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Common fixed point of multivalued graph contraction in metric spaces

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Abstract

In this paper, we introduce the (G-ψ) contraction in a metric space by using a graph. Let F, T be two multivalued mappings on X. Among other things, we obtain a common fixed point of the mappings F, T in the metric space X endowed with a graph G.

Keywords: fixed point, multivalued; common(G-ψ) contraction; directed graph.
2010 MSC: Primary 47H10; Secondary 47H09.

1. Introduction and preliminaries

For a given metric space (X, d), let T denotes a selfmap. According to Petrusel and Rus [9], T is called a Picard operator (abbr., PO) if it has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$, for all $x \in X$, and is a weakly Picard operator (abbr.WPO) if for all $x \in X$, $\lim_{n\longrightarrow\infty}(T^n x)$ exists (which may depend on x) and is a fixed point of T. Let (X, d) be a metric space and G be a directed graph with set $V(G)$ of its vertices coincides with X, and the set of its edges $E(G)$ is such that $(x, x) \notin E(G)$. Assume G has no parallel edges, we can identify G with the pair $(V(G), E(G))$, and treat it as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of the edges. Thus we can write

$$E(G^{-1}) = \{(x, y)|(y, x) \in E(G)\}. \tag{1.1}$$

Let $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually,it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}). \tag{1.2}$$

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We point out the followings:
(i) $G^{'} = (V', E')$ is called a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$ and for all $(x, y) \in E'$, $x, y \in V'$.
(ii) If x and y are vertices in a graph G, then a path in G from x to y of length N $(N \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$.
(iii) Graph G is connected if there is a path between any two vertices, and is weakly connected if $\tilde{G}$ is connected.
(iv) Assume that G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting x is called component of G, if it consists all edges and vertices which are contained in some path beginning at x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule: $y\text{R}z$ if there is a path in G from y to z. Clearly, G_x is connected.
(v) The sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, included in X, are Cauchy equivalent if each of them is a Cauchy sequence and $d(x_n, y_n) \longrightarrow 0$.
Let (X, d) be a complete metric space and let $CB(X)$ be a class of all nonempty closed and bounded subset of X. For $A, B \in CB(X)$, let

$$H(A, B) := \max\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\},$$

where

$$d(a, B) := \inf_{b\in B} d(a, b).$$

Mapping H is said to be a Hausdorff metric induced by d.

Definition 1.1. Let $T : X \longrightarrow CB(X)$ be a mappings, a point $x \in X$ is said to be a fixed point of the set-valued mapping T if $x \in T(x)$

Definition 1.2. A metric space (X, d) is called a $\epsilon-$chainable metric space for some $\epsilon > 0$ if given $x, y \in X$, there is an $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^{n}$ such that
$x_0 = x \quad x_n = y \quad \text{and} \quad d(x_{i-1}, x_i) < \epsilon \quad \text{for} \quad i = 1, \ldots, n.$

Property A ([6]). For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

Lemma 1.3. ([1]). Let (X, d) be a complete metric space and $A, B \in CB(X)$. Then for all $\epsilon > 0$ and $a \in A$ there exists an element $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.

Lemma 1.4. ([1]). Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\longrightarrow\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\longrightarrow\infty} d(x_n, x) = 0$, then $x \in A$.

Lemma 1.5. Let $A, B \in CB(X)$ with $H(A, B) < \epsilon$, then for each $a \in A$ there exists an element $b \in B$ such that $d(a, b) < \epsilon$.

Definition 1.6. Let us define the class $\Psi = \{\psi : [0, +\infty) \longrightarrow [0, +\infty) |\ \psi \text{ is nondecreasing}\}$ which satisfies the following conditions:
(i) for every $(t_n) \in \mathbb{R}^+$, $\psi(t_n) \longrightarrow 0$ *if and only if* $t_n \longrightarrow 0$;
(ii) for every t_1,$t_2 \in \mathbb{R}^+$, $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$;
(iii) for any $t > 0$ we have $\psi(t) \leq t$.

Lemma 1.7. Let $A, B \in CB(X)$, $a \in A$ and $\psi \in \Psi$. Then for each $\epsilon > 0$, there exists $b \in B$ such that $\psi(d(a, b)) \leq \psi(H(A, B)) + \epsilon$.

2. Main results

We begin with the following theorem the gives the existence of a fixed point for multivalued mappings(not necessarily unique) in metric spaces endowed with a graph.

Definition 2.1. Let (X, d) be a complete metric space and $F, T : X \longrightarrow CB(X)$ be a mappings, F and T are said to be a common(G-ψ) contraction if there exists $k \in (0, 1)$ such that

$$\psi(H(F(x), T(y)) \leq k\psi(d(x, y)) \ for \ all \ (x, y) \in E(G), (x \neq y) \tag{2.1}$$

and for all $(x, y) \in E(G)$ if $u \in F(x)$ and $v \in T(y)$ are such that
$\psi(d(u, v)) \leq k\psi(d(x, y)) + \epsilon$, for each $\epsilon > 0$ then $(u, v) \in E(G)$.

Theorem 2.2. Let (X, d) be a complete metric space and suppose that the triple (X, d, G) have the property A. Let $F, T : X \longrightarrow CB(X)$ be a (G-ψ) contraction and $X_F = \{x \in X : (x, u) \in E(G) \ for \ some \ u \in F(x)\}$. Then the following statements hold.
1. for any $x \in X_F$, $F, T|_{[x]_G}$ have a common fixed point.
2. If $X_F \neq \emptyset$ and G is weakly connected, then F,T have a common fixed point in X.
3. If $X' := \cup\{[x]_G : x \in X_F\}$, then $F, T|_{X'}$ have a common fixed point.
4. If $F \subseteq E(G)$, then F,T have a common fixed point.

Proof . Let $x_0 \in X_F$, then there is an $x_1 \in F(x_0)$ for which $(x_0, x_1) \in E(G)$. Since F, T are (G-ψ) contraction, we should have
$\psi(H(F(x_0), T(x_1))) \leq k\psi d(x_0, x_1)$.
By Lemma 1.7, it ensures that there exists an $x_2 \in T(x_1)$ such that

$$\psi(d(x_1, x_2) \leq \psi(H(F(x_0), T(x_1))) + k \leq k\psi d(x_0, x_1) + k. \tag{2.2}$$

Using the property of F, T being a (G-ψ) contraction $(x_1, x_2) \in E(G)$, since $E(G)$ is symmetric we obtain
$\psi(H(F(x_2), T(x_1))) \leq k\psi d(x_1, x_2)$
and then by Lemma 1.7 shows the existence of an $x_3 \in F(x_2)$ such that

$$\psi(d(x_2, x_3)) \leq \psi(H(T(x_1), F(x_2))) + k^2. \tag{2.3}$$

By inequality (2.2), (2.3), it results

$$\psi(d(x_2, x_3)) \leq k\psi(d(x_1, x_2)) + k^2 \leq k^2\psi(d(x_0, x_1)) + 2k^2. \tag{2.4}$$

By a similar approach, we can prove that $x_{2n+1} \in F(x_{2n})$ and $x_{2n+2} \in T(x_{2n+1})$, $n := 0, 1, 2, \cdots$ as well as $(x_n, x_{n+1}) \in E(G)$ and
$\psi(d(x_n, x_{n+1})) \leq k^n\psi(d(x_0, x_1)) + nk^n$.

We can easily show by following that (x_n) is a Cauchy sequence in X.

$$\sum_{n=0}^{\infty} \psi(d(x_n, x_{n+1})) \leq \psi(d(x_0, x_1)) \sum_{n=0}^{\infty} k^n + \sum_{n=0}^{\infty} nk^n < \infty,$$

since $\sum_{n=0}^{\infty} \psi(d(x_n, x_{n+1})) < \infty$, and $\psi(d(x_n, x_{n+1})) \longrightarrow 0$; consequently using the property of ψ we have $d(x_n, x_{n+1}) \longrightarrow 0$.

Hence (x_n) converges to some point x in X. Next step is to show that x is a common fixed point of the mapping F and T. Using the property A and the fact of F, T being a (G-ψ) contraction, since $(x_n, x) \in E(G)$, then we encounter with the following two cases:
Case 1 : for even values of n, we have

$$\psi(H(F(x_n), T(x)) \leq k\psi(d(x_n, x)).$$

Since $x_{n+1} \in F(x_n)$ and $x_n \longrightarrow x$, then by Lemma 1.4, $x \in T(x)$.
Case 2 : for odd values of n, we have

$$\psi(H(F(x), T(x_n)) \leq k\psi(d(x, x_n)).$$

Since $x_{n+1} \in T(x_n)$ and $x_n \longrightarrow x$, then by Lemma 1.4, $x \in F(x)$. Hence from $(x_n, x_{n+1}) \in E(G)$, and $(x_n, x) \in E(G)$, for $n \in \mathbb{N}$, we conclude that $(x_0, x_1, ..., x_n, x)$ is a path in G and so $x \in [x_0]_G$.

2. For $X_F \neq \emptyset$, there exists an $x_0 \in X_F$, and since G is weakly connected, then $[x_0]_G = X$ and by 1, F and T have a common fixed point.

3. From 1 and 2, the following result is now immediate.

4. $F \subseteq E(G)$ implies that all $x \in X$ are such that there exists some $u \in F(x)$ with $(x, u) \in E(G)$, so $X_F = X$ by 2 and 3. F, T have a common fixed point. □

See the following example.

Example 2.3. Let $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$. Consider the undirected graph G such that $V(G) = X$ and $E(G) = \{(\frac{1}{2^n}, 0), (0, \frac{1}{2^n}), (\frac{1}{2^n}, \frac{1}{2^{n+1}}), (\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n \in \{2, 3, 4, \cdots\}\} \cup \{(\frac{1}{2}, 0), (0, \frac{1}{2}), (1, 0), (0, 1)\}$. Let $F, T : X \longrightarrow CB(X)$ be defined by

$$F(x) = \begin{cases} 0 & x = 0, \\ \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\} & x = \frac{1}{2^n}, n \in \{2, 3, 4, \cdots\}, \\ \frac{1}{4} & x = 1, \frac{1}{2}. \end{cases} \tag{2.5}$$

$$T(y) = \begin{cases} 0 & y = 0, \\ \{\frac{1}{2^{n+1}}\} & y = \frac{1}{2^n}, n \in \{2, 3, 4, \cdots\}, \\ \frac{1}{4} & y = 1, \frac{1}{2}. \end{cases} \tag{2.6}$$

Then F, T are not a common(G-ψ) contraction where $d(x, y) = |x - y|$ and $\psi(t) = \frac{t}{t+1}$. It can be seen that if $x = \frac{1}{8}$ and $y = \frac{1}{4}$, then $T(y) = \{\frac{1}{8}\}$, $F(x) = \{\frac{1}{16}, \frac{1}{32}\}$, then we have

$$\psi(H(F(x), T(y)) \leq k\psi(d(x, y)) \ for \ all \ (x, y) \in E(G), (x \neq y),$$

and let $u = \frac{1}{32}$ and $v = \frac{1}{8}$, where $x = \frac{1}{8}$ and $y = \frac{1}{4}$, therefore $d(\frac{1}{32}, \frac{1}{8}) = \frac{3}{32}$, and $\psi(d(\frac{1}{32}, \frac{1}{8})) = \frac{3}{35}$, also we have $d(\frac{1}{8}, \frac{1}{4}) = \frac{1}{8}$, so $\psi(d(\frac{1}{8}, \frac{1}{4})) = \frac{1}{9}$. Thus there exists $k \in (0, 1)$ such that $\psi(d(\frac{1}{32}, \frac{1}{8})) \leq k\psi(d(\frac{1}{8}, \frac{1}{4})) + \epsilon$, for all $\epsilon > 0$, but $(\frac{1}{8}, \frac{1}{32}) \notin E(G)$.

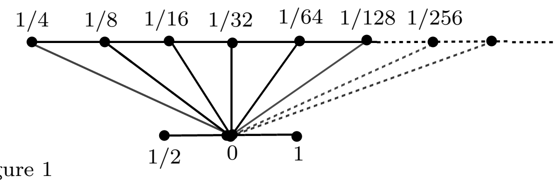

figure 1

Example 2.4. Let $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$. Consider the undirected graph G such that $V(G) = X$ and $E(G) = \{(\frac{1}{2^n}, 0), (0, \frac{1}{2^n}), (\frac{1}{2^n}, \frac{1}{2^{n+1}}), (\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n \in \mathbb{N}\} \cup \{(1, 0), (0, 1)\}$.
Let $F, T : X \longrightarrow CB(X)$ be defined by

$$F(x) = \begin{cases} 0 & x = 0, \frac{1}{2}, \\ \{\frac{1}{2}, \frac{1}{4}\} & x = 1, \\ \{\frac{1}{2^{n+1}}\} & x = \frac{1}{2^n}, n \in \{2, 3, 4, \cdots\}. \end{cases} \tag{2.7}$$

$$T(y) = \begin{cases} 0 & y = 0, \frac{1}{2}, \\ \{\frac{1}{8}, \frac{1}{16}\} & y = 1, \\ \{\frac{1}{2^{n+1}}\} & y = \frac{1}{2^n}, n \in \{2, 3, 4, \cdots\}. \end{cases} \tag{2.8}$$

Then F, T are a common(G-ψ) contraction and $0 \in F(0) \cap T(0)$, where $d(x, y) = |x - y|$ and $\psi(t) = \frac{t}{t+1}$.
PropertyA': For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \longrightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $n_k \in \mathbb{N}$. If We have property A', then improve the result of this paper as follows:

Theorem 2.5. Let (X, d) be a complete metric space and suppose that the triple (X, d, G) have the property A'. Let $F, T : X \longrightarrow CB(X)$ be a (G-ψ) contraction and $X_F = \{x \in X : (x, u) \in E(G)\ for\ some\ u \in F(x)\}$. Then the following statements hold.
1. for any $x \in X_F$, $F|_{[x]_G}$ has a fixed point.
2. If $X_F \neq \emptyset$ and G is weakly connected, then F, T have a fixed point in X.
3. If $X' := \cup\{[x]_G : x \in X_F\}$, then $F, T|_{X'}$ have a common fixed point.
4. If $F \subseteq E(G)$, then F,T have a common fixed point.

Corollary 2.6. Let (X, d) be a complete metric space and suppose that the triple (X, d, G) have the property A. If G is weakly connected, then (G-ψ) contraction mappings $F, T : X \longrightarrow CB(X)$ such that $(x_0, x_1) \in E(G)$ for some $x_1 \in Fx_0$ have a common fixed point.

Corollary 2.7. *Let (X, d) be a $\epsilon-$chainable complete metric space for some $\epsilon > 0$. Let $\psi \in \Psi$ and assume that $F, T : X \longrightarrow CB(X)$ be a such that there exists $k \in (0, 1)$ with*

$$0 < d(x, y) < \epsilon \Longrightarrow \psi(H(F(x), T(y)) \leq k\psi(d(x, y)).$$

Then T, F have a common fixed point.

Proof . Consider the G as $V(G) = X$ and

$$E(G) := \{(x, y) \in X \times X : 0 < d(x, y) < \epsilon\}.$$

The $\epsilon-$chainability of (X, d) means G is connected. If $(x, y) \in E(G)$, then

$$\psi(H(F(x), T(y)) \leq k\psi(d(x, y)) < \psi(d(x, y)) \leq d(x, y) < \epsilon$$

and by using Lemma 1.5 for each $u \in F(x)$, we have the existence of $v \in T(y)$ such that $d(u, v) < \epsilon$, which implies $(u, v) \in E(G)$. Therefore F, T are (G-ψ) contraction mappings. Also, (X, d, G) has property A. Indeed, if $x_n \longrightarrow x$ and $d(x_n, x_{n+1}) < \epsilon$ for $n \in \mathbb{N}$, then $d(x_n, x) < \epsilon$ for sufficiently large n, hence $(x_n, x) \in E(G)$. So, by Theorem 2.2, F, and T have a common fixed point. □

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