Generalized multivalued $F$-contractions on incomplete metric spaces

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Abstract

In this paper, we explain a new generalized contractive condition for multivalued mappings and prove a fixed point theorem in metric spaces (not necessary complete) which extends some well-known results in the literature. Finally, as an application, we prove that a multivalued function satisfying a general linear functional inclusion admits a unique selection fulfilling the corresponding functional equation.

Keywords: Fixed point theorem, Weakly Picard operator, O-complete metric space, Selections of multivalued functions.

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1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote, respectively, the sets of all natural numbers, rational numbers and real numbers. Also, for every nonempty set $X$ denote $\mathcal{P}^*(X)$ the set of all nonempty subsets of $X$. Let $(X, d)$ be a metric space. We denote by $B(X), CB(X)$ and $CP(X)$ collections of all bounded, closed bounded and complete members of $\mathcal{P}^*(X)$, respectively. The number

$$\text{diam}(A) := \sup\{d(a, b) : a, b \in A\},$$

is said to be the diameter of $A \in \mathcal{P}^*(X)$. For $A, B \in CB(X)$ and $x \in X$, define

$$D(x, A) := \inf\{d(x, a) : a \in A\}$$

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and

\[ H(A, B) := \max \{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \}. \]

The function \( H \) is a metric on \( CB(X) \) and is called a Pompeiu–Hausdorff metric. We can find detailed information about the Pompeiu–Hausdorff metric in \[1, 10\]. It is well known that if \( X \) is a complete metric space, then so is the metric space \( (CB(X), H) \). Let \( T : X \to CB(X) \) be a map, then \( T \) is called a multivalued contraction if there exists \( r \in (0, 1) \) such that for all \( x, y \in X \), we have

\[ H(Tx, Ty) \leq rd(x, y). \]

In 1969, Nadler \[18\] proved that every multivalued contraction on a complete metric space has a fixed point. Since then, a lot of generalizations of the result of Nadler were given (see, for example \[2, 3, 6, 11, 13, 15, 14, 24, 26\]). An interesting important generalization of it were given by Berinde et al. \[9\] where the authors introduced the concept of a multivalued weakly Picard operator as follows:

**Definition 1.1.** (Berinde and Berinde, \[9\]) Let \((X, d)\) be a metric space and \( T : X \to P^*(X) \) be a multivalued operator. \( T \) is said to be a Multivalued Weakly Picard (MWP) operator if for each \( x \in X \) and any \( y \in Tx \), there exists a sequence \( \{x_n\} \) in \( X \) such that
(i) \( x_0 = x, x_1 = y \),
(ii) \( x_{n+1} \in Tx_n \),
(iii) the sequence \( \{x_n\} \) is convergent and its limit is a fixed point of \( T \).

Then Berinde et al. \[9\] showed that the type multivalued contractions on complete metric spaces considered by Nadler \[18\], Mizoguchi and Takahashi \[17\] and Petrusel \[20\] are MWP operators. In the same paper, Berinde et al. \[9\] introduced the concepts of multivalued almost contraction (the original name was multivalued \((\delta, L)\)–weak contraction) and proved the following important fixed point theorem:

**Theorem 1.2.** (Berinde and Berinde, \[9\]) Let \((X, d)\) be a complete metric space and let \( T \) be a multivalued almost contraction from \( X \) into \( CB(X) \), that is, there exist two constant \( \delta \in (0, 1) \) and \( L \geq 0 \) such that

\[ H(Tx, Ty) \leq \delta d(x, y) + L D(y, Tx) \]

for all \( x, y \in X \). Then \( T \) is an MWP operator.

Recently, Eshaghi et al. \[12\] introduced the notion of orthogonal sets and then gave a real extension of Banach’s fixed point theorem. Then, Baghani et al. \[7\] by using the notion, proved a statement which is equivalent to the axiom of choice and explain a generalization of Theorem 3.11 of \[12\].

In this paper, by combining the ideas of Baghani et al. \[7\] and Berinde et al. \[9\], we explain a new generalized contractive condition of multivalued mappings and prove a fixed point theorem in metric spaces (not necessary complete) which improves the main result of Altun et al. \[4, 5\], Amini–Harandi \[6\], Mizoguchi et al. \[17\], Sgroi et al. \[23\] and Smajdor et al. \[25\].

**Definition 1.3.** Let \( \Lambda \) be the class of those functions \( \phi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}_+ \) which satisfy the following conditions

(\(\Lambda_1\)) \( \phi \) is increasing in \( t_2, t_3, t_4 \) and \( t_5 \);
(\(\Lambda_2\)) \( t_{n+1} < \phi(t_n, t_n, t_{n+1}, t_{n+1}, 0) \) implies that \( t_{n+1} < t_n \), for each positive sequence \( \{t_n\} \);
(\(\Lambda_3\)) If \( t_n, s_n \to 0 \) and \( u_n \to \gamma > 0 \), as \( n \to \infty \), then we have \( \limsup_{n \to \infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) < \gamma \);
(\(\Lambda_4\)) \( \phi(u, u, u, 2u, 0) \leq u \) for each \( u \in \mathbb{R}_+ := (0, +\infty) \).
Example 1.4. Let \( \phi : \mathbb{R}^+_+ \rightarrow \mathbb{R}_+ \) defined by
\[
\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + Lt_5,
\]
where \( \alpha, \beta, \gamma, \delta, L \geq 0, \alpha + \beta + \gamma + 2\delta = 1 \) and \( \gamma \neq 1 \). We claim that \( \phi \in \Lambda \). Indeed \((\Lambda_1)\) obviously holds. To show \((\Lambda_2)\), let \( \{t_n\} \) be a positive sequence such that
\[
t_{n+1} < \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0) = \alpha t_n + \beta t_n + \gamma t_{n+1} + \delta(t_n + t_{n+1}) = (\alpha + \beta + \delta) t_n + (\gamma + \delta) t_{n+1}.
\]
Since \( \alpha + \beta + \gamma + 2\delta = 1 \) and \( \gamma \neq 1 \), then we can conclude that \( 1 - (\gamma + \delta) > 0 \) and hence
\[
t_{n+1} < \frac{(\alpha + \beta + \delta)}{1 - (\gamma + \delta)} t_n = t_n.
\]
It is obvious that properties \((\Lambda_3)\) and \((\Lambda_4)\) hold for this function.

Definition 1.5. (Amini–Harandi, [6]) Let \( F : (0, +\infty) \rightarrow \mathbb{R} \) and \( \theta : (0, +\infty) \rightarrow (0, +\infty) \) be two mappings. Throughout the paper, let \( \Delta \) be the set of all pairs \((\theta, F)\) satisfying the following conditions:
\( \delta_1 \) \( \theta(t_n) \neq 0 \) for each strictly decreasing sequence \( \{t_n\} \);
\( \delta_2 \) \( F \) is a strictly increasing function;
\( \delta_3 \) For each sequence \( \{\alpha_n\} \) of positive numbers, \( \lim_{n \to \infty} \alpha_n = 0 \) if and only if \( \lim_{n \to \infty} F(\alpha_n) = -\infty \);
\( \delta_4 \) If \( t_n \downarrow 0 \) and \( \theta(t_n) \leq F(t_n) - F(t_{n+1}) \) for each \( n \in \mathbb{N} \), then we have \( \sum_{n=1}^{\infty} t_n < \infty \).

Example 1.6. (Amini–Harandi, [6]) Let \( F(t) = \ln(t) \) and \( \theta(t) = -\ln(\alpha(t)) \) for each \( t \in (0, +\infty) \), where \( \alpha : (0, \infty) \rightarrow (0, 1) \) satisfies \( \lim \sup_{s \to t^+} \alpha(s) < 1 \) for all \( t \in [0, \infty) \). Then \((\theta, F) \in \Delta\).

2. Orthogonal sets

We start our work with the following definition, which can be considered as the main definition of our paper.

Definition 2.1. (Eshaghi et al., [12]) Let \( X \neq \emptyset \) and \( \perp \subseteq X \times X \) be a binary relation. If \( \perp \) satisfies the following condition
\[
\exists x_0 \in X : (\forall y, y \perp x_0) \ or \ (\forall y, x_0 \perp y),
\]
then \( \perp \) is called an orthogonality relation and the pair \((X, \perp)\) an orthogonal set (briefly \( O\)-set).

Note that in above definition, we say that \( x_0 \) is orthogonal element. If \((X, \perp)\) has only one orthogonal element, then it is called a uniquely orthogonal set and the element is said unique orthogonal element. Also, an orthogonal element \( x_0 \) is called left orthogonal element if \( x_0 \perp x \) for each \( x \in X \). Similarly, it is called a right orthogonal element if \( x \perp x_0 \) for each \( x \in X \). Finally, we say that elements \( x, y \in X \) are \( \perp \)-comparable either \( x \perp y \) or \( y \perp x \).

As an illustration, let us consider the following examples.

Example 2.2. (Eshaghi et al., [12]) Let \( X \) be the set of all people in the world. We define \( x \perp y \) if \( x \) can give blood to \( y \). According to the following table, if \( x_0 \) is a person such that his (her) blood type is \( AB^+ \), then we have \( y \perp x_0 \) for all \( y \in X \). This means that \((X, \perp)\) is a \( O\)-set. Also, Let \( x_0 \) be a person with blood type \( O^- \), then we have \( x_0 \perp y \) for all \( y \in X \). Hence, in the \( O\)-set, \( x_0 \) is not unique.
### Example 2.3
Let $\Sigma$ be a family of nonempty subsets of $X$. Assume $\mu$ is the set of all $\sigma$–algebras containing $\Sigma$. Define $A \perp_\mu B$ iff $B \subset A$. Hence $(\mu, \perp_\mu)$ is an uniquely O–set that $\sigma$–algebra generated by $\Sigma$ is a unique orthogonal element of $\mu$.

### Example 2.4
Let $(X, \perp)$ be an O–set. Let $f$ be a choice function on $\mathcal{P}^*(X)$. For all $A, B \in \mathcal{P}^*(X)$ define $A \perp^* B$ if and only if $f(A) \perp f(B)$. It is clear that $(\mathcal{P}^*(X), \perp^*)$ is an O–set and $\{x^*\}$ is an orthogonal element of $(\mathcal{P}^*(X), \perp^*)$, where $x^*$ is an orthogonal element of $(X, \perp)$.

### Example 2.5
Let $X$ be a nonempty set. If $f$ is a choice function on $\mathcal{P}^*(X)$, then $f$ defines an equivalent relation $\perp^*$ on $\mathcal{P}^*(X)$ via

$$A \perp^* B \iff f(A) = f(B).$$

The relation $\perp^*$ in above satisfies the following.

1. The set of all equivalence classes modulo $\perp^*$ is

$$\mathcal{P}^*(X)/\perp^* = \{\{x\}/\perp^* : x \in X\},$$

where $\{x\}/\perp^*$ is the equivalence class of $\{x\}$ modulo $\perp^*$.

2. Every element of $\{x\}/\perp^*$ contains $x$.

It is easy to see that for each $x \in X$, $(\{x\}/\perp^*, \perp^*)$ is an O–set and $\{x\}$ is unique orthogonal element of $(\{x\}/\perp^*, \perp^*)$.

Let $(X, \perp)$ be a O–set and $A, B \subseteq X$. The binary relation $\perp$ between $A$ and $B$ is defined as follows.

- $A \perp B$ if $a \perp b$ for all $a \in A$ and $b \in B$.

Now, we introduce $\perp$–preserving multivalued mapping by using the relation $\perp$.

### Definition 2.6
Let $(X, \perp, d)$ be a orthogonal metric space ( $(X, \perp)$ is an O–set and $(X, d)$ is a metric space) and $T : X \rightarrow CB(X)$. Then $T$ is said to be an $\perp$–preserving multivalued mapping if

$$x, y \in X, x \perp y \Rightarrow Tx \perp Ty.$$

### Example 2.7
Let $X = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \cup \{0, 1\}$, $d(x, y) = |x - y|$ for all $x, y \in X$, and binary relation $\perp$ on $X$ be defined by

$$x \perp y \iff \begin{cases} \frac{y}{x} \in \mathbb{N}, \\ or \ x = y = 0. \end{cases}$$
Let \( T : X \to CB(X) \) be defined by

\[
T(x) = \begin{cases} 
\{ \frac{1}{2^n}, \frac{1}{2^{n+1}} \} & \text{if } x = \frac{1}{2^n}, n = 1, 2, \ldots, \\
\{0\} & \text{if } x = 0, \\
\{1, \frac{1}{2}, \frac{1}{4}\} & \text{if } x = 1.
\end{cases}
\]

It is easy to see that \( T \) is not an \( \perp \)-preserving multivalued mapping. Since \( \frac{1}{2} \perp 1 \) but \( T(\frac{1}{2}) = \{\frac{1}{2}, \frac{1}{4}\} \) is not orthogonal to \( \{1, \frac{1}{2}, \frac{1}{4}\} = T(1) \).

**Example 2.8.** Let \( X = [0, 1) \) and let the metric on \( X \) be the Euclidean metric. Define a binary relation \( \perp \) on \( X \) by \( x \perp y \) if \( xy \in \{x, y\} \) for all \( x, y \in X \). Let \( T : X \to CB(X) \) be a mapping defined by

\[
T(x) = \begin{cases} 
\{\frac{1}{2}x^2, x\}, & x \in \mathbb{Q} \cap X, \\
\{0\}, & x \in \mathbb{Q}^c \cap X.
\end{cases}
\]

It is easy to see that \( T \) is an \( \perp \)-preserving multivalued mapping.

### 3. Fixed Point Theory

In this section, we prove our main theorem. To this end, we need the following definitions.

**Definition 3.1.** (Eshaghi et al., [12]) Let \( (X, \perp) \) be an O-set. A sequence \( \{x_n\} \) is called an orthogonal sequence (briefly, O-sequence) if

\[(\forall n, x_n \perp x_{n+1}) \text{ or } (\forall n, x_{n+1} \perp x_n).\]

**Definition 3.2.** (Eshaghi et al., [12]) Let \( (X, \perp, d) \) be an orthogonal metric space. Then \( X \) is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

**Definition 3.3.** Let \( (X, \perp, d) \) be an orthogonal metric space. Then \( X \) is said to be orthogonally regular (briefly, \perp-regular) if \( X \) has the following properties

(i) for each sequence \( \{x_n\} \) such that \( x_n \perp x_{n+1} \) for all \( n \in \mathbb{N} \), and \( x_n \to x \), for some \( x \in X \), then \( x_n \perp x \) for all \( n \in \mathbb{N} \);

(ii) for each sequence \( \{x_n\} \) such that \( x_{n+1} \perp x_n \) for all \( n \in \mathbb{N} \), and \( x_n \to x \), for some \( x \in X \), then \( x \perp x_n \) for all \( n \in \mathbb{N} \).

**Example 3.4.** Let \( X = \mathbb{Q} \). Suppose that \( x \perp y \) if and only if \( x = 0 \) or \( y = 0 \). Clearly, \( \mathbb{Q} \) with the Euclidean metric is not a complete metric space, but it is O-complete. In fact, if \( \{x_k\} \) is an arbitrary Cauchy O-sequence in \( \mathbb{Q} \), then there exists a subsequence \( \{x_{k_n}\} \) of \( \{x_k\} \) for which \( x_{k_n} = 0 \) for all \( n \geq 1 \). It follows that \( \{x_{k_n}\} \) converges to 0 in \( X \). On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that \( \{x_k\} \) is convergent. It is easy to see that \( (X, \perp, d) \) is also an \perp-regular metric space.

**Example 3.5.** Let \( X = [0, 1) \). Suppose that

\[
x \perp y \iff \begin{cases} 
x \leq y \leq \frac{1}{4}, \\
or x = 0.
\end{cases}
\]
Clearly, $X$ with the Euclidian metric is not complete metric space, but it is $\perp$–complete. In fact, if $\{x_k\}$ is an arbitrary Cauchy $O$–sequence in $X$, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all $n \geq 1$ or there exists a monotone subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} \leq \frac{1}{n}$ for all $n \geq 1$. It follows that $\{x_{k_n}\}$ converges to a point $x \in [0, \frac{1}{4}] \subseteq X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. It is easy to see that $(X, \perp, d)$ is also an $\perp$–regular metric space.

**Definition 3.6.** Let $(X, \perp, d)$ be an orthogonal metric space. Then $T : X \rightarrow CB(X)$ is said to be orthogonally continuous (or $\perp$–continuous) in $a \in X$ if, for each $O$–sequence $\{a_n\}$ in $X$ with $a_n \rightarrow a$, we have $T(a_n) \rightarrow T(a)$. Also, $T$ is said to be $\perp$–continuous on $X$ if $T$ is $\perp$–continuous in each $a \in X$.

It is easy to see that every continuous mapping is $\perp$–continuous. The following example shows that the converse of the statement is not true in general.

**Example 3.7.** Let $X = \mathbb{R}$. Suppose $x \perp y$ if and only if $x = 0$ or $0 \neq y \in \mathbb{Q}$. It is easy to see that $(X, \perp)$ is an $O$–set. Define $T : X \rightarrow CB(X)$ by

$$T(x) = \begin{cases} \{x\}, & x \in \mathbb{Q}, \\ \{0\}, & x \in \mathbb{Q}^c. \end{cases}$$

The function $T$ is $\perp$–continuous at all rational numbers while it is continuous just at $x = 0$.

**Definition 3.8.** Let $(X, \perp, d)$ be an orthogonal metric space and $T : X \rightarrow \mathcal{P}^*(X)$ be a multivalued operator. $T$ is said to be an orthogonal multivalued Weakly Picard (OMWP) operator if for each orthogonal element $x \in X$ and any $y \in Tx$, there exists an orthogonal sequence $\{x_n\}$ in $X$ such that

(i) $x_0 = x, x_1 = y$,

(ii) $x_{n+1} \in Tx_n$,

(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of $T$.

Now, we are ready to prove the main theorem of this paper which can be consider as a multivalued version of Theorem 3.10 of [7].

**Theorem 3.9.** Let $(X, \perp, d)$ be an $O$–complete metric space (not necessarily a complete metric space), and $T : X \rightarrow CB(X)$ be an $\perp$–preserving multivalued mapping. Assume that there exists $(\frac{\theta}{2}, F) \in \Delta$ such that

$$\theta(d(x, y)) + F(H(Tx, Ty)) \leq F(\phi(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx))),$$

for all $\perp$–comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\phi \in \Lambda$. Also, suppose that $T$ is compact valued or $F$ is continuous from the right. If

(i) $T$ is $\perp$–continuous or;

(ii) $X$ is an $\perp$–regular metric space;

then $T$ is an OMWP operator.

**Proof.** Let $x_0$ be an orthogonal element of $X$. By the definition of orthogonality, we have

$$\forall y \in X, x_0 \perp y \quad \text{or} \quad \forall y \in X, y \perp x_0.$$
It follows that

\[ \forall y \in T(x_0), x_0 \perp y \text{ or } \forall y \in T(x_0), y \perp x_0. \]

Without loss of generality let

\[ \forall y \in T(x_0), x_0 \perp y. \]

Let \( x_1 \in T x_0 \) then \( x_0 \perp x_1 \). On the other hand, since \( T \) is \( \perp \)-preserving, then \( T x_0 \uparrow T x_1 \). If \( x_1 \in T x_1 \), then \( x_1 \) is fixed point of \( T \) and the proof is finished. Assume that \( x_1 \not\in T x_1 \), then \( T x_0 \neq T x_1 \). Since either \( T \) is compact valued or \( F \) is continuous from right, \( x_1 \in T x_0 \) and

\[ F(D(x_1, T x_1)) < F(H(T x_0, T x_1)) + \frac{\theta(d(x_0, x_1))}{2}, \]

then there exists \( x_2 \in T x_1 \) with \( x_1 \perp x_2 \) such that

\[ F(d(x_1, x_2)) \leq F(H(T x_0, T x_1)) + \frac{\theta(d(x_0, x_1))}{2}. \]

Repeating this process, we can construct an \( O \)-sequence \( \{x_n\} \) with initial point \( x_0 \) such that \( x_{n+1} \in T x_n, T x_n \neq T x_{n+1} \) and

\[ F(d(x_n, x_{n+1})) \leq F(H(T x_{n-1}, T x_n)) + \frac{\theta(d(x_{n-1}, x_n))}{2} \tag{3.2} \]

for all \( n \in \mathbb{N} \). From (3.1), (3.2), (A_1) and (\( \delta_2 \)) we have

\[
\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1}))
\leq \theta(d(x_{n-1}, x_n)) + F(H(T x_{n-1}, T x_n)) + \frac{\theta(d(x_{n-1}, x_n))}{2}
\leq F(\phi(d(x_{n-1}, x_n), D(x_{n-1}, T x_{n-1}), D(x_n, T x_n), D(x_{n-1}, T x_{n-1}), D(x_n, T x_{n-1})))
\quad + \frac{\theta(d(x_{n-1}, x_n))}{2}
\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0)) + \frac{\theta(d(x_{n-1}, x_n))}{2}
\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0))
\quad + \frac{\theta(d(x_{n-1}, x_n))}{2},
\]

and so

\[ \frac{\theta(d(x_{n-1}, x_n))}{2} + F(d(x_n, x_{n+1})) \leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)) \tag{3.3} \]

for each \( n \in \mathbb{N} \). This implies that

\[
d(x_n, x_{n+1}) < \phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)
\]
for each \( n \in \mathbb{N} \). Then by \((\Lambda_2)\), \( d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \) for each \( n \in \mathbb{N} \). Since \( \{d(x_n, x_{n+1})\} \) is a strictly decreasing sequence, then by using (3.3), \((\Lambda_1)\) and \((\Lambda_4)\), we obtain that

\[
\frac{\theta(d(x_{n-1}, x_n))}{2} + F(d(x_n, x_{n+1})) \\
\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n)), 0)) \\
\leq F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n)), 0)) \\
= F(\phi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), 2d(x_{n-1}, x_n), 0)) \\
\leq F(d(x_{n-1}, x_n),)
\]

for each \( n \in \mathbb{N} \). Let \( \lim_{n \to \infty} d(x_n, x_{n+1}) = r \), for some \( r \geq 0 \). Now, we show that \( r = 0 \). On contrary, assume that \( r > 0 \). From (3.4) we get

\[
\frac{1}{2} \sum_{i=1}^{n} \theta(d(x_i, x_{i+1})) \leq F(d(x_1, x_2)) - F(d(x_{n+1}, x_{n+2}))
\]

for each \( n \in \mathbb{N} \). Since \( \{d(x_n, x_{n+1})\} \) is strictly decreasing, then from (3.4) we obtain that \( \theta(d(x_n, x_{n+1})) \neq 0 \). Thus, \( \sum_{i=1}^{\infty} \theta(d(x_i, x_{i+1})) = +\infty \), and then from (3.5) we have \( \lim_{n \to \infty} F(d(x_n, x_{n+1})) = -\infty \). Then by (3.6), \( d(x_n, x_{n+1}) \to 0 \), as \( n \to \infty \), that a contradiction. Hence

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]

From (3.4), (3.6) and (3.1), we have \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \). Then by triangle inequality \( \{x_n\} \) is a Cauchy \( \odot \)-sequence. Since \( X \) is \( \odot \)-complete, then there exists \( x \in X \) such that \( \lim_{n \to \infty} x_n = x \). Now, we prove that \( x \) is fixed point of \( T \).

Case 1. \( T \) is \( \perp \)-continuous.

In this case, we have

\[
D(x, Tx) = \lim_{n \to \infty} D(x_{n+1}, Tx) \leq \lim_{n \to \infty} H(Tx_n, Tx) = 0.
\]

Then \( x \in Tx \) and the proof is complete.

Case 2. \( X \) is an \( \perp \)-regular metric space.

If there exists a strictly increasing sequence \( \{n_k\} \) such that \( x_{n_k} \in Tx \) for all \( k \in \mathbb{N} \), since \( Tx \) is closed and \( x_{n_k} \to x \), as \( k \to \infty \), we get that \( x \in Tx \) and the proof is complete. So, we can assume that there exists \( n_0 \in \mathbb{N} \) such that \( x_n \not\in Tx \) for each \( n > n_0 \). This implies that \( Tx_n \neq Tx \) for each \( n \geq n_0 \). Now since \( X \) is an \( \perp \)-regular metric space by using (3.1) with \( x = x_n \) and \( y = x \), we obtain

\[
F(D(x_{n+1}, Tx)) < \theta(d(x_n, x)) + F(D(x_{n+1}, Tx)) \\
\leq \theta(d(x_n, x)) + F(H(Tx_n, Tx)) \\
\leq F(\phi(d(x_n, x), D(x_n, Tx_n), D(x, Tx), D(x, Tx), D(x, Tx_n)), (x, Tx_n)) \\
\leq F(\phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x, Tx), d(x, x_{n+1})))
\]

for each \( n \geq n_0 \). Therefore

\[
D(x_{n+1}, Tx) < \phi(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x, Tx), d(x, x_{n+1}))
\]
for each \( n \geq n_0 \). Now if \( x \in Tx \), then the proof is complete. Let \( x \notin Tx \) then by using (3) and \((\Lambda_3)\) we have
\[
D(x,Tx) = \limsup_{n \to \infty} D(x_{n+1},Tx) \\
\leq \limsup_{n \to \infty} \phi(d(x,x),d(x_{n+1},x),D(x,Tx),d(x_{n+1})) \\
< D(x,Tx),
\]
which is a contradiction. Hence \( x \in Tx \) and the proof is complete. \( \square \)

Letting \( \phi(t_1,t_2,t_3,t_4,t_5) = t_1 \), we get a generalization of Theorem 2.4 of [6], Theorem 2 and Theorem 3 of [5] as follows.

**Corollary 3.10.** Let \((X , \perp , d)\) be an \( \perp \)-complete metric space (not necessarily a complete metric space), and \( T : X \to CB(X) \) be an \( \perp \)-preserving multivalued mapping. Assume that there exists \((\theta_2,F)\) \(\in \Delta\) such that
\[
\theta(d(x,y)) + F(H(Tx,Ty)) \leq F(d(x,y))
\]
for all \( \perp \)-comparable elements \( x,y \in X \) with \( Tx \neq Ty \). Also, suppose that \( T \) is compact valued or \( F \) is continuous from the right. If
(i) \( T \) is \( \perp \)-continuous or;
(ii) \( X \) is an \( \perp \)-regular metric space;
then \( T \) is an OMWP operator.

Letting
\[
\phi(t_1,t_2,t_3,t_4,t_5) = t_1 + \lambda t_5,
\]
where \( \lambda \geq 0 \), we get a generalization of Theorem 2.2 of [4] as follows.

**Corollary 3.11.** Let \((X , \perp , d)\) be an \( \perp \)-complete metric space (not necessarily a complete metric space), and \( T : X \to CB(X) \) be an \( \perp \)-preserving multivalued mapping. Assume that there exists \((\theta_2,F)\) \(\in \Delta\) such that
\[
\theta(d(x,y)) + F(H(Tx,Ty)) \leq F(d(x,y) + \lambda D(y,Tx))
\]
for all \( \perp \)-comparable elements \( x,y \in X \) with \( Tx \neq Ty \), where \( \lambda \geq 0 \). Also, suppose that \( T \) is compact valued or \( F \) is continuous from the right. If
(i) \( T \) is \( \perp \)-continuous or;
(ii) \( X \) is an \( \perp \)-regular metric space;
then \( T \) is an OMWP operator.

Letting
\[
\phi(t_1,t_2,t_3,t_4,t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + Lt_5,
\]
where \( \alpha, \beta, \gamma, \delta, L \geq 0 \), \( \alpha + \beta + \gamma + 2 \delta = 1 \) and \( \gamma \neq 1 \), we get a generalization of Theorem 3.4 of [23] as follows.

**Corollary 3.12.** Let \((X , \perp , d)\) be an \( \perp \)-complete metric space (not necessarily a complete metric space), and \( T : X \to CB(X) \) be an \( \perp \)-preserving multivalued mapping. Assume that there exists \((\theta_2,F)\) \(\in \Delta\) such that
\[
\theta(d(x,y)) + F(H(Tx,Ty)) \\
\leq F(\alpha d(x,y) + \beta D(x,Tx) + \gamma D(y,Ty) + \delta D(x,Ty) + LD(y,Tx))
\]
for all \( \perp \)-comparable elements \( x, y \in X \) with \( Tx \neq Ty \), where \( \alpha, \beta, \gamma, \delta, L \geq 0 \), \( \alpha + \beta + \gamma + 2\delta = 1 \) and \( \gamma \neq 1 \). Also, suppose that \( T \) is compact valued or \( F \) is continuous from the right. If (i) \( T \) is \( \perp \)-continuous or; (ii) \( X \) is an \( \perp \)-regular metric space; then \( T \) is an OMWP operator.

**Proof.** By using Example 2.1 of [6], we can easily show that this corollary is a generalization of Theorem 3.4 of [23]. □

In below we explain a generalization of Mizoguchi–Takahashi’s fixed point theorem [17].

**Corollary 3.13.** Let \( (X, \perp, d) \) be an \( O \)-complete metric space (not necessarily a complete metric space), and \( T : X \to CB(X) \) be an \( \perp \)-preserving multivalued mapping. Assume that

\[
H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)
\]

for all \( \perp \)-comparable elements \( x, y \in X \) with \( Tx \neq Ty \), where \( \alpha \) is a function from \((0, \infty)\) into \((0, 1)\) such that \( \limsup_{s \to t^+} \alpha(s) < 1 \) for all \( t \in [0, \infty) \). If (i) \( T \) is \( \perp \)-continuous or; (ii) \( X \) is an \( \perp \)-regular metric space; then \( T \) is an OMWP operator.

**Proof.** Let \( F(t) = \ln(t), \theta(t) = -\ln(\alpha(t)) \) for each \( t \in (0, \infty) \), and \( \phi : \mathbb{R}_+^5 \to \mathbb{R}_+ \) be defined by \( \phi(t_1, t_2, t_3, t_4, t_5) = t_1 \) then \( (\theta, F) \in \Delta \) and \( \phi \in \Lambda \). Hence by using Theorem 3.1, \( T \) has a fixed point. □

In below, we explain a new fixed point theorem for single valued mappings.

**Corollary 3.14.** Let \( (X, \perp, d) \) be an \( O \)-complete metric space (not necessarily a complete metric space), and \( f : X \to X \) be an \( \perp \)-continuous and \( \perp \)-preserving mapping. Assume that there exists \( (\theta, F) \in \Delta \) such that

\[
\theta(d(x, y)) + F(d(fx, fy)) \leq F(\phi(d(x, y), d(x, fx), d(y, fy), d(x, fy)))
\]

for all \( \perp \)-comparable elements \( x, y \in X \) with \( fx \neq fy \), where \( \phi \in \Lambda \). Then \( f \) has a fixed point.

**Open problem:** Let \( f : X \to X \) be a mapping satisfying in all conditions of Corollary 3.14, then can we conclude that \( f \) is a Picard operator? Does \( f \) have a unique fixed point?

Now we illustrate our main results by the following examples.

**Example 3.15.** Let \( (X, d) \) be a metric space, where \( X = \{1, 2, 3, 4\} \), \( d(1, 2) = d(1, 3) = 1 \), \( d(1, 4) = \frac{7}{2} \) and \( d(2, 3) = d(2, 4) = d(3, 4) = 2 \). Let \( T : X \to CB(X) \) be given by \( T1 = T4 = \{1, 4\}, T2 = T3 = \{4\} \) and \( \perp = \{(1, 1), (1, 2), (1, 3), (1, 4), (4, 1), (4, 4)\} \) be a binary relation on \( X \). Since \( X \) is finite set then every Cauchy sequence in \((X, d)\) is equivalent constant and so convergent. Then \( (X, \perp, d) \) is an \( O \)-complete metric space. It is easy to see that:

(i) \( X \) is an \( \perp \)-regular metric space;
(ii) the inequality

\[
1 + \ln(H(Tx, Ty)) \leq \ln(\alpha.d(x, y) + L.D(y, Tx)),
\]

holds for all \( \perp \)-comparable elements \( x, y \in X \) with \( Tx \neq Ty \), where \( \alpha = 1 \) and \( L = 4 \). Then by Corollary 3.12 \( T \) has a fixed point.
Example 3.16. Let $X = (0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$, for each $x, y \in X$ and suppose that $x \perp y$ if and only if $y = 1$. Let $T : X \to CB(X)$ be given by $Tx = \left[\frac{x}{2}, x\right]$ whenever $x \in (0, \frac{1}{2})$ and $Tx = \{1\}$ whenever $x \in [\frac{1}{2}, 1]$. Now we can easily show that

1. $X$ is an $O$-complete and $\perp$-regular metric space.
2. $T$ is an $\perp$-preserving multivalued mapping;
3. the inequality

$$\frac{1}{2} + \ln(H(Tx, Ty)) \leq \ln(\alpha d(x, y) + L.D(y, Tx)), $$

holds for all $\perp$-comparable elements $x, y \in X$ with $Tx \neq Ty$, where $\alpha = 1$ and $L = 11$. Then by Corollary 3.12 $T$ has a fixed point.

4. Selection of multivalued mappings in incomplete metric space

Let $(X, \|\|)$ and $(Y, \|\|)$ be real normed spaces and let $K$ be a nonempty subset of $X$. Consider a multivalued mapping $F : K \to B(Y)$. A function $f : K \to Y$ is called a selection of the $F$ if and only if $f(x) \in F(x), x \in K$. Let

$$Sel(F) := \{f : K \to Y : f(x) \in F(x), x \in K\}.$$ 

It is easy to check that if there exists a constant $M > 0$ such that $\text{diam}(F(x)) \leq M \|x\|$ for all $x \in K$, then the distance function

$$d(f, g) = \sup \left\{ \frac{\|f(x) - g(x)\|}{\|x\|}, 0 \neq x \in K, f, g \in Sel(F) \right\},$$

is a metric in $Sel(F)$. Obviously, the convergence in the space $(Sel(F), d)$ implies the point wise convergence on the set $K$.

Theorem 4.1. Let $(X, \|\|)$ and $(Y, \|\|)$ be real normed spaces and let $K$ be a nonempty subset of $X$ such that $0 \in K$. Suppose that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:

1. $|\alpha| < p$ and $K \subseteq pK$,
2. $|\beta| < q$ and $K \subseteq qK$.

Consider a multivalued function $F : K \to B(Y)$ such that $0 \in F(0)$ and

$$\text{diam}(F(x)) \leq M \|x\|, x \in K,$$

for some positive constant $M$. Also, for each $x \in K$, there exists $\perp_x \subseteq F(x) \times F(x)$ such that $(F(x), \perp_x, \|\|)$ is an $O$-complete metric space with left orthogonal element $x^*$. If

$$\alpha F(x) + \beta F(y) \subseteq F(px + qy),$$

$$\alpha \perp_x + \beta \perp_y \subseteq \perp_{px+qy},$$

(4.1)

where $x, y \in K$ and $px + qy \in K$, then there exists a unique selection $f : K \to Y$ of multivalued mapping $F$ such that

$$\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.$$
Proof. Assume that $|\alpha| < p$ and $K \subseteq pK$. Since $\text{diam} F(0) = 0$ and $0 \in F(0)$, then $F(0) = 0$ and $\bot_0 = \{(0,0)\}$. Putting $y = 0$ in (4.1), since $\bot_0 = \{(0,0)\}$, we obtain

$$\alpha F(\frac{x}{p}) \subseteq F(x),$$
$$\alpha.\bot_\frac{x}{p} \subseteq \bot_x,$$ (4.2)

for each $x \in K$. Consider the following orthogonality relation on $\text{Sel}(F)$:

$$f \perp_\ast g \iff f(x) \perp_x g(x), \ x \in K.$$ Let $f^* : K \to Y$ be defined by $f^*(x) = x^*$. It is easy to check that $(\text{Sel}(F), \perp_\ast)$ is an orthogonal set and $f^*$ is an orthogonal element of $(\text{Sel}(F), \perp_\ast)$. Let $\mathcal{F}(g)(x) := \alpha.g(\frac{x}{p})$ for each $x \in K$ and $g \in \text{Sel}(F)$. By (4.2), $\mathcal{F}(g) \in \text{Sel}(F)$ and $\mathcal{F}$ is $\perp_\ast$-preserving. Hence, $\mathcal{F} : \text{Sel}(F) \to \text{Sel}(F)$ is an $\perp_\ast$-preserving mapping. Moreover, for each $g_1, g_2 \in \text{Sel}(F)$, we obtain that

$$d(\mathcal{F}(g_1), \mathcal{F}(g_2)) = |\alpha|.\sup \left\{ \frac{\|g_1(\frac{x}{p}) - g_2(\frac{x}{p})\|}{\|x\|}, 0 \neq x \in K \right\}$$
$$= \frac{|\alpha|}{p}.\sup \left\{ \frac{\|g_1(\frac{x}{p}) - g_2(\frac{x}{p})\|}{\|x\|}, 0 \neq x \in K \right\}$$
$$\leq \frac{|\alpha|}{p}.d(g_1, g_2).$$

Since $|\alpha| < p$, then $\mathcal{F} : \text{Sel}(F) \to \text{Sel}(F)$ is a contractive mapping in $(\text{Sel}(F), d)$. Now, according to the assumptions, since for each $x \in K$, $(F(x), \perp_x, \|\|)$ is an $\text{O}$-complete metric space, then $(\text{Sel}(F), \perp_\ast, d)$ is an $\text{O}$-complete metric space. Therefore by Corollary 3.11 of [7], it has a unique fixed point $f$ and $\lim_{n \to \infty} \mathcal{F}^n(g) = f$ for each $g \in \text{Sel}(F)$. Hence $f : K \to Y$ is the unique selection of $F$ such that

$$f(x) = \alpha.f(\frac{x}{p}), \ x \in K.$$ Fix $g \in \text{Sel}(F)$ and $x, y \in K$ such that $px + qy \in K$. Then $\frac{x}{p}, \frac{y}{p}$ and $\frac{px + qy}{p}$ are belong to $K$. By (4.1), $\alpha.g(\frac{x}{p}) + \beta.g(\frac{y}{p})$ and $g(\frac{px + qy}{p})$ are elements of $F(\frac{px + qy}{p})$. Hence

$$\|\alpha.g(\frac{x}{p}) + \beta.g(\frac{y}{p}) - g(\frac{px + qy}{p})\| \leq \text{diam} F(\frac{px + qy}{p})$$
$$\leq M.\left\| \frac{px + qy}{p} \right\|.$$ Thus

$$\|\alpha.\mathcal{F}(g)(x) + \beta.\mathcal{F}(g)(y) - \mathcal{F}(g)(px + qy)\| \leq M.\frac{|\alpha|}{p}\|px + qy\|$$

for each $x, y \in K$ such that $px + qy \in K$. Repeating this process, we get

$$\|\alpha.\mathcal{F}^n(g)(x) + \beta.\mathcal{F}^n(g)(y) - \mathcal{F}^n(g)(px + qy)\| \leq M(\frac{|\alpha|}{p})^n\|px + qy\|$$

for each $n \in \mathbb{N}$ and all $x, y \in K$ with $px + qy \in K$. Letting $n \to \infty$, we obtain

$$\alpha.f(x) + \beta.f(y) = f(px + qy), x, y \in K, px + qy \in K.$$
Corollary 4.2. (Smajdor and Szczawinska, [25]) Let \((X, \|\|)\) and \((Y, \|\|)\) be real normed spaces and let \(K\) be a nonempty subset of \(X\) such that \(0 \in K\). Suppose that \(p, q > 0\) and \(\alpha, \beta \in \mathbb{R}\) are fixed and one of the following conditions holds:

1. \(|\alpha| < p\) and \(K \subseteq pK\),
2. \(|\beta| < q\) and \(K \subseteq qK\).

Consider a multivalued mapping \(F : K \to CP(Y)\) such that \(0 \in F(0)\) and

\[
\text{diam}(F(x)) \leq M \|x\|, x \in K,
\]

for some positive constant \(M\). If

\[
\alpha F(x) + \beta F(y) \subseteq F(px + qy),
\]

where \(x, y \in K\) and \(px + qy \in K\), then there exists a unique selection \(f : K \to Y\) of multivalued mapping \(F\) such that

\[
\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K, px + qy \in K.
\]

Corollary 4.3. Let \((X, \|\|)\) and \((Y, \|\|)\) be real normed spaces and let \(K\) be a convex cone in \(X\). Suppose that \(p, q > 0\) and \(\alpha, \beta \in \mathbb{R}\) are fixed and one of the following conditions holds:

1. \(|\alpha| < p\) and \(K \subseteq pK\),
2. \(|\beta| < q\) and \(K \subseteq qK\).

Consider a multivalued mapping \(F : K \to B(Y)\) such that \(0 \in F(0)\) and

\[
\text{diam}(F(x)) \leq M \|x\|, x \in K,
\]

for some positive constant \(M\). Also, for each \(x \in K\), there exists \(\perp_x \subseteq F(x) \times F(x)\) such that \((F(x), \perp_x, \|\|)\) is an \(O\)-complete metric space with left orthogonal element \(x^*\). If

\[
\alpha F(x) + \beta F(y) \subseteq F(px + qy),
\]

\[
\alpha \perp_x + \beta \perp_y \subseteq \perp_{px+qy},
\]

where \(x, y \in K\), then there exists a unique selection \(f : K \to Y\) of multivalued mapping \(F\) such that

\[
\alpha f(x) + \beta f(y) = f(px + qy), x, y \in K.
\]

References


