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A new approximation method for common fixed points of a finite family of nonexpansive non-self mappings in Banach spaces

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Abstract

In this paper, we introduce a new iterative scheme to approximate a common fixed point for a finite family of nonexpansive non–self mappings. Strong convergence theorems of the proposed iteration in Banach spaces.

Keywords: nonexpansive non-self mappings; common fixed points; Banach spaces.

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1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of a Banach space X and let $P : X \to C$ be the *nonexpansive retraction* of X onto C, $T : C \to X$ a given mapping. T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T denoted by $F(T)$ such that $F(T) = \{x \in C : x = Tx\}$ and f is called *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$.

In 1953, Mann [7] introduced *Mann iteration process* define as follows: $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$. Later, in 1974, Ishikawa [5] proposed the following two–step iteration: $x_1 \in C$ and

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad \forall n \geq 1,$$

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where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. This method is often called the *Ishikawa iteration process.*

Very recently, Agarwal et al. [2] introduced a new iteration process as follows: $x_1 \in C$ and

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n T y_n, \quad \forall n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. This method is called the *S–iteration process.*

Motivated by Agarwal et al. [2], we have the aim to introduce and study a new mapping defined by the following definition.

Definition 1.1. *Let X be a real Banach space, C a nonempty closed convex subset of a real Banach space X and let $P : X \to C$ be the nonexpansive retraction of X onto C. Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive non-self mappings of C onto X, and let $\lambda_1, \lambda_2, \ldots, \lambda_N \in [0, 1]$ for all $i = 1, 2, \ldots, N$. Define the mapping $Y : X \to X$ as follows:*

$$\begin{aligned} U_1 &= \lambda_1 PT_1 + (1-\lambda_1)I, \\ U_2 &= \lambda_2 PT_2U_1 + (1-\lambda_2)PT_1, \\ U_3 &= \lambda_3 PT_3U_2 + (1-\lambda_3)PT_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} PT_{N-1}U_{N-2} + (1-\lambda_{N-1})PT_{N-2}, \\ Y = U_N &= \lambda_N PT_NU_{N-1} + (1-\lambda_N)PT_{N-1}, \end{aligned} \tag{1.1}$$

such that a mapping Y is called the Y–mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $I : X \to X$ be identity mapping.

First, we use the definition above, study weak convergence of the following Mann–type iteration process in a uniformly convex Banach space with a Fréchet differentiable norm: $x_1 \in C$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Y_n x_n, \quad \forall n \geq 1, \tag{1.2}$$

where Y_n is a $Y-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.

Finally, we discuss strong convergence of the iteration scheme involving the modified viscosity approximation method [8] define as follows: $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \lambda_n Y_n x_n, \quad \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$ and $f \in \Sigma_C$.

The aim of this paper is to obtain weak and strong convergence results for the iterative process (1.1) of a nonexpansive non-self mappings in Banach spaces. This paper, we use the notation as follows:

i $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;

ii $\omega_\omega(x_n) = \{x : x_{n_i} \rightharpoonup\}$ denote the weak ω–limit set of $\{x_n\}$.

2. Preliminaries

In this section, we give some definitions and lemmas used in the main results.[1]

Let X be a real Banach space and let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. A Banach space X is said to be *strictly convex* if for any $x, y \in U$,

$$x \neq y \qquad \text{implies} \qquad \left\| \frac{x+y}{2} \right\| < 1.$$

It also said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \qquad \text{implies} \qquad \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0, 2] \to [0, 1]$ called the *modulus of convexity* of X as follows:

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \right\}.$$

Then X is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. A Banach space X is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exist for all $x, y \in U$. The norm is said to be *uniformly Gâteaux differentiable*, if for $y \in U$, the limit is attained uniformly for $x \in U$. It is said to be *Fŕechet differentiable*, if for $x \in U$, the limit is attained uniformly for $y \in U$. It is said to be *uniformly smooth* or *uniformly Fŕechet differentiable* if the limit (2.1) is attained uniformly for $x, y \in U$. The *normalized duality mapping* $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.2}$$

for all $x \in X$. It is know that X is smooth if and only if the duality mapping J is single valued and that if X has a uniformly Gâteaux differentiable norm, J is uniformly norm–to–weak continuous on each bounded subset of X. A Banach space X is said to satisfy Opial's condition [9]. If $x \in X$ and $x_n \rightharpoonup x$, then

$$\lim_{n \to \infty} \sup \|x_n - x\| < \lim_{n \to \infty} \sup \|x_n - y\|, \quad \forall y \in X, x \neq y. \tag{2.3}$$

Let $T : C \to C$. Then $I - T$ is demiclosed at 0 if for all sequence $\{x_n\}$ in C, $x_n \rightharpoonup q$ and $\|x_n - T_n\| \to 0$ imply $q = Tq$. It is known that if X is uniformly convex, C is nonempty closed and convex, and T is nonexpansive, then $I - T$ is demiclosed at 0 [3].

The following lemmas are needed for proving our main results.

Lemma 2.1. (Agarwal et al. [1]) Let X be a Banach space. Then the following hold:

1. $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, J(x) \rangle$ for all $x, y \in X$;
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$ for all $x, y \in X$.

Lemma 2.2. (Takahashi [11]) In a strictly convex Banach space X, if

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$$

for all $x, y \in X$ and $\lambda \in (0, 1)$, then $x = y$

Lemma 2.3. (Suzuki [10]) Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space E such that

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \quad n \geq 1,$$

where $\{\beta_n\}$ satisfies the condition $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
If $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\|z_n - x_n\| \to 0$ as $n \to \infty$.

Lemma 2.4. (Tan and Xu [12]) Let X be a uniformly convex Banach space with a Frećhet differentiable norm. Let G be a closed convex subset of X and let $\{S_n\}_{n=1}^{\infty}$ be a family of L_n-Lipschitzian self-mappings on C such that $\Sigma_{n=1}^{\infty}(L_n - 1) < \infty$ and $F \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. For arbitrary $x_1 \in C$, define $x_{n+1} = S_n x_n$ for all $n \geq 1$. Then, for every $p, q \in F$, $\lim_{n\to\infty}\langle x_n, J(p-q)\rangle$ exists, in particular, for all $u, v \in \omega_\omega(x_n)$ and $p, q \in F$, $\langle u - v, J(p-q)\rangle = 0$.

Lemma 2.5. (Jung and Sahu [6]) Let X be a reflexive and strictly uniformly convex Banach space with a uniformly Gâcteaux differentiable norm, let C be a closed convex subset of X and let $A : C \to C$ be a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$, and let $T : C \to X$ be a continuous pseudocontractive mapping satisfying the weakly inward condition. If T has a fixed point in C, then the path $\{x_t\}$ defined by

$$x_t = tAx_t + (1-t)Tx_t,$$

converges strongly to a fixed point q of T as $t \to 0$, which is a unique solution of the variational inequality

$$\langle (I - A)q, J(q-p)\rangle \leq 0, \quad \forall p \in F(T).$$

Lemma 2.6. (Xu [13]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-c_n)a_n + b_n, \quad \forall n \geq 1,$$

where $\{c_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

1. $\Sigma_{n=1}^{\infty} c_n = \infty$;
2. $\limsup\limits_{n\to\infty} \dfrac{b_n}{c_n} \leq 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Weak convergence in Banach spaces

In this section, we use the concept of Y–mapping and study weak convergence of the sequence generated by Mann–type iteration process (1.2).

Lemma 3.1. *Let C be a nonempty, closed and convex subset of a strictly convex Banach space X and let $P : X \to C$ be a nonexpansive retraction of X onto C. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive non–self mappings of C into X such that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be s real numbers such that $0 < \lambda_i < 1$ for all $i = 1, 2, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let Y be the $Y-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then the following hold:*

1. $F(Y) = \bigcap_{i=1}^{N} F(T_i)$;
2. Y *is nonexpansiave.*

Proof .

1. Since $\bigcap_{i=1}^{N} F(T_i) \subset F(Y)$ is trivial, it suffices to show that $F(Y) \subset \bigcap_{i=1}^{N} F(T_i)$. To this end, let $p \in F(Y)$ and $p^* \in \bigcap_{i=1}^{N} F(T_i)$. Then we have

$$\begin{aligned}
\|p - p^*\| &= \|Yp - p^*\| \\
&= \|\lambda_N(PT_N U_{N-1}p - p^*) + (1-\lambda_N)\lambda_N(PT_N p - p^*)\| \\
&\leq \lambda_N\|U_{N-1}p - p^*\| + (1-\lambda_N)\|p - p^*\| \\
&= \lambda_N\|\lambda_{N-1}(PT_{N-1}U_{N-2}p - p^*) + (1-\lambda_{N-1})(PT_{N-2}p - p^*)\| \\
&\quad +(1-\lambda_N)\|p - p^*\| \\
&\leq \lambda_N\lambda_{N-1}\|U_{N-2}p - p^*\| + (1-\lambda_N\lambda_{N-1})\|p - p^*\| \\
&= \lambda_N\lambda_{N-1}\|\lambda_{N-2}(PT_{N-2}U_{N-3}p - p^*) + (1-\lambda_{N-2}(PT_{N-3}p - p^*)\| \\
&\quad +(1-\lambda_N\lambda_{N-1})\|p - p^*\| \\
&\vdots \\
&= \lambda_N\lambda_{N-1}\ldots\lambda_3\|\lambda_2(PT_2U_1p - p^*) + (1-\lambda_2)(PT_1p - p^*)\| \\
&\quad +(1-\lambda_N\lambda_{N-1}\ldots\lambda_3)\|p - p^*\| \\
&\leq \lambda_N\lambda_{N-1}\ldots\lambda_2\|PT_2U_1p - p^*\| + (1-\lambda_N\lambda_{N-1}\ldots\lambda_2)\|p - p^*\| \\
&\leq \lambda_N\lambda_{N-1}\ldots\lambda_2\|U_1p - p^*\| + (1-\lambda_N\lambda_{N-1}\ldots\lambda_2)\|p - p^*\| \\
&= \lambda_N\lambda_{N-1}\ldots\lambda_2\|\lambda_1(PT_1p - p^*) + (1-\lambda_1)(p - p^*)\| \\
&\quad +(1-\lambda_N\lambda_{N-1}\ldots\lambda_2)\|p - p^*\| \\
&\leq \lambda_N\lambda_{N-1}\ldots\lambda_2\lambda_1\|PT_1p - p^*\| + (1-\lambda_N\lambda_{N-1}\ldots\lambda_2\lambda_1)\|p - p^*\| \\
&\leq \lambda_N\lambda_{N-1}\ldots\lambda_2\lambda_1\|p - p^*\| + (1-\lambda_N\lambda_{N-1}\ldots\lambda_2\lambda_1)\|p - p^*\| \\
&= \|p - p^*\|. \qquad (3.1)
\end{aligned}$$

This show that

$$\|p - p^*\| = \lambda_N\lambda_{N-1}\ldots\lambda_2\|\lambda_1(PT_1p - p^*) + (1-\lambda_1)(p - p^*)\| + (1-\lambda_N\lambda_{N-1}\ldots\lambda_2)\|p - p^*\|,$$

which turns out to be

$$\|p - p^*\| = \|\lambda_1(PT_1p - p^*) + (1-\lambda_1)(p - p^*)\|.$$

By (3.1), we see that

$$\|p - p^*\| = \|PT_1p - p^*\|$$

and thus

$$\|p - p^*\| = \|PT_1p - p^*\| = \|\lambda_1(PT_1p - p^*) + (1-\lambda_1)(p - p^*)\|.$$

Using Lemma 2.2, we get that $PT_1 = p$ and hence $U_1p = p$. Again by (3.1), we have

$$\|p - p^*\| = \lambda_N\lambda_{N-1}\ldots\lambda_3\|\lambda_2(PT_2U_1p - p^*) + (1-\lambda_2)(PT_1p - p^*)\| + (1-\lambda_N\lambda_{N-1}\ldots\lambda_3)\|p - p^*\|,$$

which implies that

$$\|p - p^*\| = \|\lambda_2(PT_2U_1p - p^*) + (1-\lambda_2)(PT_1p - p^*)\|.$$

From (3.1), we see that

$$\|U_1p - p^*\| = \|PT_2U_1p - p^*\|.$$

Since $U_1p = p$ and $PT_1p = p$,

$$\|p - p^*\| = \|PT_2p - p^*\| = \|\lambda_2(PT_2p - p^*) + (1 - \lambda_2)(p - p^*)\|.$$

Again by (2.2), we get that $PT_2p = p$ and hence $U_2p = p$. By continuing this process, we can show that $PT_ip = p$ and $U_ip = p$ for all $i = 1, 2, \ldots, N-1$. Finally, we obtain

$$\begin{aligned}\|p - T_Np\| &\leq \|p - Yp\| + \|Yp - T_Np\| \\ &= \|p - Yp\| + (1 - \lambda_N)\|p - T_Np\|,\end{aligned}$$

which yields that $p = PT_Np$, since $p \in F(Y)$. Hence $p = PT_1p = PT_2p = \cdots = PT_Np$ and thus $p \in \bigcap_{i=1}^N F(T_i)$.

2. The proof follows directly from (1).

□

Lemma 3.2. *Let C be a nonempty closed and convex subset of a strictly convex Banach space X and let $P : X \to C$ be a nonexpansive retraction of X onto C. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive non–self mappings of C into X such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let Y be the Y–mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be real sequence in $(0, 1)$. For every $n \in \mathbb{N}$, let y_n be the $Y-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ as follows;*

$$\begin{aligned}U_{n,1} &= \lambda_{n,1}PT_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}PT_2U_1 + (1 - \lambda_{n,2})PT_1, \\ U_{n,3} &= \lambda_{n,3}PT_3U_2 + (1 - \lambda_{n,3})PT_2, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}PT_{N-1}U_{N-2} + (1 - \lambda_{n,N-1})PT_{N-2}, \\ Y_n = U_{n,N} &= \lambda_{n,N}PT_NU_{N-1} + (1 - \lambda_{n,N})PT_N.\end{aligned}$$

If $\lambda_{n,i} \to \lambda_i \in (0, 1)$ for all $i = 1, 2, \ldots, N$ then

1. *$\lim_{n\to\infty} Y_nx = Yx$ for all $x \in C$,*
2. *Y_n is nonexpansive.*

Proof .

1. Let $x \in C$,U_k be generated by $T_1, T_2, \ldots, T_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_k$ and let $U_{n,k}$ be generated by $T_1, T_2, \ldots, T_k$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,k}$, respectively. Then

$$\|U_{n,1}x - U_1x\| = \|(\lambda_{n,1} - \lambda_1)(PT_1x - x)\| \leq |\lambda_{n-1} - \lambda_1|\|PT_1x - x\|.$$

Let $k \in \{2, 3, \ldots, N\}$ and $M = \max\{\|PT_kU_{k-1}x\| : k = 2, 3, \ldots, N\}$. Then

$$\begin{aligned}\|U_{n,k}x - U_kx\| &= \|\lambda_{n,k}PT_kU_{n,k-1}x + (1 - \lambda_{n,k})PT_{k-1}x - \lambda_kPT_kU_{k-1} - (1 - \lambda_k)PT_{k-1}x\| \\ &= \|\lambda_{n,k}PT_kU_{n,k-1}x - \lambda_{n,k}PT_{k-1}x - \lambda_kPT_kU_{k-1} + \lambda_kPT_{k-1}x\| \\ &\leq \lambda_{n,k}\|PT_kU_{n,k-1}x - PT_kU_{k-1}x\| + |\lambda_{n,k} - \lambda_k|\|PT_kU_{k-1}x\| \\ &\quad + |\lambda_{n,k} - \lambda_k|\|PT_{k-1}x\| \\ &\leq \|U_{n,k-1}x - U_{k-1}x\| + |\lambda_{n,k} - \lambda_k|M.\end{aligned}$$

It follows that

$$\begin{aligned}\|Y_nx - Yx\| &= \|U_{n,N}x - U_N\| \\ &= \|U_{n,N-1}x - U_{N-1}x\| + |\lambda_{n,N} - \lambda_N|M \\ &\le \|U_{n,N-2}x - U_{N-2}x\| + |\lambda_{n,N-1} - \lambda_{N-1}|M + |\lambda_{n,N} - \lambda_N|M \\ &\vdots \\ &\le \|U_{n,1}x - U_1x\| + |\lambda_{n,2} - \lambda_2|M + \ldots + |\lambda_{n,N-1} - \lambda_{N-1}|M + |\lambda_{n,N} - \lambda_N|M \\ &\le |\lambda_{n,1} - \lambda_1|\|PT_1x - x\| + |\lambda_{n,2} - \lambda_2|M + \ldots + |\lambda_{n,N-1} - \lambda_{N-1}|M \\ &\quad + |\lambda_{n,N} - \lambda_N|M.\end{aligned}$$

Since $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$ $(i = 1, 2, \ldots, N)$, we thus complete the proof.

2. It is easily see that for all $n \in \mathbb{N}, Y_n$ is nonexpansive.

□

Lemma 3.3. *Let C be a nonempty closed and convex subset of a real Banach space X and let $P : X \to C$ be a nonexpansive retraction of X onto C. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive non–self mappings of C into X such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0, 1)$, for all $n \in \mathbb{N}$, let Y_n be the Y–mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. If $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ for all $i = 1, 2, \ldots, N$, then*

$$\lim_{n\to\infty} \|Y_{n+1}z_n - Y_nz_n\| = 0$$

for each bounded sequence $\{z_n\} \in C$.

Proof . Let $\{z_n\}$ be a bounded sequence in C. For $j \in \{0, 1, \ldots, N - 2\}$ and for some $M > 0$, we have that

$$\begin{aligned}\|U_{n+1,N-j}z_n - U_{n,N-1}z_n\| &= \|\lambda_{n+1,N-j}PT_{N-j}U_{n+1,N-j-1}z_n + 1 - \lambda_{n+1,N-j}PT_{N-j-1}z_n \\ &\quad - \lambda_{n,N-j}PT_{N-j}U_{n,N-j-1}z_n - 1 - \lambda_{n,N-j}PT_{n-j-1}z_n\| \\ &\le \lambda_{n+1,N-j}\|PT_{N-j}U_{n+1,N-j-1}z_n - PT_{N-j}U_{n,N-j-1}z_n\| \\ &\quad + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|PT_{N-j}U_{n,N-j-1}z_n\| \\ &\quad + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|PT_{N-j-1}z_n\| \\ &\le \|U_{n+1,N-j-1}z_n - U_{n,N-j-1}z_n\| + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|M.\end{aligned}$$

Using the relation above, we can show that

$$\begin{aligned}\|Y_{n+1}z_n - Y_nz_n\| &= \|U_{n+1,N}z_n - U_{n,N}z_n\| \\ &\le M\sum_{j=2}^{N} |\lambda_{n+1,j} - \lambda_{n,j}| + |\lambda_{n+1,1} - \lambda_{n,1}|(\|z_n\| + \|T_1z_n\|).\end{aligned}$$

Since $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ for all $i = 1, 2, \ldots, N$, we obtain the desired result. □

Theorem 3.4. *Let X be a uniformly convex Banach space having a Fréchet differentiable norm. Let C be a nonempty, closed and convex subset of X and let $P : X \to C$ be a nonexpansive retraction of*

X onto C. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive non–self mappings of C into X such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0,1)$ such that $\lambda_{n,i} \to \lambda_i (i = 1, 2, \dots, N)$. For every $n \in \mathbb{N}$, let Y_n be the Y–mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Let $\{x_n\}$ be generated by $x_1 \in C$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Y_n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges weakly to $x^* \in \bigcap_{i=1}^N F(T_i)$.

Proof . Let $p \in \bigcap_{i=1}^N F(T_i)$. Then $p = Y_n p$ for all $n \geq 1$ and hence

$$\|x_{n+1} - p\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|Y_n x_n - p\| \leq \|x_n - p\|.$$

It follows that $\{\|x_n - p\|\}$ is nonincreasing; consequently, $\lim_{n\to\infty} \|x_n - p\|$ exists. Assume that $\|x_n - p\| > 0$. Since X is uniformly convex, it follows (see, for example, [4]) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \Big\{ 1 - 2\min\{\alpha_n, 1-\alpha_n\} \delta_X \Big(\frac{\|x_n - Y_n x_n\|}{\|x_n - p\|} \Big) \Big\},$$

which implies that

$$\begin{aligned} \alpha_n(1-\alpha_n)\|x_n - p\| \delta_X \Big(\frac{\|x_n - Y_n x_n\|}{\|x_n - p\|} \Big) &\leq \min\{\alpha_n, 1-\alpha_n\} \|x_n - p\| \delta_X \Big(\frac{\|x_n - Y_n x_n\|}{\|x_n - p\|} \Big) \\ &\leq \frac{1}{2}(\|x_n - p\| - \|x_{n+1} - p\|). \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, by the continuity of δ_X, we have

$$\lim_{n\to\infty} \|x_n - Y_n x_n\| = 0.$$

Since $\delta_{n,i} \to \lambda_i (i = 1, 2, \dots, \lambda_N)$, let the mapping $Y : X \to X$ be generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$. Then, by Lemma 3.2, we have $\lim_{n\to\infty} \|Y_n x - Yx\| = 0$ for all $x \in X$. So we have

$$\begin{aligned} \|x_n - Yx_n\| &\leq \|x_n - Y_n x_n\| + \|Y_n x_n - Yx_n\| \\ &\leq \|x_n - Y_n x_n\| + \sup_{z \in \{x_n\}} \|Y_n z - Yz\| \\ &\to 0. \end{aligned}$$

Since Y is nonexpansive adn X is uniformly convex, by the demiclosedness principle, $\omega_\omega(x_n) \subset F(Y)$. Morever, $F(Y) = \bigcap_{i=1}^N F(T_i)$ by Lemma 3.1 (i). Next, we show that $\omega_\omega(x_n)$ is a singleton. Indeed, suppose that $x^*, y^* \in \omega_\omega(x_n) \subset \bigcap_{i=1}^N F(T_i)$. Define $S_n; X \to X$ by

$$S_n x = (1-\alpha_n)x + \alpha_n Y_n x, \quad x \in X.$$

Then S_n is nonexpansive and $x^*, y^* \in \bigcap_{i=1}^\infty F(S_n)$. Using Lemma 2.4, we have $\lim_{n\to\infty} \langle x_n, J(x^* - y^*) \rangle$ exists. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. Then

$$\|x^* - y^*\|^2 = \langle x^* - y^*, J(x^* - y^*) \rangle = \lim_{k\to\infty} \langle x_{n_k} - x_{m_k}, J(x^* - y^*) \rangle = 0.$$

This shows that $x^* = y^*$. The proof is completes. □

4. Strong convergence in Banach spaces

In this section, strong convergence results for the iterative process (1.1) on strictly convex and reflexive Banach space having a uniformly Gáteaux differentiable norm involving the modified viscosity approximation method [8].

Theorem 4.1. *Let X be a strictly convex and reflexive Banach space having a uniformly Gáteaux differentiable norm. Let C be a nonempty, closed and convex subset of X and let $P : X \to C$ be a nonexpansive retraction of X onto C. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive non–self mappings of C into X such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=1}^N$ be a real sequence in $(0,1)$ such that $\lambda_{n,i} \to \lambda_i$ $(i = 1, 2, \ldots, N)$. For every $n \in \mathbb{N}$, let Y_n be the Y–mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequence in $(0,1)$ which satisfy the conditions:*

(A1) $\alpha_n + \beta_n + \gamma_n = 1$;
(A2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(A3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $f \in \sum_C$ and define the sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Y_n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to $q \in \cap_{i=1}^N F(T_i)$, where q also the unique solution of the variational inequality

$$\langle (I - f)(q), J(q - p)\rangle \leq 0, \quad \forall p \in \cap_{i=1}^N F(T_i). \tag{4.1}$$

Proof . We divide the proof into the following steps.
Step 1. We show that $\{x_n\}$ is bounded. Let $p \in \cap_{i=1}^N F(T_i)$. Then $p = Y_n p$ for all $n \geq 1$ and hence, by the nonexpansiveness of $\{Y_n\}_{n=1}^\infty$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(Y_n x_n - p)\| \\
&\leq \alpha_n\|(f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\
&\leq \alpha_n\|(f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\left\{\|x_n - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\right\}.
\end{aligned}$$

By induction, we can conclude that $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{Y_n x_n\}$.
Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To this end, we define $z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$. From $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Y_n x_n$, $\forall n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, and $f \in \sum_C$, we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \left\|\frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} Y_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n Y_n x_n}{1 - \beta_n}\right\| \\
&= \left\|\frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - Y_n x_n) + \frac{\alpha_n}{1 - \beta_n}(Y_n x_n - f(x_n))\right. \\
&\quad \left. + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(Y_{n+1} x_{n+1} - Y_n x_n)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}M + \frac{\alpha_n}{1-\beta_n}M + \|Y_{n+1}x_{n+1} - Y_nx_n\| \\
\leq\ & \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M + \|Y_{n+1}x_{n+1} - Y_nx_n\| \\
\leq\ & \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M + \|x_{n+1} - x_n\| + \|Y_{n+1}x_{n+1} - Y_nx_n\|
\end{aligned}
$$

for some $M > 0$. It turns out that

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M + \|Y_{n+1}x_n - y_nx_n\|.$$

From conditions $(A2), (A3)$ and Lemma 3.3, we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Lemma 2.3 yields that $\|z_n - x_n\| \to 0$ and hence

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|z_n - x_n\| \to 0.$$

Step 3. We show that $\lim_{n\to\infty}\|Yx_n - x_n\| = 0$. Indeed, noting that

$$Y_nx_n - x_n = \frac{1}{\gamma_n}(x_{n+1} - x_n) + \alpha_n(x_n - f(x_n)),$$

we have, by $(A2)$ and $(A3)$,

$$\lim_{n\to\infty}\|Y_nx_n - x_n\| = 0.$$

Let $Y : C \to C$ be the Y–mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. So, by Lemma 3.2, we have $Y_nx \to Yx$ for all $x \in C$. It also follows that

$$
\begin{aligned}
\|Yx_n - x_n\| &\leq \|Yx_n - Y_nx_n\| + \|Y_nx_n - x_n\| \\
&\leq \sup_{z\in\{x_n\}}\|Yz - Y_nz\| + \|Y_nx_n - x_n\| \to 0.
\end{aligned}
$$

For $t \in (0, 1)$, we define a contraction as follows:

$$S_tx = tf(x) + (1-t)Yx.$$

Then there exists a unique path $x_t \in C$ such that

$$x_t = tf(x_t) + (1-t)Yx_t.$$

From Lemma 2.4, we know that $x_t \to q$ as $t \to 0$, where $q \in F(Y)$. Lemma 3.1 (i) also yields that $q \in F(Y) = \cap_{i=1}^N F(T_i)$. Moreover, q is the unique solution of variational inequality (4.1).

Step 4. We show that $\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \leq 0$. We see that

$$x_t - x_n = (1-t)(Yx_t - x_n) + t(f(x_t) - x_n).$$

It follows, by Lemma 2.1 (ii) that

$$
\begin{aligned}
\|x_t - x_n\|^2 &\leq (1-t)^2\|Yx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\leq (1 - 2t + t^2)(\|x_t - x_n\| \\
&\quad + \|Yx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,
\end{aligned}
$$

which gives

$$\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq \frac{(1+t^2)\|x_n - Yx_n\|}{2t}(2\|x_t - x_n\| + \|x_n - Yx_n\|) + \frac{t}{2}\|x_t - x_n\|^2.$$

So we have

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq \frac{t}{2}M \tag{4.2}$$

for some $M > 0$. Since X has a uniformly Gáteaux differentiable norm, J is norm–to–weak* uniformly continuous in bounded subsets of E. So have

$$\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \to 0 \tag{4.3}$$

and

$$\langle f(q) - f(x_t) + x_t - q, J(x_n - x_t)\rangle \to 0 \tag{4.4}$$

as $t \to 0$. On the other hand, we have

$$\begin{aligned}\langle f(q) - q, J(x_n - q)\rangle &= \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \langle f(q) - f(x_t) + x_t - q, J(x_n - x_t)\rangle \\ &\quad + \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.\end{aligned} \tag{4.5}$$

Since $\limsup_{n\to\infty}$ and $\limsup_{t\to 0}$ are interchangeable, using (4.2)–(4.5), we obtain

$$\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \leq 0.$$

Step 5. We show that $x_n \to q$ as $n \to \infty$. In fact, we have

$$\begin{aligned}\|x_{n+1}\| &= \alpha_n\langle f(x_n) - q, J(x_{n+1} - q)\rangle + \beta_n\langle x_n - q, J(x_{n+1} - q)\rangle + \gamma_n\langle Y_n x_n - q, J(x_{n+1} - q)\rangle \\ &\leq \alpha_n\alpha\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\quad + \beta_n\|x_n - q\|\|x_{n+1} - q\| + \gamma_n\|x_n - q\|\|x_{n+1} - q\| \\ &= (1 - \alpha_n(1-\alpha))\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\leq \frac{1}{2}(1 - \alpha_n(1-\alpha))(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \frac{1 - \alpha_n(1-\alpha)}{1 + \alpha_n(1-\alpha)}\|x_n - q\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1-\alpha)}\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &= \left(1 - \frac{2\alpha_n(1-\alpha)}{1 + \alpha_n(1-\alpha)}\right)\|x_n - q\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1-\alpha)}\langle f(q) - q, J(x_{n+1} - q)\rangle.\end{aligned}$$

Put $c_n = \frac{2\alpha_n(1-\alpha)}{1+\alpha_n(1-\alpha)}$ and $b_n = \frac{2\alpha_n}{1+\alpha_n(1-\alpha)}\langle f(q) - q, J(x_{n+1} - q)\rangle$. So it is easy to check that $\{c_n\}$ is a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} c_n = \infty$ and $\limsup_{n\to\infty}\frac{b_n}{c_n} \leq 0$. Hence, By Lemma 2.6, we conclude that $x_n \to q$ as $n \to \infty$. The proof is completes. □

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