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Convergence Theorems of Iterative Approximation for Finding Zeros of Accretive Operator and Fixed Points Problems

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Abstract

In this paper we propose and studied a new composite iterative scheme with certain control conditions for viscosity approximation for a zero of accretive operator and fixed points problems in a reflexive Banach space with weakly continuous duality mapping. Strong convergence of the sequence $\{x_n\}$ defined by the new introduced iterative sequence is proved. The main results improve and complement the corresponding results of [1, 4, 10].

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1. Introduction and preliminaries

Let C be a nonempty closed convex subset of a real Banach space E with E^* be dual space of E and the value of $x^* \in E^*$ will be denoted by $\langle x^*, x\rangle$. The *normalized duality mapping* J from E into the family of nonempty w^*-compact subsets of its dual E^* is defined by

$$J(x) = \{x^* \in E^* \ : \ \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\} \tag{1.1}$$

for each $x \in E$ [5]. Recall that a mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and a self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T)$, that is $F(T) = \{x \in C \mid x = Tx\}$ and we use Π_C to denote the collection of all contractions on C, that is $\Pi_C = \{f : C \to C \mid f \ \textit{ is a contraction with a constant } \alpha\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C, and for any fixed element $x_0 \in C$, Picard's iteration

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$x_{n+1} = f^n(x_0)$ converges strongly to a unique fixed point of f. However, a simple example shows that Picard's iteration cannot be used in the case of nonexpansive mappings.

An operator $A : E \to E$ is said to be *accretive* if for each $(x_1, y_1), (x_2, y_2) \in Gph(A)$ there exists a $j \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j\rangle \geq 0$. An accretive operator A is *m-accretive* if $R(I + rA) = E$ for each $r \geq 0$. The set of zeros of A is denoted by $N(A) = A^{-1}(0) = \{z \in D(A) \ : \ 0 \in Az\}$ it is always assumed that A is accretive and $N(A)$ is nonempty. For each $r \geq 0$, we denote by J_r the *resolvent* of A, that is $J_r = (I + rA)^{-1}$. Note that, if A is m-accretive, then $J_r : E \to E$ is a nonexpansive mapping and $F(J_r) = N(A)$ for all $r \geq 0$.

In 2008 Jung [9] introduced a new composite iterative scheme for a nonexpansive mapping T as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 1, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $f \in \Pi_C$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. He proved the strong convergence of the sequence $\{x_n\}$ defined by (1.2) under suitable conditions of the control parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and asymptotic regularity on $\{x_n\}$ in a reflexive Banach space with a uniformly Gateaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

On the other hand, He, Xu and He [8] introduced an iteration scheme for viscosity approximation for a zero of accretive operator and fixed points problems in a reflexive Banach space with weakly continuous duality mapping as follows:

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n TJ_{r_n} x_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $f \in \Pi_C$, J_{r_n} is the resolvent of A and T is nonexpansive mapping. They proved that $\{x_n\}$ strongly convergence to a zero of accretive operator and fixed points problems under some control conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$.

In this paper, inspired and motivated by the above iterative schemes, we introduced and studied a new composite iterative scheme as follows:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)TJ_{r_n} x_n, \\ x_{n+1} = \beta_n Ty_n + (1-\beta_n)y_n, \end{cases} \tag{1.4}$$

where $f \in \Pi_C$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, J_{r_n} is the resolvent of A and T is nonexpansive mapping. The main results improve and complement the corresponding results of [1, 4, 10].

By a gauge function φ we mean a continuous strictly increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to +\infty$. Let E^* be the dual space of E. The *duality mapping* $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \{x^* \in E^* \ : \ \langle x^*, x\rangle = \|x\|\varphi(\|x\|), \ \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in E.$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x)$ for all $x \neq 0$. Browder [2] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [2], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < +\infty$. Set

$$\phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \forall t \geq 0, \tag{1.5}$$

then

$$J_\varphi(x) = \partial\phi(\|x\|), \quad \forall x \in E, \tag{1.6}$$

where ∂ denotes the subdifferential in the sense of convex analysis that for each $x \in X$ such that $f(x) \in \mathbb{R}$, the *subdifferential* of f at x defined by $\partial f(x) = \{x^* \in X^* \mid f(y) - f(x) \geq \langle x^*, y - x\rangle \forall x \in X\}$. The next lemma is an immediate consequence of the subdifferential inequality.

Lemma 1.1. *Assume that E has a weakly continuous duality map J_φ with gauge φ. Then, for each $x, y \in E$, one has*

$$\phi(\| x + y\|) \leq \phi(\| x\|) + \langle y, J_\varphi(x + y)\rangle. \tag{1.7}$$

Lemma 1.2. [11] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \sigma_n\gamma_n,\ n \geq 1$, where $\{\gamma_n\} \subseteq (0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\lim\limits_{n\to\infty} \gamma_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup\limits_{n\to\infty} \sigma_n \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\sigma_n\gamma_n| < \infty$.

Then $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 1.3. [3] *For $\lambda > 0$, $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right).$$

Lemma 1.4. [6] *Let E be a reflexive Banach space, C a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive mapping. Suppose that E admits a weakly sequentially continuous duality mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping, i.e., if $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, then $x = Tx$.*

Let D be a subset of C. Then $Q : C \to D$ is called a *retraction* from C onto D if $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$ whenever $Qx + t(x - Qx) \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is known ([7] p. 48) that $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Q(x), J(z - Q(x))\rangle \leq 0 \quad x \in C,\ x \in D.$$

Lemma 1.5. [12] *Let E be a reflexive Banach space and have a weakly continuous duality map J with gauge φ. Let C be a closed convex subset of E and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C to equation $x_t = tu + (1 - t)Tx_t$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T. If we define $Q : C \to F(T)$ by $Q(u) := \lim\limits_{t\to 0} x_t,\ u \in C$, then $Q(u)$ solves the variational inequality*

$$\langle u - Q(u), J(Q(u) - p)\rangle \leq 0 \quad u \in C,\ p \in F(T).$$

where Q is the sunny nonexpansive retraction from C onto $F(T)$.

Lemma 1.6. [8] *Let E be a reflexive Banach space and have a weakly continuous duality map J with gauge φ. Let C be a closed convex subset of E and let $T : C \to C$ be a nonexpansive mapping, $f \in \Pi_C$. Let $z_t \in C$ be the unique solution in C to equation $z_t = tf(z_t) + (1-t)Tz_t,\ t \in (0,1)$. Then T has a fixed point if and only if $\{z_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{z_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T. If we define $Q : \Pi_C \to F(T)$ by $Q(f) := \lim_{t\to 0} z_t,\ f \in \Pi_C$; then $Q(f)$ is a solution of the variational inequality*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad p \in F(T),$$

where Q is the sunny nonexpansive retraction from C onto $F(T)$.

2. Main Results

In this section, we prove several strong convergence theorems of the iterative scheme (1.4).

Theorem 2.1. *Let E be a real reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ and A a m-accretive maps in E such that $C = \overline{D(A)}$ is convex. let $T : C \to C$ be a nonexpansive mapping with $F = F(T) \cap N(A) \neq \emptyset$ and $f : C \to C$ a fixed contraction mapping with contract constant α. Suppose that $\{\alpha_n\},\ \{\beta_n\} \subset (0,1)$, $r_n \in \mathbb{R}^+$ which satisfy the following conditions:*

(C1) $\lim_{\mathrm{n}\to\infty} \alpha_\mathrm{n} = 0,\ \sum_{\mathrm{n}=1}^{\infty} \alpha_\mathrm{n} = \infty,$

(C2) $\lim_{\mathrm{n}\to\infty} \frac{\beta_\mathrm{n}}{\alpha_\mathrm{n}} = 0,$

(C3) $\lim_{\mathrm{n}\to\infty} \mathrm{r_n} = \mathrm{r}, \mathrm{r} \in \mathbb{R}^+.$

Let $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (1.4) Suppose that $\sum_{n=1}^{\infty} \sup\{\|TJ_{r_{n+1}}z - TJ_{r_n}z\|\ ;\ z \in B\} < \infty$ for any bounded subset B of C. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to $p \in F$, where p is the unique solution of the variational inequality

$$\langle (I-f)(p), J(p-q)\rangle \leq 0, \quad q \in F. \tag{2.1}$$

Proof . First, we note that by Lemma 1.6 with the contraction f and $TJ_{r_n} : E \to C$ nonexpansive mapping instead of a mapping T, there exists the unique solution p of a variational inequality

$$\langle (I-f)(p), J(p-q)\rangle \leq 0, \quad q \in F.$$

where $p = \lim_{t\to 0} z_t$ and z_t is defined by $z_t = tf(z_t) + (1-t)\,TJ_r(z_t)$ for each $r > 0$ and $0 < t < 1$. Second, we claim that $\{x_n\}$ is bounded. Indeed, take an arbitrary fixed $p \in F$ so using the definition of $\{x_n\}$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\beta_n)y_n + \beta_n Ty_n - p\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n\|Ty_n - p\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n\|y_n - p\| \\
&= \|y_n - p\|.
\end{aligned}$$

and hence by the definition of $\{y_n\}$, we obtain

$$\begin{aligned}\|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)TJ_{r_n}x_n - p\| \\ &= \|\alpha_n(f(x_n) - f(p)) + \alpha_n(f(p) - p) + (1-\alpha_n)(TJ_{r_n}x_n - p)\| \\ &\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|TJ_{r_n}x_n - p\| \\ &\le \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= (1-(1-\alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\le \max\{\| x_n - p\| + \frac{1}{1-\alpha}\|f(p) - p\|\}.\end{aligned}$$

By induction on n, we obtain that $\|x_n - p\| \le \max\{\frac{\|f(p)-p\|}{1-\alpha}, \|x_1 - p\|\}$ for all $n \in \mathbb{N}$ and all $p \in F(T)$. Hence, the sequence $\{x_n\}$ is bounded and so $\{y_n\}$, $\{Tx_n\}$, and $\{f(x_n)\}$ are bounded sequences. From (C2), we can assume, without loss of generality, that $\beta_n \le \alpha_n$ for each $n \ge 1$. By (C1) and the definition of $\{x_n\}$, we have

$$\|x_{n+1} - y_n\| = \beta_n \|Ty_n - y_n\| \to 0,\ n \to \infty,$$

and hence asymptotic regularity of $\{x_n\}$ implies that

$$\| x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0,\ n \to \infty. \tag{2.2}$$

Then, from (C1) and (2.2) we obtain

$$\begin{aligned}\|y_n - TJ_{r_n}y_n\| &= \| \alpha_n f(x_n) + (1-\alpha_n)TJ_{r_n}x_n - TJ_{r_n}y_n\| \\ &= \| \alpha_n (f(x_n) - TJ_{r_n}x_n) + TJ_{r_n}x_n - TJ_{r_n}y_n\| \\ &\le \alpha_n \| f(x_n) - TJ_{r_n}x_n\| + \| x_n - y_n\| \to 0\ , n \to \infty.\end{aligned} \tag{2.3}$$

From Lemma 1.3 and (C3), we get

$$\begin{aligned}\| TJ_{r_n}y_n - TJ_r y_n\| &\le \| J_{r_n}y_n - J_r y_n\| \\ &= \left\| J_r\left(\frac{r}{r_n}y_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}y_n\right) - J_r y_n\right\| \\ &\le \left\|\left(\frac{r}{r_n}y_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}y_n\right) - y_n\right\| \\ &= \left|1 - \frac{r}{r_n}\right| \|J_{r_n}y_n - y_n\| \to 0, \quad n \to \infty.\end{aligned} \tag{2.4}$$

Therefore, (2.3) and (2.4) imply that

$$\|y_n - TJ_r y_n\| \le \|y_n - TJ_{r_n}y_n\| + \| TJ_{r_n}y_n - TJ_r y_n\| \to 0 \quad , \quad n \to \infty.$$

Now, we prove that

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(y_n - p)\rangle \le 0, \qquad p \in F. \tag{2.5}$$

Take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(y_n - p)\rangle = \lim_{k\to\infty} \langle f(p) - p, J_\varphi(y_{n_k} - p)\rangle .$$

Since E is reflexive, we may further assume that $y_{n_k} \rightharpoonup \overline{y}$. Moreover, since $\|y_n - TJ_r y_n\| \to 0$ and demicloseness of $I - TJ_r y_n$ and using Lemma 1.4 we know that $\overline{y} \in F(TJ_r)$. Hence, by Lemma 1.5 we get

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(y_n - p)\rangle = \langle f(p) - p, J_\varphi(\overline{y} - p)\rangle \leq 0.$$

Finally, we claim that $\{x_n\}$ strongly convergence to p. Indeed, we have

$$\begin{aligned}
\phi(\|y_n - p\|) &= \phi(\|\alpha_n f(x_n) + (1-\alpha_n) TJ_{r_n} x_n - p\|) \\
&= \phi(\|\alpha_n (f(x_n) - f(p)) + \alpha_n (f(p) - p) + (1-\alpha_n)(TJ_{r_n} x_n - p)\|) \\
&\leq \phi(\alpha_n \| f(x_n) - f(p)\| + \alpha_n \| f(p) - p\| + (1-\alpha_n) \| TJ_{r_n} x_n - p\|) \\
&\leq \phi(\alpha\alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n) \|TJ_{r_n} x_n - p\|) \\
&\leq \phi(\alpha\alpha_n \|x_n - p\| + (1-\alpha_n) \|x_n - p\|) + \alpha_n \langle f(p) - p, J_\varphi(y_n - p)\rangle \\
&= (1 - (1-\alpha)\alpha_n) \phi(\|x_n - p\|) + \alpha_n \langle f(p) - p, J_\varphi(y_n - p)\rangle
\end{aligned} \tag{2.6}$$

and also

$$\begin{aligned}
\phi(\|x_{n+1} - p\|) &= \phi(\|(1-\beta_n) y_n + \beta_n T y_n - p\|) \\
&= \phi(\|\beta_n (Ty_n - p) + (1-\beta_n)(y_n - p)\|) \\
&= \phi(\|\beta_n (Ty_n - T(p)) + \beta_n (T(p) - p) + (1-\beta_n)(y_n - p)\|) \\
&\leq \phi(\|\beta_n (Ty_n - T(p)) + (1-\beta_n)(y_n - p)\|) + \beta_n \langle T(p) - p, J_\varphi(x_{n+1} - p)\rangle \\
&\leq \phi(\beta_n \|y_n - p\| + (1-\beta_n)\|y_n - p\|) + \beta_n \langle T(p) - p, J_\varphi(x_{n+1} - p)\rangle \\
&= \phi(\|y_n - p\|) + \beta_n \langle T(p) - p, J_\varphi(x_{n+1} - p)\rangle .
\end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7), we obtain

$$\begin{aligned}
\phi(\|x_{n+1} - p\|) &\leq (1 - \alpha_n(1-\alpha)) \phi(\|x_n - p\|) + \alpha_n \langle f(p) - p, J_\varphi(y_n - p)\rangle \\
&\quad + \beta_n \langle T(p) - p, J_\varphi(x_{n+1} - p)\rangle \\
&= (1 - \alpha_n(1-\alpha)) \phi(\|x_n - p\|) \\
&\quad + \alpha_n (1-\alpha) \left[\frac{\langle f(p) - p, J_\varphi(y_n - p)\rangle}{1-\alpha} + \frac{\beta_n}{\alpha_n} \frac{\langle T(p) - p, J_\varphi(x_{n+1} - p)\rangle}{1-\alpha} \right] \\
&= (1-\gamma_n)\phi(\|x_n - p\|) + \sigma_n \gamma_n,
\end{aligned}$$

where $\gamma_n = \alpha_n(1-\alpha)$ and $\sigma_n = \left[\frac{\langle f(p)-p, J_\varphi(y_n-p)\rangle}{1-\alpha} + \frac{\beta_n}{\alpha_n}\frac{\langle T(p)-p, J_\varphi(x_{n+1}-p)\rangle}{1-\alpha}\right]$. Then, (C2) and (2.5) imply that

$$\begin{aligned}
\limsup_{n\to\infty} \sigma_n &\leq \limsup_{n\to\infty} \frac{\langle f(p) - p, J_\varphi(y_n - p)\rangle}{1-\alpha} + \limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \frac{\langle T(p) - p, J_\varphi(x_{n+1} - p)\rangle}{1-\alpha} \\
&\leq \limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \frac{\|T(p) - p\| \|x_{n+1} - p\|}{1-\alpha} = 0,
\end{aligned}$$

and using Lemma 1.2, $\{x_n\}$ convergence strongly to $p \in F$. □

Theorem 2.2. *Let E be a real reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ and A a m-accretive maps in E such that $C = \overline{D(A)}$ is convex. Let $T : C \to C$ be a nonexpansive mapping with $F = F(T) \cap N(A) \neq \emptyset$ and $f : C \to C$ a fixed contraction mapping with* *contract constant α. Suppose that $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, $r_n \in \mathbb{R}^+$ which satisfy in conditions* (C1), (C2), (C3) and

$$\text{(C4)} \sum_{\mathrm{n}=1}^{\infty} |\alpha_{\mathrm{n}+1} - \alpha_{\mathrm{n}}| < \infty,$$

$$\text{(C5)} \sum_{\mathrm{n}=1}^{\infty} |\beta_{\mathrm{n}+1} - \beta_{\mathrm{n}}| < \infty.$$

Let $x_1 \in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by (1.4). Suppose that $\sum_{n=1}^{\infty} \sup\{\|TJ_{r_{n+1}}z - TJ_{r_n}z\| \ ; \ z \in B\} < \infty$ for any bounded subset B of C, then $\{x_n\}$ converges strongly to $p \in F$, where p is the unique solution of the variational inequality (2.1).

Proof . From the definition of $\{y_n\}$ for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\alpha_n f(x_n) + (1-\alpha_n) TJ_{r_n} x_n - \alpha_{n-1} f(x_{n-1}) \ - (1-\alpha_{n-1}) TJ_{r_{n-1}} \, x_{n-1}\| \\
&\le \alpha_n \|f(x_n) - f(x_{n-1})\| + (1-\alpha_n) \left\|TJ_{r_n} x_n - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\quad + \left\|f(x_{n-1})(\alpha_n - \alpha_{n-1}) - (\alpha_n - \alpha_{n-1}) TJ_{r_{n-1}} x_{n-1}\right\| \\
&\le \alpha\alpha_n \|x_n - x_{n-1}\| + (1-\alpha_n) \left\|TJ_{r_n} x_n - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1}) - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\le \alpha\alpha_n \|x_n - x_{n-1}\| + (1-\alpha_n) \|TJ_{r_n} x_n - TJ_{r_n} x_{n-1}\| \\
&\quad + (1-\alpha_n) \left\|TJ_{r_n} x_{n-1} - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1}) - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\le \alpha\alpha_n \|x_n - x_{n-1}\| + (1-\alpha_n) \|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n) \left\|TJ_{r_n} x_{n-1} - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1}) - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\le (1 - (1-\alpha)\alpha_n) \|x_n - x_{n-1}\| + \left\|TJ_{r_n} x_{n-1} - TJ_{r_{n-1}} x_{n-1}\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1}) - TJ_{r_{n-1}} x_{n-1}\right\|,
\end{aligned} \tag{2.8}$$

and from the definition of $\{x_n\}$ for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|((1-\beta_n) y_n + \beta_n T y_n) - ((1-\beta_{n-1}) y_{n-1} + \beta_{n-1} T y_{n-1})\| \\
&\le (1-\beta_n) \|y_n - y_{n-1}\| + \beta_n \|Ty_n - Ty_{n-1}\| \\
&\quad + \|(\beta_{n-1} - \beta_n) y_{n-1} + Ty_{n-1}(\beta_n - \beta_{n-1})\| \\
&= (1-\beta_n) \|y_n - y_{n-1}\| + \beta_n \|Ty_n - Ty_{n-1}\| + |\beta_n - \beta_{n-1}| \|Ty_{n-1} - y_{n-1}\| \\
&\le (1-\beta_n) \|y_n - y_{n-1}\| + \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|Ty_{n-1} - y_{n-1}\| \\
&= \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|Ty_{n-1} - y_{n-1}\|.
\end{aligned} \tag{2.9}$$

Substituting (2.8) into (2.9), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le (1 - (1-\alpha)\alpha_n) \|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\
&\quad + \left\|TJ_{r_n} x_{n-1} - TJ_{r_{n-1}} x_{n-1}\right\| \\
&= (1-\gamma_n) \|x_n - x_{n-1}\| + \mu_n,
\end{aligned}$$

where

$$M=\max\left\{\sup_n\left\|f\left(x_{n-1}\right)-TJ_{r_{n-1}}x_{n-1}\right\|,\sup_n\left\|Ty_{n-1}-y_{n-1}\right\|\right\},$$

and

$$\mu_n=M\left(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)+\left\|TJ_{r_n}x_{n-1}-TJ_{r_{n-1}}x_{n-1}\right\|,\quad n\geq 2.$$

Hence

$$\sum_{n=2}^{\infty}\mu_n\leq M\sum_{n=2}^{\infty}\left(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)+\sum_{n=2}^{\infty}\sup\left\{\left\|TJ_{r_n}z-TJ_{r_{n-1}}z\right\|:z\in\{x_k\}\right\}<\infty.$$

Therefore Lemma 1.2 implies that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Hence $\{x_n\}$ is asymptotic regular, then by Theorem 2.1 the proof is complete. □

Corollary 2.3. *Let E be a real reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ and A a m-accretive maps in E such that $C=\overline{D(A)}$ is convex. Let $T:C\to C$ be a nonexpansive mapping with $F=F(T)\cap N(A)\neq\emptyset$ and $f:C\to C$ a fixed contraction mapping with contract constant α. Suppose that $\{\alpha_n\}\subset[0,1]$, $r_n\in\mathbb{R}^+$ which satisfy in conditions* (C1), (C3) and (C4). Let $x_1\in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)TJ_{r_n}x_n.$$

Suppose that $\sum_{n=1}^{\infty}\sup\{\|TJ_{r_{n+1}}z-TJ_{r_n}z\|\ ;\ z\in B\}<\infty$ for any bounded subset B of C, then $\{x_n\}$ converges strongly to $p\in F$, where p is the unique solution of the variational inequality (2.1).

Proof . It is sufficient that assume $\beta_n=0$ in Theorem 2.2. □

Corollary 2.4. *Let E be a real reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ and A a m-accretive maps in E such that $C=\overline{D(A)}$ is convex. Let $N(A)\neq\emptyset$ and $f:C\to C$ a fixed contraction mapping with contract constant α. Suppose that $\{\alpha_n\}$, $\{\beta_n\}\subset[0,1]$, $r_n\in\mathbb{R}^+$ which satisfy in conditions* (C1)-(C5). Let $x_1\in C$ be chosen arbitrarily and $\{x_n\}$ be a sequence generated by

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)J_{r_n}x_n.$$

Suppose that $\sum_{n=1}^{\infty}\sup\{\|J_{r_{n+1}}z-J_{r_n}z\|\ ;\ z\in B\}<\infty$ for any bounded subset B of C, then $\{x_n\}$ converges strongly to $p\in F$, where p is the unique solution of the variational inequality

$$\langle (I-f)(p),J(p-q)\rangle\leq 0,\quad q\in N(A).$$

Proof . It is sufficient that assume $T=I$ in Theorem 2.2. □

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References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, *Nonlinear Anal.*, **67**(2007), 2350-2360.

[2] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, *Mathematische Zeitschrift*, **100**(1967), 201-225.

[3] R.E. Bruck and G.B. Passty, *Almost convergence of the infinite product of resolvents in Banach spaces, Nonlinear Anal.*, **3**(1979), 279-282.

[4] R. Chen and Z. Zhu, *Viscosity approximation method for accretive operator in Banach spaces, Nonlinear Anal.*, **69**(2008), 1356–1363.

[5] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Mathematics and Its Applications of Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990, vol 62.

[6] K. Goebeland W.A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, UK, 1990.

[7] K. Goebel and S. Reich, *Uniform Convexity, in: Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York, Basel, 1984.

[8] X.F. He, Y.C. Xu and Z. He, *Iterative approximation for a zero of accretive operator and fixed points problems in Banach space, Applied Mathematics and Computation*, **217**(2011), 4620-4626.

[9] J.S. Jung. *Convergence on composite iterative schemes for nonexpansive mappings in Banach space, Fixed Point Theory and Appl.* **14**(2008), Article ID 167535.

[10] A. Moudafi, *Viscosity approximation methods for fixed point problems, J. Math. Anal. Appl.*, **241**(2000), 46-55.

[11] H. K. Xu, *Iterative algorithm for nonlinear operators, J. London. Math. Soc.*, **2**(2002), 240–256.

[12] H.K. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators, J. Math. Anal. Appl.*, **314**(2006), 631-643.