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Dynamical behavior of a stage structured prey-predator model

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Abstract

In this paper, a new stage structured prey-predator model with linear functional response is proposed and studied. The stages for prey have been considered. The proposed mathematical model consists of three nonlinear ordinary differential equations to describe the interaction among juvenile prey, adult prey and predator populations. The model is analyzed by using linear stability analysis to obtain the conditions for which our model exhibits stability around the possible equilibrium points. Besides this a rigorous global stability analysis has been performed for our proposed model by using Li and Muldowney approach (geometric approach). Global stability conditions for the proposed model are described in the form of theorem. This is not a case study, hence the real parameters are not available for this model. However, model may be simulated by using hypothetical set of parameters. Investigation of real parameters for the proposed model is an open problem.

Keywords: Stage Structured Population; Global Stability; Local Stability.
2010 MSC: Primary 34XX; Secondary 34C23.

1. Motivation

Mathematical study of prey-predator models has been observed in recent studies. The prey-predator model with stages are also available. The researchers from both applied mathematical modeling and ecology are working in this area. Study of the interaction among different species in an ecosystem is a great concern of the researchers. A lot of studies have been emerged in recent time. For ready reference we cite the recent references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Authors studied, prey predator interaction by means of mathematical modeling. It is well known that living species grow

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in stages. Two such stages are juvenile (immature) and adult (mature). In mathematical ecology, authors considered the stages of prey and/or predator depending on the complexity of the system. The systems with stages of one specie, will lead to the three dimensional model. Further, if stages of both the species are considered, it will lead to four dimensional model(s). In this way the system considered in [12] is three dimensional. In [12], the predator population splits into two stages. As mentioned earlier, for more models reader can refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In all these studies, prey and/or population have been considered as stage structured. In this paper a new stage structured prey-predator system by means of mathematical modeling is proposed and studied. To the best of my knowledge, proposed model is new and nobody studied it.

Rest of the paper is structured as follows: In Section 2, mathematical model is formulated followed by Section 3 in which Mathematical preliminaries of the model has been investigated. Local and global stability analysis have been performed in Section 4 and 5 respectively. The paper ends with brief discussion.

2. Mathematical Model

Let at any time t, $x_1(t), x_2(t), x_3(t)$ be the population densities of juvenile prey, adult prey and predator respectively. The proposed model takes the form:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= rx_1(t)(1-\frac{x_1(t)}{k}) - \beta x_1(t) - ax_1(t)x_3(t), \\ \frac{dx_2(t)}{dt} &= \beta x_1(t) - bx_2(t)x_3(t) - d_1x_2(t), \\ \frac{dx_3(t)}{dt} &= k_1ax_1(t)x_3(t) + k_2bx_2(t)x_3(t) - d_2x_3(t), \end{aligned} \tag{2.1}$$

with initial conditions:

$$x_1(0) > 0, x_2(0) > 0, x_3(0) > 0. \tag{2.2}$$

The constants r and k are intrinsic growth rate and carrying capacity of juvenile prey respectively. The constant β is the rate of converting the juvenile prey to adult prey. The capturing rates of adult predator for juvenile and adult prey respectively are the constants a and b. The coefficients in conversing the juvenile and adult prey to predator are denoted by k_1 and k_2. The death rates of adult prey and predator are denoted by d_1, d_2 respectively.

Model (2.1) is derived under the following assumptions:

(H1) We assume that prey population is divided into two stages viz. juvenile(or immature) and adult(or mature). The growth rate of prey population follows the logistic curve.

(H2) We also assume that predator consumes both preys. The interaction among predators have been ignored and therefore we take linear functional response.

3. Mathematical preliminaries of the model

This section is related to the positivity and boundedness for the system (2.1). The necessity for this is due to the fact that the variables x_1, x_2 and x_3 represents the living species. Positivity guarantee that they can not assume negative values and population always survive. Boundedness explains that there is a natural restrictions to growth of population as the sources are limited.

3.1. Positivity

We state and prove a theorem which proves that the proposed system (2.1) is positive on $[0, +\infty[$.

Theorem 3.1. *Every solution of* (2.1) *with initial conditions* (2.2) *which exists in* $[0, +\infty[$, *remain positive for all* $t > 0$.

Proof . We follow the procedure as discussed in [12, 13]. The proposed model (2.1) with the imposed initial conditions (2.2) can be written in the matrix equation form

$$\frac{dH}{dt} = G(H(t)), \tag{3.1}$$

where,

$$H(t) = (x_1, x_2, x_3)^T, H(0) = (x_1(0), x_2(0), x_3(0))^T \in R^3_+$$

and

$$G(H(t)) = \begin{pmatrix} G_1(H(t)) \\ G_2(H(t)) \\ G_3(H(t)) \end{pmatrix}$$
$$= \begin{pmatrix} rx_1(t)(1 - \frac{x_1(t)}{k}) - \beta x_1(t) - ax_1(t)x_3(t) \\ \beta x_1(t) - bx_2(t)x_3(t) - d_1x_2(t) \\ k_1ax_1(t)x_3(t) + k_2bx_2(t)x_3(t) - d_2x_3(t) \end{pmatrix},$$

where $G : R^3 \to R^3_+$ and $G \in C^\infty(R^3)$. It is clear that, whenever $H(0) \in R^3_+$ such that $G_i(H_i) |_{H_i=0} \geq 0$, for $i = 1, 2, 3$. By [14], the solution of matrix equation (3.1) with initial condition $H_0 \in R^3_+$, for instance $H(t) = H(t, H_0)$ such that $G(t) \in R^3_+$ for all finite and positive time t. □

3.2. Boundedness

Theorem 3.2. *All the solutions of the mathematical model system* (2.1) *with initial conditions* (2.2) *in* R^3_+ *are uniformly bounded.*

Proof .

Case I. $k_1 = k_2 = 1$.

By the positivity of the solution of (2.1), from equation (2.1), we have

$$\frac{dx_1}{dt} \leq rx_1(1 - \frac{x_1}{k}). \tag{3.2}$$

By the standard result in ordinary differential equations(ODEs), we have

$$\limsup x_1(t) \leq k. \tag{3.3}$$

Let $W = x_1 + x_2 + x_3$,

$$\frac{dW}{dt} = rx_1(1 - \frac{x_1}{k}) - d_1x_1 - d_2x_3. \tag{3.4}$$

Now we choose a positive constant $\eta > 0$, such that

$$\frac{dW}{dt} + \eta W = x_1(r(1 - \frac{x_1}{k}) + \eta) - (d_1 - \eta)x_2 - (d_2 - \eta)x_3. \tag{3.5}$$

If we choose $\eta = \min\{d_1, d_2\}$, we have

$$\frac{dW}{dt} + \eta W \leq x_1[r(1 - \frac{x_1}{k}) + \eta]. \tag{3.6}$$

The supremum of $(x_1[r(1-\frac{x_1}{k})+\eta])$ is $\frac{k(r+\eta)^2}{4r}$. Therefore $\frac{dW}{dt}+\eta W \leq \frac{k(r+\eta)^2}{4r}=\Omega>0$ (say). Now by the differential inequality introduced by Birkoff and Rota [15], we have

$$0 \leq W \leq \frac{\Omega(1-e^{-\eta t})}{\eta}+W(x_1(0),x_2(0),x_3(0))e^{-\eta t}. \tag{3.7}$$

For $t\to\infty$, $0<W<\frac{\Omega}{\eta}$. Hence, the solutions of system (2.1) in R^3_+ are confined in the region

$$A=\{(x_1,x_2,x_3)\in R^3_+ : 0<W<\tfrac{\Omega}{\eta}+\Phi, \Phi>0, \Omega=\tfrac{k(r+\eta)^2}{4r}\}.$$

Case II. $k_1=k_2\neq 1$.
In this case, by assuming $W=k_1(x_1+x_2)+x_3$, the theorem may be proved same way.
Case III. $k_1\neq k_2$.
In this case, by assuming $W=k_1x_1+k_2x_2+x_3$, the theorem may be proved same way. □

3.3. Existence of equilibrium points

System (2.1) possesses the following equilibrium points:
The trivial equilibrium points $E_0=(0,0,0)$ exists always.
The axial equilibrium points E_{x_1}, E_{x_2} and E_{x_3} do not exist.
The planar equilibrium point $E_{x_1x_2}=(\tilde{x_1},\tilde{x_2},0)=(\frac{k(r-\beta)}{r},\frac{\beta k(r-\beta)}{d_1 r},0)$ exists provided $(r-\beta)>0$.
Another planar equilibrium points $E_{x_2x_3}$ and $E_{x_1x_3}$ do not exist.
The interior equilibrium point (positive equilibrium point) $E^*=(x_1^*,x_2^*,x_3^*)$ is the solution of the following system of equations:

$$r(1-\tfrac{x_1}{k})-\beta-ax_3=0, \beta x_1-bx_2x_3-d_1x_2=0, k_1ax_1x_3+k_2bx_2x_3-d_2x_3=0.$$

After solving this system, we have
$x_1^*=\frac{k(r-\beta-ax_3^*)}{r}$, $x_2^*=\frac{d_2}{k_2 b}-\frac{k_1ak(r-\beta-ax_3^*)}{rk_2b}$ and x_3^* is the solution of the quadratic equation

$$\mu_1x_3^2+\mu_2x_3+\mu_3=0,$$

where

$$\mu_1=\tfrac{k_1a^2k}{rk_2}, \mu_2=\tfrac{d_2}{k_2}-\tfrac{k_1a^2kd_1}{rk_2b}-\tfrac{k_1ak(r-\beta)}{rk_2}+\tfrac{a\beta k}{r},$$
$$\mu_3=\tfrac{k_1ak(r-\beta)d_1}{rk_2b}-\tfrac{d_1d_2}{k_2b}-\tfrac{\beta k(r-\beta)}{r}.$$

4. Local stability analysis

The Jacobian matrix $J(x_1,x_2,x_3)$ for the system (2.1) at an arbitrary point (x_1,x_2,x_3) is given by

$$\begin{pmatrix} r-\beta-\frac{2r}{k}x_1-ax_3 & 0 & -ax_1 \\ \beta & -bx_3-d_1 & -bx_2 \\ k_1ax_3 & k_2bx_3 & k_1ax_1+k_2bx_2-d_2 \end{pmatrix}. \tag{4.1}$$

Remark 4.1. The eigenvalues about E_0 are $(r-\beta),-d_1,-d_2$. Hence E_0 is locally stable provided $(r-\beta)<0$.

Theorem 4.2. *The planar equilibrium $E_{x_1x_2}=(\tilde{x_1},\tilde{x_2},0)$ if exists, is locally asymptotically stable, if the following single condition is satisfied:*

$$\left(\frac{k_1ka(r-\beta)}{r}+\frac{k_2kb(r-\beta)}{d_1r}-d_2\right)<0. \tag{4.2}$$

Proof . Let the planar equilibrium $E_{x_1x_2}$ exists. The Jacobian at $E_{x_1x_2}$ is

$$\begin{pmatrix} r-\beta-\frac{2r}{k}\tilde{x_1} & 0 & -a\tilde{x_1} \\ \beta & -d_1 & -b\tilde{x_2} \\ 0 & 0 & k_1a\tilde{x_1}+k_2b\tilde{x_2}-d_2 \end{pmatrix}. \tag{4.3}$$

This has three eigenvalues viz. $r-\beta-\frac{2r}{k}\tilde{x_1}=-(r-\beta)$, $-d_1$ and

$$k_1a\tilde{x_1}+k_2b\tilde{x_2}-d_2=\left(\frac{k_1ka(r-\beta)}{r}+\frac{k_2kb(r-\beta)}{d_1r}-d_2\right).$$

By existence condition that $(r-\beta)>0$, it is observed that all the eigenvalues are negative except $\left(\frac{k_1ka(r-\beta)}{r}+\frac{k_2kb(r-\beta)}{d_1r}-d_2\right)$. Hence the theorem follows. □

Remark 4.3. It is observed that, if $E_{x_1x_2}$ exists, by Remark 4.1, the trivial equilibrium point E_0 is a saddle point and has unstable manifold along x_1 axis.

Theorem 4.4. *The positive interior equilibrium point $E^*=(x_1^*,x_2^*,x_3^*)$ is locally stable if the following conditions holds:*

$$m_i>0, i=1,2,3, m_1m_2-m_3>0. \tag{4.4}$$

The values of m_i are given in the proof.

Proof . The Jacobian matrix at E^* takes the form

$$\begin{pmatrix} r-\beta-\frac{2r}{k}x_1^*-ax_3^* & 0 & -ax_1^* \\ \beta & -bx_3^*-d_1 & -bx_2^* \\ k_1ax_3^* & k_2bx_3^* & k_1ax_1^*+k_2bx_2^*-d_2 \end{pmatrix}. \tag{4.5}$$

The characteristics equation around the equilibrium point E^* is

$$\lambda^4+m_1\lambda^3+m_2\lambda^2+m_3\lambda+m_4=0, \tag{4.6}$$

where

$$\begin{aligned} m_1 &= -\{r-\beta-\alpha-d_1-d_2-d_3-\tfrac{2rx_1^*}{k}-ax_4-bx_4\} \\ m_2 &= -\alpha(k_1ax_1^*+k_2bx_2^*)+(r-\beta-\tfrac{2rx_1^*}{k}-ax_4^*)(\alpha+d_2) \\ m_3 &= \alpha(r-\beta-\tfrac{2rx_1^*}{k}-ax_4^*)(k_1ax_1^*+k_2bx_2^*) \\ m_4 &= -\alpha\beta(k_1ax_1^*+k_2bx_2^*)(bx_4^*+d_1). \end{aligned}$$

Hence, by Routh-Hurwitz criteria the theorem follows. □

5. Global stability

To study the global stability of the positive interior equilibrium, geometric approach (GA) is used. For new readers, we shall give the basic flavor of the GA approach introduced by Li and Muldowney (1996)[16] to show an n-dimensional autonomous system $f: D\to R^n, D\subset R^n$, a simply connected and open set and also $f\in C^1(D)$, where the dynamical system is given by

$$\frac{dx}{dt}=f(x), \tag{5.1}$$

is globally stable under certain parametric conditions. Now we list three conditions.

(H_3) System (5.1) has a unique interior (positive)equilibrium x^* in D.

(H_4) The domain D is simply connected.

(H_5) There is a compact absorbing set $\Omega \subset D$.

This approach has also been used in recent literature. Kunal Chakraborty et al (2012)[18], Mainul Haque et al (2008)[17] studied the three dimensional models and proved the global stability. B. Buonomo and D. Lacitignola (2010) [19] proved the global stability for a four dimensional epidemic model.

Definition 5.1. The unique interior (positive) equilibrium x^* of the dynamical system (5.1) is globally asymptotically stable in the domain D if it is locally asymptotically stable and all the trajectories in D converges to its positive equilibrium point x^*.

Definition 5.2. Let $J = (J_{ij})_n$ be the variational matrix of the system (5.1) and $J^{[2]}$ be the second additive compound matrix with order $\begin{pmatrix} n \\ 2 \end{pmatrix} \times \begin{pmatrix} n \\ 2 \end{pmatrix}$.

To understand the form of $J^{[2]}$, we list two examples.

Example 5.3. Let us consider the following square matrix of order 3

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$J^{[2]}$ of this matrix is given by

$$J^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}.$$

Example 5.4. Let us consider the following square matrix of order 4

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$J^{[2]}$ of this matrix is given by

$$J^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11} + a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22} + a_{24} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33} + a_{44} \end{pmatrix}.$$

Let $M(x) \in C^1(D)$ be the $\begin{pmatrix} n \\ 2 \end{pmatrix} \times \begin{pmatrix} n \\ 2 \end{pmatrix}$ matrix valued function. Also we choose a matrix B such that

$$B = M_f M^{-1} + M J^{[2]} M^{-1}, \tag{5.2}$$

where the matrix M_f is represented by

$$(M_{ij}(x))_f = (\frac{\partial M_{ij}}{\partial x})^t . f(x) = \nabla M_{ij} . f(x). \tag{5.3}$$

Definition 5.5. Let matrix B is given by (5.2), we consider the Lozinskii measure Γ of B with respect to a vector norm $\mid . \mid$ in R^N,

$$N = \begin{pmatrix} n \\ 2 \end{pmatrix},$$

then we have

$$\Gamma(B) = \lim_{h \to 0^+} \frac{\mid l + hB \mid -1}{h}. \tag{5.4}$$

Definition 5.6. If the set of conditions viz. (H_3), (H_4) and (H_5) satisfied then Li and Muldowney investigated that if the following condition

$$\lim \sup \sup \frac{1}{t} \int_0^t \Gamma(B(x(s, x_0)))ds < 0, \tag{5.5}$$

is satisfied, then there are no orbits (i,e, homoclinic orbits, hetroclinic cycles and periodic orbits), generated to a simple closed rectifiable curve in D, which is invariant for the system (5.2). It is also a robust Benixson criterion.

Model system (2.1) can be written as

$$\frac{dX}{dt} = f(X), \tag{5.6}$$

where

$$f(X) = \begin{pmatrix} rx_1(1 - \frac{x_1}{k}) - \beta x_1 - a x_3 x_1 \\ \beta x_1 - b x_2 x_3 - d_1 x_2 \\ k_1 a x_1 x_3 + k_2 b x_2 x_3 - d_2 x_3 \end{pmatrix}$$

and

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and recall that the variational matrix is given by Eq. (4.1) viz.

$$J = \begin{pmatrix} r - \beta - \frac{2r}{k} x_1 - a x_3 & 0 & -a x_1 \\ \beta & -b x_3 - d_1 & -b x_2 \\ k_1 a x_3 & k_2 b x_3 & k_1 a x_1 + k_2 b x_2 - d_2 \end{pmatrix}. \tag{5.7}$$

The second compounded matrix is given by

$$J^{[2]} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}, \tag{5.8}$$

where

$$v_{11} = r - \beta - \frac{2r}{k}x_1 - ax_3 - bx_3 - d_1,\ v_{12} = -bx_2,\ v_{13} = ax_1,\ v_{21} = k_2bx_3,$$
$$v_{22} = r - \beta - \frac{2r}{k}x_1 - ax_3 + k_1ax_1 + k_2bx_2 - d_2,\ v_{23} = 0,\ v_{31} = -k_1ax_3,\ v_{32} = \beta,$$
$$v_{33} = -bx_3 - d_1 + k_1ax_1 + k_2bx_2 - d_2.$$

Now we consider $M(X) \in C^1(D)$ such that $M = diag(\frac{x_1}{x_3}, \frac{x_1}{x_3}, \frac{x_1}{x_3})$. Then we have

$$M^{-1} = diag(\frac{x_3}{x_1}, \frac{x_3}{x_1}, \frac{x_3}{x_1}),\ M_f = \frac{dM}{dX} = diag(\frac{\dot{x_1}}{x_3} - \frac{x_1}{x_3^2}\dot{x_3}, \frac{\dot{x_1}}{x_3} - \frac{x_1}{x_3^2}\dot{x_3}, \frac{\dot{x_1}}{x_3} - \frac{x_1}{x_3^2}\dot{x_3})$$

and

$$M_fM^{-1} = diag(\frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3}, \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3}, \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3})$$

and $MJ^{[2]}M^{-1} = J^{[2]}$. Therefore,

$$B = M_fM^{-1} + MJ^{[2]}M^{-1} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{5.9}$$

where

$$V_{11} = \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + v_{11} = \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + r - \beta - \frac{2r}{k}x_1 - ax_3 - bx_3 - d_1,$$
$$V_{12} = \begin{pmatrix} v_{12} & -v_{13} \end{pmatrix} = \begin{pmatrix} -bx_2 & ax_1 \end{pmatrix},$$
$$V_{21} = \begin{pmatrix} v_{21} & v_{31} \end{pmatrix}^t = \begin{pmatrix} k_2bx_3 & -k_1ax_3 \end{pmatrix}^t,$$

and

$$V_{22} = \begin{pmatrix} \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + v_{22} & v_{23} \\ v_{32} & \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + v_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + r - \beta - \frac{2r}{k}x_1 - ax_3 + k_1ax_1 + k_2bx_2 - d_2 & 0 \\ \beta & \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} - bx_3 - d_1 + k_1ax_1 + k_2bx_2 - d_2 \end{pmatrix}.$$

Now let us define the vector norm in R^3 for $(x, y, z) \in R^3$ as follows

$$\mid (x, y, z) \mid = \max\{\mid x \mid, \mid y \mid + \mid z \mid\} \tag{5.10}$$

for defining the Lozinskii measure with respect to this norm. Therefore,

$$\Gamma(B) \leq \sup\{p_1, p_2\}, \tag{5.11}$$

where

$$p_i = \Gamma_1(V_{ii}) + \mid V_{ij} \mid, \tag{5.12}$$

for $i = 1, 2$ and $i \neq j$, where $\mid V_{12} \mid$ and V_{21} are matrix norms with respect to the L^1 vector norm and Γ_1 is represented as Lozinskii measure with respect to that norm. Therefore, we can obtain the following terms

$$\begin{aligned} \Gamma_1(V_{11}) &= \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + r - \beta - \frac{2r}{k}x_1 - ax_3 - bx_3 - d_1, \\ \mid V_{12} \mid &= \max\{\mid -bx_2 \mid, ax_1\}, \\ \mid V_{21} \mid &= -k_1ax_3, \\ \Gamma_1(V_{22}) &= \frac{\dot{x_1}}{x_1} - \frac{\dot{x_3}}{x_3} + k_1ax_1 + k_2bx_2 + \max\{r - \beta - \frac{2r}{k}x_1 - ax_3 - d_2, -bx_3 - d_1 - d_2\}. \end{aligned} \tag{5.13}$$

The last equation of system (2.1) gives us

$$\frac{\dot{x_3}}{x_3} = (k_1ax_1 + k_2bx_2 - d_2). \tag{5.14}$$

Using equations (5.12) and (5.13), we have

$$p_1 = \frac{\dot{x_1}}{x_1} - (k_1ax_1 + k_2bx_2 - d_2) + r - \beta - \frac{2r}{k}x_1 - ax_3 - bx_3 - d_1 + \max\{| -bx_2 |, ax_1\},$$

$$p_2 = \frac{\dot{x_1}}{x_1} + d_2 + \max\{r - \beta - \frac{2r}{k}x_1 - ax_3 - d_2, -bx_3 - d_1 - d_2\} - k_1ax_3 = \frac{\dot{x_1}}{x_1} + d_2 - \min\{-r + \beta + \frac{2r}{k}x_1 + ax_3 + d_2, bx_3 + d_1 + d_2\} - k_1ax_3.$$

From the values of p_1 and p_2 as calculated above and Eq. (5.11), we obtain the following inequality;

$$\Gamma(B) \leq \frac{\dot{x_1}}{x_1} + d_2 - \min\Bigg(\frac{2r}{k}x_1 + k_1ax_1 + k_2bx_2 + k_1ax_3 - ax_3 - bx_3 + \beta - r + d_1 - \max\{| -bx_2 | , ax_1\}, -r + \beta + \frac{2r}{k}x_1 + ax_3 + d_2, bx_3 + d_1 + d_2\Bigg).$$

It is assumed that there exists a positive $\eta_1 \in R$ and $t_1 > 0$ such that $\eta_1 = \inf\{x_1, x_2, x_3\}$ for $t > t_1$. We also denote

$$\eta_2 = \max\{| -bx_2 |, ax_1\}$$

and

$$\eta_3 = \min\Bigg(\frac{2r}{k}\eta_1 + k_1a\eta_1 + k_2b\eta_1 + k_1a\eta_1 - a\eta_1 - b\eta_1 + \beta - r + d_1 - \eta_2, -r + \beta + \frac{2r}{k}\eta_1 + a\eta_1 + d_2, b\eta_1 + d_1 + d_2\Bigg).$$

Therefore, we can write

$$\Gamma(B) \leq \frac{\dot{x_1}}{x_1} + d_2 - \eta_3,$$

or

$$\Gamma(B) \leq \frac{\dot{x_1}}{x_1} - (\eta_3 - d_2),$$

i.e.

$$\frac{1}{t}\int_0^t \Gamma(B)dl \leq \frac{1}{t}\log\frac{x_1(t)}{x_1(0)} - (\eta_3 - d_2),$$

therefore we have

$$\lim\sup\sup\frac{1}{t}\int_0^t \Gamma(B)dl < -(\eta_3 - d_2) < 0. \tag{5.15}$$

provided $(\eta_3 - d_2) > 0$.

By the above discussion, we can state the main result of this section giving global stability of the positive equilibrium E^* in the form of the following theorem:

Theorem 5.7. The system (2.1) with initial data (2.2) is globally asymptotically stable around its positive equilibrium E^* if the following condition is satisfied

$$\eta_3 > d_2, \tag{5.16}$$

where,

$$\eta_3 = \min\Bigg(\frac{2r}{k}\eta_1 + k_1a\eta_1 + k_2b\eta_1 + k_1a\eta_1 - a\eta_1 - b\eta_1 + \beta - r + d_1 - \eta_2, -r + \beta + \frac{2r}{k}\eta_1 + a\eta_1 + d_2, b\eta_1 + d_1 + d_2\Bigg).$$

$\eta_2 = \max\{| -bx_2 |, ax_1\}$ and $\eta_1 = \inf\{x_1, x_2, x_3\}$, for $t > t_1$.

Remark 5.8. The global stability may be proved similarly for other non trivial equilibrium point viz. $E_{x_1x_2}$.

Remark 5.9. The global stability of the axial equilibrium $E_x = (1, 0, 0, 0)$ has been proved in [13] only. They have not considered the global stability for other equilibrium points including the positive equilibrium point. In this paper we have proved the global stability of positive equilibrium E^*.

Remark 5.10. All the parameters used in model (2.1) are time independent.

6. Discussion

In this paper, a stage structured prey-predator model is proposed and studied. Stages for prey have been considered, therefore the prey population is bifurcated into two populations viz. immature prey and mature prey. Local and global stability analysis have been investigated and results are listed in the form of theorems. As mentioned in the Remark 5.10, all the parameters are time independent. In real life situations the parameters are changing with time. Hence, models with time dependent parameters may be included in the future scope. In future the stages for both the populations may be considered. As a matter of fact, this study is not a case study hence real data/parameters are not available. Real parameters investigation is also a concern of future study. In the literature many prey-predator models are simulated by using hypothetical set of parameters [12, 13]. The model so proposed can also simulated by using the hypothetical set of parameters. The proposed model is simple, lucid and easy to understand. It is very much useful for the researchers who are new in this area. The basic mathematical investigations are done is lucid manner. The proposed model may be useful for any real ecosystem.

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