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Stochastic Differential Equations and Integrating Factor

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Abstract

The aim of this paper is the analytical solutions the family of first-order nonlinear stochastic differential equations. We define an integrating factor for the large class of special nonlinear stochastic differential equations. With multiply both sides with the integrating factor, we introduce a deterministic differential equation. The results showed the accuracy of the present work.

Keywords: Stochastic Differential Equation, Analytical Solution, Integrating Factor.
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1. Introduction and Preliminaries

Stochastic and deterministic differential equations are fundamentals for the modeling in science, engineering and mathematical finance. As the computational power increases, it becomes feasible to use more accurate differential equation models and solve more demanding problems. The model can be stochastic by two reasons: if calibration of data implies this, as in financial mathematical, or, if fundamental microscopic laws generate stochastic behavior when coarse-grained, as in molecular dynamics for chemistry, material science and biology.
Fluctuations in statistical mechanics are usually modeled by adding a stochastic term to the deterministic differential equation. By doing this one obtains what is called stochastic differential equations (SDEs), and the term stochastic called noise [1]. Then, a SDE is a differential equation in which one or more of the terms is a stochastic process, and resulting in a solution which is itself a stochastic process. Every unwanted signal that adds to the information called noise. Noise in dynamical system is usually considered a nuisance. Noise has the most important role in the SDE [2].

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When a differential equations model for some physical phenomenon is formulated preferably the exact solution can be obtained. However, even for ordinary differential equations, this is generally not possible [3]. Rezaeyan and et al ([4]-[6]) discussed the application of the SDEs for the molding electrical circuits. In this paper, we will present an application of the integrating factor for analytical solution to the family of SDEs.
For attention, in the next section, we describe stochastic calculus and SDEs. In Section 3, we define the integrating factor for the class of SDEs. Finally, the paper ends with an example and a conclusion.

2. Stochastic Calculus

In many physical applications one has to deal with random quantities that depend on a parameter. This phenomenon is termed Brownian motion. Each coordinate of the Brownian particle is a random variable that depends on a parameter. A stochastic process $X_t(\omega)$, is a family of random variables $\{X_t(\omega) : t \in T, \omega \in \Omega\}$ depending upon the parameter t and defined on the probability space $(\Omega, \Im, P)$ ($\Im$, is a σ- algebra of subsets of Ω and P is probability measure defined on all elements of $\Im$)[3].
Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. The best-known stochastic process to which stochastic calculus is applied the Wiener process.
The main part of stochastic calculus is the Ito calculus and Stratonovich. Ito calculus extends the methods of calculus to stochastic processes such as Brownian motion. We go back to the definition of an integral:

$$\int_0^T f(t)dt = \lim_{n\to+\infty} \sum_{j=1}^{n} f(\tau_j)(t_{j+1} - t_j), \tag{2.1}$$

where τ_j is in the interval $[t_j, t_{j+1}]$. More generally have Riemann-Stieltjes integral:

$$\int_0^T f(t)dt = \lim_{n\to+\infty} \sum_{j=1}^{n} f(\tau_j)(g(t_{j+1}) - g(t_j)). \tag{2.2}$$

For a smooth measure $g(t)$, limit converges to a unique value regardless where τ_j , taken in interval $[t_j, t_{j+1}]$.
The Ito and Stratonovich calculus follows the same rules as for the regular Riemann-Stieltjes calculus. If our choose is lower end point, of the partition $[t_j, t_{j+1}]$, we have the case Ito integral, but if we choose midpoint $\frac{t_{j+1}+t_j}{2}$, we got Stratonovich case.
Let $t_0 = 0 \le t_1 \le t_2 \le \cdots \le t_n = T$ be a partition of the interval $[0, T]$ and $\delta = \max(t_i - t_{i-1})$. The Ito integral $\int_0^T h(t, X_t))dW_t$ is defined as the limit in the quadratic mean

$$\int_0^T h(t, X_t)dW_t = \lim_{\delta_n\to 0} \sum_{i=1}^{n} h(\tau_{i-1}, X_{\tau_{i-1}})(W_{t_i} - W_{t_{i-1}}). \tag{2.3}$$

If the integrand h is jointly measurable and

$$\int_0^T E(|h(s,X_s)|^2)ds < \infty, \tag{2.4}$$

the stochastic integral in (2.1) is defined as the limit in probability. The Stratonovich integral is defined by

$$\int_0^T h(t,X_t)odW_t = \lim_{\delta_n \to 0} \sum_{i=1}^{n} h(\tau_{i-1}, \frac{X_{\tau_{i-1}} + X_{\tau_i}}{2})(W_{t_i} - W_{t_{i-1}}), \tag{2.5}$$

(where the symbol o, is employed).
In addition to the conditions on the existence of the Ito integral, it is required for the existence of the Stratonovich integral in (2.3) that the $h(t,X_t)$ function be continuous in t and have continuous partial derivatives (see [1]-[5]). Moreover

$$\int_0^T h(t,X_t)odW_t = \int_0^T h(t,X_t)dW_t + \frac{1}{2}\int_0^T g(t,X_t)\frac{\partial h}{\partial x}(t,X_t)dt \tag{2.6}$$

or, equivalently[1]

$$h(t,X_t)odW_t = h(t,X_t)dW_t + \frac{1}{2}g(t,X_t)\frac{\partial h}{\partial x}(t,X_t)dt. \tag{2.7}$$

Consider a SDE,

$$dX_t = f(t,X_t)dt + g(t,X_t)dW_t, \tag{2.8}$$

where f is an n-vector valued function, g is an $n \times p$ matrix valued function, W_t is an p-dimensional Brownian motion process or Wiener process, and the solution X_t of the stochastic differential equation (2.6), is meant a process X_t for all t, in some interval $[0,T]$.
We also assume that the distribution of X_0 is known and independent of W_t. There is an explicit several-dimensional formula which expresses the Stratonovich interpretation of (2.6)

$$dX_t = \tilde{f}(t,X_t)dt + g(t,X_t)odW_t, \tag{2.9}$$

where:

$$\tilde{f}(t,X_t) = f(t,X_t) + \frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{n}\frac{\partial g_{ij}}{\partial x_j}g_{kj}, \tag{2.10}$$

(see Oksendal (2000)).

$$(I)\int_0^t W_s dW_s = \frac{1}{2}[W_t^2 - W_0^2 - t], \tag{2.11}$$

while

$$(S)\int_0^t W_s o dW_s = \frac{1}{2}[W_t^2 - W_0^2], \tag{2.12}$$

Ito integral and Stratonovich integral have applications in the SDEs. In this paper, we consider SDE of Ito kind . A SDE is given by

$$X_t' = f(t, X_t) + g(t, X_t)\xi_t, X_0 = x_0, t \geq 0, \tag{2.13}$$

where f is the deterministic part, $g\xi_t$ is the stochastic part, and ξ_t denotes a generalized stochastic process [1]-[3].
An example of generalized stochastic processes is white noise. For a generalized stochastic process, derivatives of any order can be defined. Suppose that W_t is a generalized version of a Wiener process which is used to model the motion of stock prices.
A Wiener process is a time continuous process with the property $W_t \sim N(0, t)$, $(0 \leq t \leq T)$, usually it is differentiable almost nowhere. White noise ξ_t is defined as $\xi_t = \frac{dW_t}{dt}$[1].
If we replace $\xi_t dt$ by dW_t in equation (2. 13), an Ito SDE can be rewritten as:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \tag{2.14}$$

where f and g are drift and diffusion term, respectively, and X_t is a solution which we try to find based on the integrating factor [3].

3. Main Results

Consider the following nonlinear SDE,

$$dX_t = f(t, X_t)dt + C(t)X_t dW_t, \ X_0 = x, \tag{3.1}$$

where $f : R \times R \ \to R$ and $C : R \to R$ are given continuous (deterministic) functions.

Lemma 3.1. *Let X_t and Y_t be Ito processes in R. Then:*

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t dY_t. \tag{3.2}$$

Proof . Let X_t and Y_t be Ito processes given by

$$dX_t = g_0(t)dt + g_1(t)dW_t, \tag{3.3}$$

$$dY_t = h_0(t)dt + h_1(t)dW_t. \tag{3.4}$$

By Ito Lemma (See Theorem 4. 1. 2 of [1]), we have

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + g_1(t)h_1(t), \tag{3.5}$$

such as

$$dt.dt = dt.dW_t = dW_t.dt = 0, dW_t.dW_t = dt, \tag{3.6}$$

then

$$dX_tdY_t = g_0(t)h_0(t)(dt)^2 + g_1(t)h_0(t)dtdW_t \qquad (3.7)$$
$$+g_0(t)h_1(t)dW_tdt + g_1(t)h_1(t)(dW_t)^2 = g_1(t)h_1(t)dt,$$

so

$$d(X_tY_t) = X_tdY_t + Y_tdX_t + dX_tdY_t. \qquad (3.8)$$

□

Remark 3.2. *If X_t and Y_t be Ito processes in R, then deduce the following general integration by parts formula*

$$\int_0^t X_sdY_s = X_tY_t - X_0Y_0 - \int_0^t Y_sdX_s - \int_0^t dX_sdY_s. \qquad (3.9)$$

Lemma 3.3. *If*

$$F_t = \exp(-\int_0^t C(s)dW_s + \frac{1}{2}\int_0^t C^2(s)ds), \qquad (3.10)$$

then,

$$d(F_tX_t) = F_tf(t, X_t)dt. \qquad (3.11)$$

Proof . Suppose $Y_t(\omega) = F_t(\omega)X_t(\omega)$. Then

$$X_t = F_t^{-1}Y_t, \qquad (3.12)$$

and

$$dY_t = d(F_tX_t) = F_tf(t, X_t)dt = F_tf(t, F_t^{-1}Y_t)dt. \qquad (3.13)$$

From (3. 13) we obtain

$$\frac{dY_t}{dt} = F_tf(t, F_t^{-1}Y_t). \qquad (3.14)$$

Note that this is just a deterministic differential equation in the function $t \to Y_t(\omega)$ for each $\omega \in \Omega$. □

Example 3.4. *Let*

$$dX_t = \frac{1}{X_t}dt + \alpha X_tdB_t, X_0 = x > 0, \qquad (3.15)$$

where α is constant. Where $f(t, X_t) = \frac{1}{X_t}$ and $g(t, X_t) = \alpha X_t$, equation (3.15) is a SDE to form (2.14). So

$$F_t = \exp(-\int_0^t C(s)dW_s + \frac{1}{2}\int_0^t C^2(s)ds) = \exp(-\alpha B_t + \frac{1}{2}\alpha^2 t), \qquad (3.16)$$

and

$$X_t = F_t^{-1} Y_t = \exp(-\alpha B_t + \frac{1}{2}\alpha^2 t) Y_t, \tag{3.17}$$

$$\frac{dY_t}{dt} = F_t f(t, F_t^{-1} Y_t) = \exp(-\alpha B_t + \frac{1}{2}\alpha^2 t) \frac{1}{\exp(\alpha B_t - \frac{1}{2}\alpha^2 t) Y_t}, \tag{3.18}$$

$$Y_t dY_t = \exp(2(\alpha B_t + \frac{1}{2}\alpha^2 t)) dt. \tag{3.19}$$

Then,

$$X_t = \exp(\alpha B_t - \frac{1}{2}\alpha^2 t)[x^2 + 2\int_0^t \exp(2(-\alpha B_t + \frac{1}{2}\alpha^2 t)) dt]^{\frac{1}{2}}. \tag{3.20}$$

4. Conclusion

We introduced first-order nonlinear SDEs and it's applications. Also, we defined an integrating factor for the large class of special nonlinear SDEs. By multiplying both sides of the integrating factor, we obtain a deterministic differential equation. By an example, we show that the accuracy of this method.

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