} \left(\frac{1}{(b-t)^{\frac{\alpha}{k}-n}} - \frac{1}{(t-a)^{\frac{\alpha}{k}-n}}\right)(b-t)dt = \int_{\frac{a+b}{2}}^b \left(\frac{1}{(t-a)^{\frac{\alpha}{k}-n}} - \frac{1}{(b-t)^{\frac{\alpha}{k}-n}}\right)(t-a)dt$$
$$= \frac{(b-a)^{n-\frac{\alpha}{k}+2}}{n-\frac{\alpha}{k}+1}\left(\frac{n-\frac{\alpha}{k}+1}{n-\frac{\alpha}{k}+2} - \frac{1}{2^{n-\frac{\alpha}{k}+1}}\right) \tag{3.9}$$

and

$$\int_a^{\frac{a+b}{2}} \left(\frac{1}{(b-t)^{\frac{\alpha}{k}-n}} - \frac{1}{(t-a)^{\frac{\alpha}{k}-n}}\right)(t-a)dt = \int_{\frac{a+b}{2}}^b \left(\frac{1}{(t-a)^{\frac{\alpha}{k}-n}} - \frac{1}{(b-t)^{\frac{\alpha}{k}-n}}\right)(b-t)dt$$
$$= \frac{(b-a)^{n-\frac{\alpha}{k}+2}}{n-\frac{\alpha}{k}+1}\left(\frac{1}{n-\frac{\alpha}{k}+2} - \frac{1}{2^{n-\frac{\alpha}{k}+1}}\right). \tag{3.10}$$

Using (3.9) and (3.10) in (3.8), we get the required result. □

Corollary 3.7. In Theorem 3.6 if we put $k = 1$, we get the following result for Caputo fractional derivatives [4]

$$\left|\left(\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right)[({}^C D_{a+}^{\alpha} g)(b) + (-1)^n ({}^C D_{b-}^{\alpha} g)(a)]\right.$$
$$\left. -[({}^C D_{a+}^{\alpha} f*g)(b) + (-1)^n ({}^C D_{b-}^{\alpha} f*g)(a)]\right|$$
$$\leq \frac{(b-a)^{n-\alpha+1}\|g^{(n)}\|_\infty}{(n-\alpha+1)\Gamma(n-\alpha+1)}\left(1 - \frac{1}{2^{n-\alpha}}\right)\left[|f^{(n+1)}(a)| + |f^{(n+1)}(b)|\right].$$

Theorem 3.8. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in AC^{n+1}[a, b]$, $a < b$. Also let $|f^{(n+1)}|^q$, $q \geq 1$ be convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ be such that $g \in AC^n[a, b]$. If $g^{(n)}$ is symmetric to $\frac{a+b}{2}$, then the following inequality for Caputo k–fractional derivatives holds*

$$\left|\frac{f^{(n)}(a) + f^{(n)}(b)}{2}\left[({}^C D^{\alpha,k}_{a+}g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}g)(a)\right] - \left[({}^C D^{\alpha,k}_{a+}f * g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}f * g)(a)\right]\right| \tag{3.11}$$

$$\leq \frac{2(b-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_\infty}{(n-\frac{\alpha}{k}+1)k\Gamma_k(n-\frac{\alpha}{k}+k)(b-a)^{\frac{1}{q}}}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\left(\frac{|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Proof . By Using Lemma 3.5, power mean inequality, inequality (3.7) and convexity of $|f^{(n+1)}|^q$ respectively we have

$$\left|\frac{f^{(n)}(a) + f^{(n)}(b)}{2}\left[({}^C D^{\alpha,k}_{a+}g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}g)(a)\right] - \left[({}^C D^{\alpha,k}_{a+}f * g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}f * g)(a)\right]\right|$$

$$\leq \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\int_a^b\left|\int_t^{a+b-t}\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right|dt\right]^{1-\frac{1}{q}}\left[\int_a^b\left|\int_t^{a+b-t}\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right||f^{(n+1)}(t)|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right|ds\right)dt + \int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right|ds\right)dt\right]^{1-\frac{1}{q}}$$

$$\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right|ds\right)|f^{(n+1)}(t)|^q dt + \int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right|ds\right)|f^{(n+1)}(t)|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{\|g^{(n)}\|_\infty}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\left(\frac{2(b-a)^{n-\frac{\alpha}{k}+1}}{(n-\frac{\alpha}{k})(n-\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\right)^{1-\frac{1}{q}}\right.$$

$$\left.\times\left(\frac{(|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q)(b-a)^{n-\frac{\alpha}{k}+2}}{(n-\frac{\alpha}{k})(n-\frac{\alpha}{k}+1)(b-a)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\right)^{\frac{1}{q}}\right].$$

From which after a little computation we have the required result. □

Theorem 3.9. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in AC^{n+1}[a, b]$, $a < b$. Also let $|f^{(n+1)}|^q$, $q > 1$ be convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ be such that $g \in AC^n[a, b]$. If $g^{(n)}$ is symmetric to $\frac{a+b}{2}$, then the following inequalities for Caputo k–fractional derivatives hold*

(i) $$\left|\frac{f^{(n)}(a) + f^{(n)}(b)}{2}\left[({}^C D^{\alpha,k}_{a+}g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}g)(a)\right] - \left[({}^C D^{\alpha,k}_{a+}f * g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}f * g)(a)\right]\right|$$

$$\leq \frac{2^{\frac{1}{p}}(b-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_\infty}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}k\Gamma_k(n-\frac{\alpha}{k}+k)}\left(1-\frac{1}{2^{np-\frac{\alpha p}{k}}}\right)^{\frac{1}{p}}\left(\frac{|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(ii) $$\left|\frac{f^{(n)}(a) + f^{(n)}(b)}{2}\left[({}^C D^{\alpha,k}_{a+}g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}g)(a)\right] - [({}^C D^{\alpha,k}_{a+}f * g)(b) + (-1)^n({}^C D^{\alpha,k}_{b-}f * g)(a)]\right|$$

$$\leq \frac{(b-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_\infty}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}k\Gamma_k(n-\frac{\alpha}{k}+k)}\left(\frac{|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}},$$

with $0 < \alpha \leq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof . (i) By Using Lemma 3.5, Hölder's inequality, inequality (3.7) and convexity of $|f^{(n+1)}|^q$, we have

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^CD^{\alpha,k}_{a+}g)(b)+(-1)^n({}^CD^{\alpha,k}_{b-}g)(a)\right]-[({}^CD^{\alpha,k}_{a+}f*g)(b)+(-1)^n({}^CD^{\alpha,k}_{b-}f*g)(a)]\right|$$
$$\leq \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\left(\int_a^b\left|\int_t^{a+b-t}\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|f^{(n+1)}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right|^p ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|\frac{g^{(n)}(s)}{(b-s)^{\frac{\alpha}{k}-n+1}}\right|^p ds\right)dt\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b\left(\frac{b-t}{b-a}|f^{(n+1)}(a)|^q+\frac{t-a}{b-a}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\times\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^CD^{\alpha,k}_{a+}g)(b)+(-1)^n({}^CD^{\alpha,k}_{b-}g)(a)\right]-[({}^CD^{\alpha,k}_{a+}f*g)(b)+(-1)^n({}^CD^{\alpha,k}_{b-}f*g)(a)]\right| \tag{3.12}$$
$$\leq \frac{\|g^{(n)}\|_\infty}{k\Gamma_k(n-\frac{\alpha}{k})}\left[\int_a^{\frac{a+b}{2}}\left(\frac{(b-t)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}-\frac{(t-a)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}\right)^p dt+\int_{\frac{a+b}{2}}^b\left(\frac{(t-a)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}-\frac{(b-t)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}\right)^p dt\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b\left(\frac{b-t}{b-a}|f^{(n+1)}(a)|^q+\frac{t-a}{b-a}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}.$$

Now

$$(A-B)^q\leq A^q-B^q \quad A\geq B\geq 0,$$

gives

$$\left[\frac{(b-t)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}-\frac{(t-a)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}\right]^p\leq\frac{(b-t)^{np-\frac{\alpha p}{k}}}{(n-\frac{\alpha}{k})^p}-\frac{(t-a)^{np-\frac{\alpha p}{k}}}{(n-\frac{\alpha}{k})^p} \tag{3.13}$$

for $t\in[a,\frac{a+b}{2}]$ and

$$\left[\frac{(t-a)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}-\frac{(b-t)^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}}\right]^p\leq\frac{(t-a)^{np-\frac{\alpha p}{k}}}{(n-\frac{\alpha}{k})^p}-\frac{(b-t)^{np-\frac{\alpha p}{k}}}{(n-\frac{\alpha}{k})^p} \tag{3.14}$$

for $t\in[\frac{a+b}{2},b]$. Using (3.13) and (3.14) in inequality (3.12) and then solving, we get inequality (i).
(ii) Here one can use inequality (3.12) and Lemma 3.1 in order to prove inequality (ii). □

Corollary 3.10. If we take $k=1$ in above theorem, we get the following result for Caputo fractional derivatives [4]

$$(i)\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^CD^{\alpha}_{a+}g)(b)+(-1)^n({}^CD^{\alpha}_{b-}g)(a)\right]-\left[({}^CD^{\alpha}_{a+}f*g)(b)+(-1)^n({}^CD^{\alpha}_{b-}f*g)(a)\right]\right|$$
$$\leq\frac{2^{\frac{1}{p}}(b-a)^{n-\alpha+1}\|g^{(n)}\|_\infty}{(np-\alpha p+1)^{\frac{1}{p}}\Gamma(n-\alpha+1)}\left(1-\frac{1}{2^{np-\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}}.$$

$$(ii)\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[({}^{C}D^{\alpha}_{a+}g)(b)+(-1)^n({}^{C}D^{\alpha}_{b-})g(a)\right]-[({}^{C}D^{\alpha}_{a+}f*g)(b)+(-1)^n({}^{C}D^{\alpha}_{b-}f*g)(a)]\right|$$
$$\leq \frac{(b-a)^{n-\alpha+1}\|g^{(n)}\|_\infty}{(np-\alpha p+1)^{\frac{1}{p}}\Gamma(n-\alpha+1)}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}},$$

with $0<\alpha\leq 1$, where $\frac{1}{p}+\frac{1}{q}=1$.

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