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Generalized multivalued F-contractions on incomplete metric spaces

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Abstract

In this paper, we explain a new generalized contractive condition for multivalued mappings and prove a fixed point theorem in metric spaces (not necessary complete) which extends some well–known results in the literature. Finally, as an application, we prove that a multivalued function satisfying a general linear functional inclusion admits a unique selection fulfilling the corresponding functional equation.

Keywords: Fixed point theorem, Weakly Picard operator, O–complete metric space, Selections of multivalued functions.
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1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote, respectively, the sets of all natural numbers, rational numbers and real numbers. Also, for every nonempty set X denote $\mathcal{P}^*(X)$ the set of all nonempty subsets of X. Let (X, d) be a metric space. We denote by $B(X)$, $CB(X)$ and $CP(X)$ collections of all bounded, closed bounded and complete members of $\mathcal{P}^*(X)$, respectively. The number

$$diam(A) := \sup\{d(a, b) : a, b \in A\},$$

is said to be the diameter of $A \in \mathcal{P}^*(X)$. For $A, B \in CB(X)$ and $x \in X$, define

$$D(x, A) := \inf\{d(x, a); a \in A\}$$

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and

$$H(A,B) := \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}.$$

The function H is a metric on $CB(X)$ and is called a Pompeiu–Hausdorff metric. We can find detailed information about the Pompeiu–Hausdorff metric in [1, 10]. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$. Let $T : X \to CB(X)$ be a map, then T is called a multivalued contraction if there exists $r \in (0,1)$ such that for all $x, y \in X$, we have

$$H(Tx,Ty) \leq rd(x,y).$$

In 1969, Nadler [18] proved that every multivalued contraction on a complete metric space has a fixed point. Since then, a lot of generalizations of the result of Nadler were given (see, for example [2, 3, 6, 11, 13, 15, 14, 24, 26]). An interesting important generalization of it were given by Berinde et al. [9] where the authors introduced the concept of a multivalued weakly Picard operator as follows:

Definition 1.1. (Berinde and Berinde, [9]) Let (X,d) be a metric space and $T : X \to \mathcal{P}^*(X)$ be a multivalued operator. T is said to be a Multivalued Weakly Picard (MWP) operator if for each $x \in X$ and any $y \in Tx$, there exists a sequence $\{x_n\}$ in X such that
(i) $x_0 = x, x_1 = y$,
(ii) $x_{n+1} \in Tx_n$,
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

Then Berinde et al. [9] showed that the type multivalued contractions on complete metric spaces considered by Nadler [18], Mizoguchi and Takahashi [17] and Petrusel [20] are MWP operators. In the same paper, Berinde et al. [9] introduced the concepts of multivalued almost contraction (the original name was multivalued (δ, L)–weak contraction) and proved the following important fixed point theorem:

Theorem 1.2. (Berinde and Berinde, [9]) Let (X,d) be a complete metric space and let T be a multivalued almost contraction from X into $CB(X)$, that is, there exist two constant $\delta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx,Ty) \leq \delta.d(x,y) + L.D(y,Tx)$$

for all $x, y \in X$. Then T is an MWP operator.

Recently, Eshaghi et al. [12] introduced the notion of orthogonal sets and then gave a real extension of Banach's fixed point theorem. Then, Baghani et al. [7] by using the notion, proved a statement which is equivalent to the axiom of choice and explain a generalization of Theorem 3.11 of [12].

In this paper, by combining the ideas of Baghani et al. [7] and Berinde et al. [9], we explain a new generalized contractive condition of multivalued mappings and prove a fixed point theorem in metric spaces (not necessary compete) which improves the main result of Altun et al. [4, 5], Amini–Harandi [6], Mizoguchi et al. [17], Sgroi et al. [23] and Smajdor et al. [25].

Definition 1.3. Let Λ be the class of those functions $\phi(t_1,t_2,t_3,t_4,t_5) : \mathbb{R}_+^5 \to \mathbb{R}_+$ which satisfy the following conditions
(Λ_1) ϕ is increasing in t_2, t_3, t_4 and t_5;
(Λ_2) $t_{n+1} < \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0)$ implies that $t_{n+1} < t_n$, for each positive sequence $\{t_n\}$;
(Λ_3) If $t_n, s_n \to 0$ and $u_n \to \gamma > 0$, as $n \to \infty$, then we have $\limsup_{n\to\infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) < \gamma$;
(Λ_4) $\phi(u,u,u,2u,0) \leq u$ for each $u \in \mathbb{R}^+ := (0,+\infty)$.

Example 1.4. Let $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + L t_5,$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. We claim that $\phi \in \Lambda$. Indeed (Λ_1) obviously holds. To show (Λ_2), let $\{t_n\}$ be a positive sequence such that

$$\begin{aligned} t_{n+1} &< \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0) = \alpha t_n + \beta t_n + \gamma t_{n+1} + \delta(t_n + t_{n+1}) \\ &= (\alpha + \beta + \delta) t_n + (\gamma + \delta) t_{n+1}. \end{aligned}$$

Since $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$, then we can conclude that $1 - (\gamma + \delta) > 0$ and hence

$$t_{n+1} < \frac{(\alpha + \beta + \delta)}{1 - (\gamma + \delta)} t_n = t_n.$$

It is obvious that properties (Λ_3) and (Λ_4) hold for this function.

Definition 1.5. (Amini–Harandi, [6]) Let $F : (0, +\infty) \to \mathbb{R}$ and $\theta : (0, +\infty) \to (0, +\infty)$ be two mappings. Throughout the paper, let Δ be the set of all pairs (θ, F) satisfying the following conditions:
(δ_1) $\theta(t_n) \not\to 0$ for each strictly decreasing sequence $\{t_n\}$;
(δ_2) F is a strictly increasing function;
(δ_3) For each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(δ_4) If $t_n \downarrow 0$ and $\theta(t_n) \leq F(t_n) - F(t_{n+1})$ for each $n \in \mathbb{N}$, then we have $\sum_{n=1}^{\infty} t_n < \infty$.

Example 1.6. (Amini–Harandi, [6]) Let $F(t) = \ln(t)$ and $\theta(t) = -\ln(\alpha(t))$ for each $t \in (0, +\infty)$, where $\alpha : (0, \infty) \to (0, 1)$ satisfies $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$. Then $(\theta, F) \in \Delta$.

2. Orthogonal sets

We start our work with the following definition, which can be considered as the main definition of our paper.

Definition 2.1. (Eshaghi et al., [12]) Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 \in X : (\forall y, y \perp x_0) \ \ or \ \ (\forall y, x_0 \perp y),$$

then $\perp$ is called an orthogonality relation and the pair $(X, \perp)$ an orthogonal set (briefly *O–set*).

Note that in above definition, we say that x_0 is orthogonal element. If $(X, \perp)$ has only one orthogonal element, then it is called a uniquely orthogonal set and the element is said unique orthogonal element. Also, an orthogonal element x_0 is called left orthogonal element if $x_0 \perp x$ for each $x \in X$. Similarly, it is called a right orthogonal element if $x \perp x_0$ for each $x \in X$. Finally, we say that elements $x, y \in X$ are $\perp$–comparable either $x \perp y$ or $y \perp x$.

As an illustration, let us consider the following examples.

Example 2.2. (Eshaghi et al., [12]) Let X be the set of all people in the world. We define $x \perp y$ if x can give blood to y. According to the following table, if x_0 is a person such that his (her) blood type is $AB+$, then we have $y \perp x_0$ for all $y \in X$. This means that $(X, \perp)$ is a O-set. Also, Let x_0 be a person with blood type $O-$, then we have $x_0 \perp y$ for all $y \in X$. Hence, in the O–set, x_0 is not unique.

Type	You can give blood to	You can receive blood from
A+	A+ AB+	A+ A- O+ O-
O+	O+ A+ B+ AB+	O+ O-
B+	B+ AB+	B+ B- O+ O-
AB+	AB+	Everyone
A-	A+ A- AB+ AB-	A- O-
O-	Everyone	O-
B-	B+ B- AB+ AB-	B- O-
AB-	AB+ AB-	AB- B- O- A-

Example 2.3. Let Σ be a family of nonempty subsets of X. Assume μ is the set of all σ–algebras containing Σ. Define $A \perp_\mu B$ iff $B \subset A$. Hence $(\mu, \perp_\mu)$ is an uniquely O–set that σ–algebra generated by Σ is a unique orthogonal element of μ.

Example 2.4. Let $(X, \perp)$ be an O–set. Let f be a choice function on $\mathcal{P}^*(X)$. For all $A, B \in \mathcal{P}^*(X)$ define $A \perp^* B$ if and only if $f(A) \perp f(B)$. It is clear that $(\mathcal{P}^*(X), \perp^*)$ is an O–set and $\{x^*\}$ is an orthogonal element of $(\mathcal{P}^*(X), \perp^*)$, where x^* is an orthogonal element of $(X, \perp)$.

Example 2.5. Let X be a nonempty set. If f is a choice function on $\mathcal{P}^*(X)$, then f defines a equivalent relation $\perp^*$ on $\mathcal{P}^*(X)$ via

$$A \perp^* B \Longleftrightarrow f(A) = f(B).$$

The relation $\perp^*$ in above satisfies the following.

1. The set of all equivalence classes modulo $\perp^*$ is

$$\mathcal{P}^*(X)/\perp^* = \{\{x\}/\perp^* : x \in X\},$$

   where $\{x\}/\perp^*$ is the equivalence class of $\{x\}$ modulo $\perp^*$.
2. Every element of $\{x\}/\perp^*$ contains x.

It is easy to see that for each $x \in X$, $(\{x\}/\perp^*, \perp^*)$ is an O–set and $\{x\}$ is unique orthogonal element of $(\{x\}/\perp^*, \perp^*)$.

Let $(X, \perp)$ be a O–set and $A, B \subseteq X$. The binary relation $\hat{\perp}$ between A and B is defined as follows.

- $A \ \hat{\perp} \ B$ if $a \perp b$ for all $a \in A$ and $b \in B$.

Now, we introduce $\perp$–preserving multivalued mapping by using the relation $\hat{\perp}$.

Definition 2.6. Let $(X, \perp, d)$ be a orthogonal metric space ($(X, \perp)$ is an O–set and (X, d) is a metric space) and $T : X \to CB(X)$. Then T is said to be an $\perp$–preserving multivalued mapping if

$$x, y \in X, x \ \perp \ y \Rightarrow Tx \ \ \hat{\perp} \ \ Ty.$$

Example 2.7. Let $X = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \cup \{0, 1\}$, $d(x, y) = |x - y|$ for all $x, y \in X$, and binary relation $\perp$ on X be defined by

$$x \ \perp \ y \quad \Longleftrightarrow \quad \begin{cases} \frac{y}{x} \in \mathbb{N}, \\ or \ x = y = 0. \end{cases}$$

Let $T : X \to CB(X)$ be defined by

$$Tx = \begin{cases} \{\frac{1}{2^n}, \frac{1}{2^{n+1}}\} & \text{if} \quad x = \frac{1}{2^n}, n = 1, 2, \dots, \\ \{0\}, & \text{if} \quad x = 0, \\ \{1, \frac{1}{2}, \frac{1}{4}\}, & \text{if} \quad x = 1. \end{cases}$$

It is easy to see that T is not an $\perp$–preserving multivalued mapping. Since $\frac{1}{2} \perp 1$ but $T(\frac{1}{2}) = \{\frac{1}{2}, \frac{1}{4}\}$ is not orthogonal to $\{1, \frac{1}{2}, \frac{1}{4}\} = T(1)$.

Example 2.8. Let $X = [0, 1)$ and let the metric on X be the Euclidean metric. Define a binary relation $\perp$ on X by $x \perp y$ if $xy \in \{x, y\}$ for all $x, y \in X$. Let $T : X \to CB(X)$ be a mapping defined by

$$T(x) = \begin{cases} \{\frac{1}{2}x^2, x\}, & x \in \mathbb{Q} \cap X, \\ \{0\}, & x \in \mathbb{Q}^c \cap X. \end{cases}$$

It is easy to see that T is an $\perp$–preserving multivalued mapping

3. Fixed Point Theory

In this section, we prove our main theorem. To this end, we need the following definitions.

Definition 3.1. (Eshaghi et al., [12]) Let $(X, \perp)$ be an O–set. A sequence $\{x_n\}$ is called an *orthogonal sequence* (briefly, *O–sequence*) if

$$(\forall n, x_n \perp x_{n+1}) \ \ or \ \ (\forall n, x_{n+1} \perp x_n).$$

Definition 3.2. (Eshaghi et al., [12]) Let $(X, \perp, d)$ be an orthogonal metric space. Then X is said to be *orthogonally complete* (briefly, *O–complete*) if every Cauchy O–sequence is convergent.

Definition 3.3. Let $(X, \perp, d)$ be an orthogonal metric space. Then X is said to be orthogonally regular (briefly, $\perp$*–regular) if X has the following properties*

(i) for each sequence $\{x_n\}$ such that $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}$, and $x_n \to x$, for some $x \in X$, then $x_n \perp x$ for all $n \in \mathbb{N}$;

(ii) for each sequence $\{x_n\}$ such that $x_{n+1} \perp x_n$ for all $n \in \mathbb{N}$, and $x_n \to x$, for some $x \in X$, then $x \perp x_n$ for all $n \in \mathbb{N}$.

Example 3.4. Let $X = \mathbb{Q}$. Suppose that $x \perp y$ if and only if $x = 0$ or $y = 0$. Clearly, $\mathbb{Q}$ with the Euclidean metric is not a complete metric space, but it is O–complete. In fact, if $\{x_k\}$ is an arbitrary Cauchy O–sequence in $\mathbb{Q}$, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all $n \geq 1$. It follows that $\{x_{k_n}\}$ converges to $0 \in X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. It is easy to see that $(X, \perp, d)$ is also an $\perp$–regular metric space.

Example 3.5. Let $X = [0, 1)$. Suppose that

$$x \ \perp \ y \quad \Longleftrightarrow \quad \begin{cases} x \leq y \leq \frac{1}{4}, \\ or \ x = 0. \end{cases}$$

Clearly, X with the Euclidian metric is not complete metric space, but it is O–complete. In fact, if $\{x_k\}$ is an arbitrary Cauchy O–sequence in X, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all $n \geq 1$ or there exists a monotone subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} \leq \frac{1}{4}$ for all $n \geq 1$. It follows that $\{x_{k_n}\}$ converges to a point $x \in [0, \frac{1}{4}] \subseteq X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. It is easy to see that $(X, \perp, d)$ is also an $\perp$–regular metric space.

Definition 3.6. *Let $(X, \perp, d)$ be an orthogonal metric space. Then $T : X \to CB(X)$ is said to be orthogonally continuous (or $\perp$–continuous) in $a \in X$ if, for each O–sequence $\{a_n\}$ in X with $a_n \to a$, we have $T(a_n) \to T(a)$. Also, T is said to be $\perp$–continuous on X if T is $\perp$–continuous in each $a \in X$.*

It is easy to see that every continuous mapping is $\perp$–continuous. The following example shows that the converse of the statement is not true in general.

Example 3.7. Let $X = \mathbb{R}$. Suppose $x \perp y$ if and only $x = 0$ or $0 \neq y \in \mathbb{Q}$. It is easy to see that $(X, \perp)$ is an O-set. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{x\}, x \in \mathbb{Q}, \\ \{0\}, x \in \mathbb{Q}^c. \end{cases}$$

The function T is $\perp$–continuous at all rational numbers while it is continuous just at $x = 0$.

Definition 3.8. Let $(X, \perp, d)$ be an orthogonal metric space and $T : X \to \mathcal{P}^*(X)$ be a multivalued operator. T is said to be an orthogonal multivalued Weakly Picard (OMWP) operator if for each orthogonal element $x \in X$ and any $y \in Tx$, there exists an orthogonal sequence $\{x_n\}$ in X such that
(i) $x_0 = x, x_1 = y$,
(ii) $x_{n+1} \in Tx_n$,
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

Now, we are ready to prove the main theorem of this paper which can be consider as a multivalued version of Theorem 3.10 of [7].

Theorem 3.9. *Let $(X, \perp, d)$ be an O–complete metric space (not necessarily a complete metric space), and $T : X \to CB(X)$ be an $\perp$–preserving multivalued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(H(Tx, Ty)) \leq F(\phi(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx))), \tag{3.1}$$

for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\phi \in \Lambda$. Also, suppose that T is compact valued or F is continuous from the right. If
(i) T is $\perp$–continuous or;
(ii) X is an $\perp$–regular metric space;
then T is an OMWP operator.

Proof . Let x_0 be an orthogonal element of X. By the definition of orthogonality, we have

$$\forall y \in X, x_0 \perp y \quad or \quad \forall y \in X, y \perp x_0.$$

It follows that

$$\forall y \in T(x_0), x_0 \perp y \quad or \quad \forall y \in T(x_0), y \perp x_0.$$

Without loss of generality let

$$\forall y \in T(x_0), x_0 \perp y.$$

Let $x_1 \in Tx_0$ then $x_0 \perp x_1$. On the other hand, since T is $\perp$-preserving, then $Tx_0 \hat{\perp} Tx_1$. If $x_1 \in Tx_1$, then x_1 is fixed point of T and the proof is finished. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since either T is compact valued or F is continuous from right, $x_1 \in Tx_0$ and

$$F(D(x_1, Tx_1)) < F(H(Tx_0, Tx_1)) + \frac{\theta(d(x_0, x_1))}{2},$$

then there exists $x_2 \in Tx_1$ with $x_1 \perp x_2$ such that

$$F(d(x_1, x_2)) \leq F(H(Tx_0, Tx_1)) + \frac{\theta(d(x_0, x_1))}{2}.$$

Repeating this process, we can construct an O–sequence $\{x_n\}$ with initial point x_0 such that $x_{n+1} \in Tx_n$, $Tx_n \neq Tx_{n+1}$ and

$$F(d(x_n, x_{n+1})) \leq F(H(Tx_{n-1}, Tx_n)) + \frac{\theta(d(x_{n-1}, x_n))}{2} \tag{3.2}$$

for all $n \in \mathbb{N}$. From (3.1), (3.2), (Λ_1) and (δ_2)we have

$$\begin{aligned}
&\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \\
&\leq \theta(d(x_{n-1}, x_n)) + F(H(Tx_{n-1}, Tx_n)) + \frac{\theta(d(x_{n-1}, x_n))}{2} \\
&\leq F(\phi(d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_n), D(x_n, Tx_{n-1}))) \\
&\quad + \frac{\theta(d(x_{n-1}, x_n))}{2} \\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0)) + \frac{\theta(d(x_{n-1}, x_n))}{2} \\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)) \\
&\quad + \frac{\theta(d(x_{n-1}, x_n))}{2},
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{\theta(d(x_{n-1}, x_n))}{2} + F(d(x_n, x_{n+1})) \\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0))
\end{aligned} \tag{3.3}$$

for each $n \in \mathbb{N}$. This implies that

$$\begin{aligned}
&d(x_n, x_{n+1}) \\
&< \phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)
\end{aligned}$$

for each $n \in \mathbb{N}$. Then by (Λ_2), $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence, then by using (3.3), (Λ_1) and (Λ_4), we obtain that

$$\begin{aligned}
&\frac{\theta(d(x_{n-1}, x_n))}{2} + F(d(x_n, x_{n+1})) \\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)) \\
&\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_{n-1}, x_n), 0)) \\
&= F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), 2d(x_{n-1}, x_n), 0)) \\
&\leq F(d(x_{n-1}, x_n)),
\end{aligned} \tag{3.4}$$

for each $n \in \mathbb{N}$. Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$, for some $r \geq 0$. Now, we show that $r = 0$. On contrary, assume that $r > 0$. From (3.4) we get

$$\frac{1}{2}\sum_{i=1}^{n} \theta(d(x_i, x_{i+1})) \leq F(d(x_1, x_2)) - F(d(x_{n+1}, x_{n+2})) \tag{3.5}$$

for each $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is strictly decreasing, then from (δ_1) we obtain that $\theta(d(x_n, x_{n+1})) \not\to 0$. Thus, $\sum_{i=1}^{\infty} \theta(d(x_i, x_{i+1})) = +\infty$, and then from (3.5) we have $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Then by (δ_3), $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$, that a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

From (3.4), (3.6) and (δ_4), we have $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Then by triangle inequality $\{x_n\}$ is a Cauchy O–sequence. Since X is O–complete, then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we prove that x is fixed point of T.

Case 1. T is $\perp$–continuous.
In this case, we have

$$D(x, Tx) = \lim_{n\to\infty} D(x_{n+1}, Tx) \leq \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Then $x \in Tx$ and the proof is complete.

Case 2. X is an $\perp$–regular metric space.
If there exists a strictly increasing sequence $\{n_k\}$ such that $x_{n_k} \in Tx$ for all $k \in \mathbb{N}$, since Tx is closed and $x_{n_k} \to x$, as $k \to \infty$, we get that $x \in Tx$ and the proof is complete. So, we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tx$ for each $n > n_0$. This implies that $Tx_n \neq Tx$ for each $n \geq n_0$. Now since X is an $\perp$–regular metric space by using (3.1) with $x = x_n$ and $y = x$, we obtain

$$\begin{aligned}
&F(D(x_{n+1}, Tx)) < \theta(d(x_n, x)) + F(D(x_{n+1}, Tx)) \\
&\leq \theta(d(x_n, x)) + F(H(Tx_n, Tx)) \\
&\leq F(\phi(d(x_n, x), D(x_n, Tx_n), D(x, Tx), D(x_n, Tx), D(x, Tx_n))) \\
&\leq F(\phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), d(x, x_{n+1})))
\end{aligned}$$

for each $n \geq n_0$. Therefore

$$D(x_{n+1}, Tx) < \phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), d(x, x_{n+1}))$$

for each $n \geq n_0$. Now if $x \in Tx$, then the proof is complete. Let $x \notin Tx$ then by using (3) and (Λ_3) we have

$$\begin{aligned} D(x, Tx) &= \limsup_{n\to\infty} D(x_{n+1}, Tx) \\ &\leq \limsup_{n\to\infty} \phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), d(x, x_{n+1})) \\ &< D(x, Tx), \end{aligned}$$

which is a contradiction. Hence $x \in Tx$ and the proof is complete. □

Letting $\phi(t_1, t_2, t_3, t_4, t_5) = t_1$, we get a generalization of Theorem 2.4 of [6], Theorem 2 and Theorem 3 of [5] as follows.

Corollary 3.10. Let $(X, \perp, d)$ be an O–complete metric space (not necessarily a complete metric space), and $T : X \to CB(X)$ be an $\perp$–preserving multivalued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that

$$\theta(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y))$$

for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$. Also, suppose that T is compact valued or F is continuous from the right. If
(i) T is $\perp$–continuous or;
(ii) X is an $\perp$–regular metric space;
then T is an OMWP operator.

Letting

$$\phi(t_1, t_2, t_3, t_4, t_5) = t_1 + \lambda.t_5,$$

where $\lambda \geq 0$, we get a generalization of Theorem 2.2 of [4] as follows.

Corollary 3.11. Let $(X, \perp, d)$ be an O–complete metric space (not necessarily a complete metric space), and $T : X \to CB(X)$ be an $\perp$–preserving multivalued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that

$$\theta(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y) + \lambda.D(y, Tx))$$

for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\lambda \geq 0$. Also, suppose that T is compact valued or F is continuous from the right. If
(i) T is $\perp$–continuous or;
(ii) X is an $\perp$–regular metric space;
then T is an OMWP operator.

Letting

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + L t_5,$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$, we get a generalization of Theorem 3.4 of [23] as follows.

Corollary 3.12. *Let $(X, \perp, d)$ be an O–complete metric space (not necessarily a complete metric space), and $T : X \to CB(X)$ be an $\perp$–preserving multivalued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that*

$$\begin{aligned} &\theta(d(x, y)) + F(H(Tx, Ty)) \\ &\leq F(\alpha d(x, y) + \beta D(x, Tx) + \gamma D(y, Ty) + \delta D(x, Ty) + L D(y, Tx)) \end{aligned}$$

for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Also, suppose that T is compact valued or F is continuous from the right. If
(i) T is $\perp$–continuous or;
(ii) X is an $\perp$–regular metric space;
then T is an OMWP operator.

Proof . By using Example 2.1 of [6], we can easily show that this corollary is a generalization of Theorem 3.4 of [23]. □

In below we explain a generalization of Mizoguchi–Takahashi's fixed point theorem [17].

Corollary 3.13. *Let $(X, \perp, d)$ be an O–complete metric space (not necessarily a complete metric space), and $T : X \to CB(X)$ be an $\perp$–preserving multivalued mapping. Assume that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where α is a function from $(0, \infty)$ into $(0, 1)$ such that $\limsup_{s \to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$. If
(i) T is $\perp$–continuous or;
(ii) X is an $\perp$-regular metric space;
then T is an OMWP operator.

Proof . Let $F(t) = \ln(t)$, $\theta(t) = -\ln(\alpha(t))$ for each $t \in (0, \infty)$, and $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be defined by $\phi(t_1, t_2, t_3, t_4, t_5) = t_1$ then $(\theta, F) \in \Delta$ and $\phi \in \Lambda$. Hence by using Theorem 3.1, T has a fixed point. □

In below, we explain a new fixed point theorem for single valued mappings.

Corollary 3.14. Let $(X, \perp, d)$ be an O–complete metric space (not necessarily a complete metric space), and $f : X \to X$ be an $\perp$–continuous and $\perp$–preserving mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that

$$\theta(d(x, y)) + F(d(fx, fy)) \leq F(\phi(d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)))$$

for all $\perp$–comparable elements $x, y \in X$ with $fx \neq fy$, where $\phi \in \Lambda$. Then f has a fixed point.

Open problem: Let $f : X \to X$ be a mapping satisfying in all conditions of Corollary 3.14, then can we conclude that f is a Picard operator? Does f have a unique fixed point?

Now we illustrate our main results by the following examples.

Example 3.15. Let (X, d) be a metric space, where $X = \{1, 2, 3, 4\}$, $d(1, 2) = d(1, 3) = 1$, $d(1, 4) = \frac{7}{4}$ and $d(2, 3) = d(2, 4) = d(3, 4) = 2$. Let $T : X \to CB(X)$ be given by $T1 = T4 = \{1, 4\}$, $T2 = T3 = \{4\}$ and $\perp = \{(1, 1), (1, 2), (1, 3), (1, 4), (4, 1), (4, 4)\}$ be a binary relation on X. Since X is finite set then every Cauchy sequence in (X, d) is equivalent constant and so convergent. Then $(X, \perp, d)$ is an O–complete metric space. It is easy to see that:
(i) X is an $\perp$–regular metric space;
(ii) the inequality

$$1 + \ln(H(Tx, Ty)) \leq \ln(\alpha.d(x, y) + L.D(y, Tx)),$$

holds for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\alpha = 1$ and $L = 4$. Then by Corollary 3.12, T has a fixed point.

Example 3.16. Let $X = (0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$, for each $x, y \in X$ and suppose that $x \perp y$ if and only if $y = 1$. Let $T : X \to CB(X)$ be given by $Tx = [\frac{x}{2}, x]$ whenever $x \in (0, \frac{1}{2})$ and $Tx = \{1\}$ whenever $x \in [\frac{1}{2}, 1]$. Now we can easily show that
(1) X is an O–complete and $\perp$–regular metric space.
(2) T is an $\perp$–preserving multivalued mapping;
(3) the inequality

$$\frac{1}{2} + \ln(H(Tx, Ty)) \leq \ln(\alpha.d(x, y) + L.D(y, Tx)),$$

holds for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\alpha = 1$ and $L = 11$. Then by Corollary 3.12, T has a fixed point.

4. Selection of multivalued mappings in incomplete metric space

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a nonempty subset of X. Consider a multivalued mapping $F : K \to B(Y)$. A function $f : K \to Y$ is called a selection of the F if and only if $f(x) \in F(x), x \in K$. Let

$$Sel(F) := \{f : K \to Y : f(x) \in F(x), x \in K\}.$$

It is easy to check that if there exists a constant $M > 0$ such that $diam(F(x)) \leq M.\|x\|$ for all $x \in K$, then the distance function

$$d(f, g) = \sup\left\{\frac{\|f(x) - g(x)\|}{\|x\|}, 0 \neq x \in K, f, g \in Sel(F)\right\},$$

is a metric in $Sel(F)$. Obviously, the convergence in the space $(Sel(F), d)$ implies the point wise convergence on the set K.

Theorem 4.1. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a nonempty subset of X such that $0 \in K$. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:*

1. $|\alpha| < p$ *and* $K \subseteq pK$,
2. $|\beta| < q$ *and* $K \subseteq qK$.

Consider a multivalued function $F : K \to B(Y)$ such that $0 \in F(0)$ and

$$diam(F(x)) \leq M.\|x\|, x \in K,$$

for some positive constant M. Also, for each $x \in K$, there exists $\perp_x \subseteq F(x) \times F(x)$ such that $(F(x), \perp_x, \|.\|)$ is an O–complete metric space with left orthogonal element x^*. If

$$\begin{aligned} &\alpha F(x) + \beta F(y) \subseteq F(px + qy), \\ &\alpha.\perp_x + \beta.\perp_y \subseteq \perp_{px+qy}, \end{aligned} \tag{4.1}$$

where $x, y \in K$ and $px + qy \in K$, then there exists a unique selection $f : K \to Y$ of multivalued mapping F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.$$

Proof . Assume that $|\alpha| < p$ and $K \subseteq pK$. Since $diamF(0) = 0$ and $0 \in F(0)$, then $F(0) = 0$ and $\perp_0 = \{(0,0)\}$. Putting $y = 0$ in (4.1), since $\perp_0 = \{(0,0)\}$, we obtain

$$\begin{aligned} &\alpha F(\frac{x}{p}) \subseteq F(x), \\ &\alpha.\perp_{\frac{x}{p}} \subseteq \perp_x, \end{aligned} \tag{4.2}$$

for each $x \in K$. Consider the following orthogonality relation on $Sel(F)$:

$$f \ \perp_* \ g \iff f(x) \ \perp_x \ g(x), \quad x \in K.$$

Let $f^* : K \to Y$ be defined by $f^*(x) = x^*$. It is easy to check that $(Sel(F), \perp_*)$ is an orthogonal set and f^* is an orthogonal element of $(Sel(F), \perp_*)$. Let $\mathcal{F}(g)(x) := \alpha.g(\frac{x}{p})$ for each $x \in K$ and $g \in Sel(F)$. By (4.2), $\mathcal{F}(g) \in Sel(F)$ and $\mathcal{F}$ is $\perp_*$–preserving. Hence, $\mathcal{F} : Sel(F) \to Sel(F)$ is an $\perp_*$-preserving mapping. Moreover, for each $g_1, g_2 \in Sel(F)$, we obtain that

$$\begin{aligned} d(\mathcal{F}(g_1), \mathcal{F}(g_2)) &= |\alpha|. \sup \left\{ \frac{\|g_1(\frac{x}{p}) - g_2(\frac{x}{p})\|}{\|x\|}, 0 \neq x \in K \right\} \\ &= \frac{|\alpha|}{p}. \sup \left\{ \frac{\|g_1(\frac{x}{p}) - g_2(\frac{x}{p})\|}{\frac{\|x\|}{p}}, 0 \neq x \in K \right\} \\ &\leq \frac{|\alpha|}{p}.d(g_1, g_2). \end{aligned}$$

Since $|\alpha| < p$, then $\mathcal{F} : Sel(F) \to Sel(F)$ is a contractive mapping in $(Sel(F), d)$. Now, according to the assumptions, since for each $x \in K$, $(F(x), \perp_x, \|.\|)$ is an O–complete metric space, then $(Sel(F), \perp_*, d)$ is an O–complete metric space. Therefore by Corollary 3.11 of [7], it has a unique fixed point f and $\lim_{n\to\infty} \mathcal{F}^n(g) = f$ for each $g \in Sel(F)$. Hence $f : K \to Y$ is the unique selection of F such that

$$f(x) = \alpha.f(\frac{x}{p}), \qquad x \in K.$$

Fix $g \in Sel(F)$ and $x, y \in K$ such that $px + qy \in K$. Then $\frac{x}{p}, \frac{y}{p}$ and $\frac{px+qy}{p}$ are belong to K. By (4.1), $\alpha.g(\frac{x}{p}) + \beta.g(\frac{y}{p})$ and $g(\frac{px+qy}{p})$ are elements of $F(\frac{px+qy}{p})$. Hence

$$\begin{aligned} \left\| \alpha.g(\frac{x}{p}) + \beta.g(\frac{y}{p}) - g(\frac{px+qy}{p}) \right\| &\leq diamF(\frac{px+qy}{p}) \\ &\leq M. \left\| \frac{px+qy}{p} \right\|. \end{aligned}$$

Thus

$$\|\alpha.\mathcal{F}(g)(x) + \beta.\mathcal{F}(g)(y) - \mathcal{F}(g)(px+qy)\| \leq M\frac{|\alpha|}{p}\|px+qy\|$$

for each $x, y \in K$ such that $px + qy \in K$. Repeating this process, we get

$$\|\alpha.\mathcal{F}^n(g)(x) + \beta.\mathcal{F}^n(g)(y) - \mathcal{F}^n(g)(px+qy)\| \leq M\Big(\frac{|\alpha|}{p}\Big)^n\|px+qy\|$$

for each $n \in \mathbb{N}$ and all $x, y \in K$ with $px + qy \in K$. Letting $n \to \infty$, we obtain

$$\alpha f(x) + \beta f(y) = f(px+qy), x, y \in K, px+qy \in K.$$

□

Corollary 4.2. (Smajdor and Szczawinska, [25]) Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a nonempty subset of X such that $0 \in K$. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:

1. $|\alpha| < p$ and $K \subseteq pK$,
2. $|\beta| < q$ and $K \subseteq qK$.

Consider a multivalued mapping $F : K \to CP(Y)$ such that $0 \in F(0)$ and

$$diam(F(x)) \leq M.\|x\|, x \in K,$$

for some positive constant M. If

$$\alpha F(x) + \beta F(y) \subseteq F(px + qy),$$

where $x, y \in K$ and $px + qy \in K$, then there exists a unique selection $f : K \to Y$ of multivalued mapping F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.$$

Corollary 4.3. Let $(X, \|.\|)$ and $(Y, \|.\|)$ be real normed spaces and let K be a convex cone in X. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:

1. $|\alpha| < p$ and $K \subseteq pK$,
2. $|\beta| < q$ and $K \subseteq qK$.

Consider a multivalued mapping $F : K \to B(Y)$ such that $0 \in F(0)$ and

$$diam(F(x)) \leq M.\|x\|, x \in K,$$

for some positive constant M. Also, for each $x \in K$, there exists $\perp_x \subseteq F(x) \times F(x)$ such that $(F(x), \perp_x, \|.\|)$ is an O–complete metric space with left orthogonal element x^*. If

$$\alpha F(x) + \beta F(y) \subseteq F(px + qy),$$
$$\alpha.\perp_x + \beta.\perp_y \subseteq \perp_{px+qy},$$

where $x, y \in K$, then there exists a unique selection $f : K \to Y$ of multivalued mapping F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K.$$

References

[1] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Fixed Point Theory for Lipschitzian–Type Mappings with Applications*, Vol. 6, New York: Springer, 2009.

[2] R.P. Agarwal, D. O'Regan and N.S. Papageorgiou, *Common fixed point theory for multivalued contractive maps of Reich type in uniform spaces*, Appl. Anal. 83 (2004) 37–47.

[3] R. P. Agarwal, D. O'Regan and N. Shahzad, *Fixed point theorems for generalized contractive maps of Meir-Keeler type*, Math. Nachr. 276 (2004) 3–22.

[4] I. Altun, G. Durmaza, G. Minak and S. Romaguera, *Multivalued almost F–contractions on complete metric spaces*, Filomat 30 (2016) 441–448.

[5] I. Altun, G. Minak and H. Dag, *Multivalued F–contractions on complete metric spaces*, J. Nonlinear Convex Anal. 16 (2015) 659–666.

[6] A. Amini–Harandi, *Fixed and coupled fixed points of a new type set–valued contractive mappings in complete metric spaces*, Fixed Point Theory and Appl. 2012 (2012): 215.

[7] H. Baghani, M. Eshaghi Gordji and M. Ramezani, *Orthogonal sets: The axiom of choice and proof of a fixed point theorem*, J. Fixed Point Theory and Appl. 18 (2016) 465–477.

[8] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. 3 (1922) 133–181.

[9] M. Berinde and V. Berinde, *On a general class of multivalued weakly Picard mappings*, J. Math. Anal. Appl. 326 (2007) 772–782.

[10] V. Berinde and M. Pacurar, *The role of the Pompeiu–Hausdorff metric in fixed point theory*, Creat. Math. Inform. 22 (2013) 143–150.

[11] L. Čirič, *Multi–valued nonlinear contraction mappings*, Nonlinear Anal. 71 (2009) 2716–2723.

[12] M. Eshaghi Gordji, M. Ramezani, M. De La Sen and Y.J. Cho, *On orthogonal sets and Banach fixed Point theorem*, Fixed Point Theory 18 (2017) 569–578.

[13] Y. Feng and S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi–valued Caristi type mappings*, J. Math. Anal. Appl. 317 (2006) 103–112.

[14] D. Klim and D. Wardowski, *Fixed point theorems for set–valued contractions in complete metric spaces*, J. Math. Anal. Appl. 334 (2007) 132–139.

[15] T. Kamran and Q. Kiran, *Fixed point theorems for multi–valued mappings obtained by altering distances*, Math. Comput. Model. 54 (2011) 2772–2777.

[16] H. Kaneko, *Generalized contractive multi-valued mappings and their fixed points*, Math. Japon. 33 (1988) 57–64.

[17] N. Mizogochi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric space*, J. Math. Anal. Appl. 141 (1989) 177–188.

[18] S.B. Nadler, *Multi–valued contraction mappings*, Pacific J. Math. 30 (1969) 475–488.

[19] A. Nicolae, *Fixed point theorems for multivalued mappings of Feng-Liu type*, Fixed Point Theory 12 (2011) 145–154.

[20] A. Petrusel, *On Frigon–Granas–type multifunctions*, Nonlinear Anal. Forum 7 (2002) 113–122.

[21] S. Reich, *Some problems and results in fixed point theory*, Contemp. Math. 21 (1983) 179–187.

[22] S. Reich, *Fixed points of contractive functions*, Boll. Unione Mat. Ital. 5 (1972) 26–42.

[23] M. Sgroi and C. Vetro, *Multi–valued F–contractions and the solution of certain functional and integral equations*, Filomat 27 (2013) 1259–1268.

[24] M. Shoaib, M. Sarwar and C. Tunc, *Coupled fixed point theorem for multi–valued mapping via generalized contraction in partially ordered metric spaces with applications*, J. Math. Anal. 8 (2017) 27–39.

[25] A. Smajdor and J. Szczawinska, *Selections of set–valued functions satisfying the general linear inclusion*, J. Fixed Point Theory and Appl. 18 (2016) 133–145.

[26] T. Suzuki, *Mizoguchi and Takahashi's fixed point theorem is a real generalization of Nadler's*, J. Math. Anal. Appl. 340 (2008) 752–755.

[27] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Appl. 2012 (2012): 94.