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Strict fixed points of Ćirić-generalized weak quasicontractive multi-valued mappings of integral type

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Abstract

Many authors such as Amini–Harandi, Rezapour *et al.*, Kadelburg *et al.*, have tried to find at least one fixed point for quasi–contractions when $\alpha \in [\frac{1}{2}, 1)$ but no clear answer exists right now and many of them either have failed or changed to a lighter version. In this paper, we introduce some new strict fixed point results in the set of multi–valued Ćirić–generalized weak quasi–contraction mappings of integral type. We consider a necessary and sufficient condition on such mappings which guarantees the existence of unique strict fixed point of such mappings. Our result is a partial positive answer for the mentioned problem which has remained open for many years. Also, we give an strict fixed point result of α–ψ–quasicontractive multi–valued mappings of integral type. Our results generalize and improve many existing results on multi–valued mappings in literature. Moreover, some examples are presented to support our new class of multi–valued contractions.

Keywords: Strict fixed point, Ćirić–generalized weak quasi–contraction, Multi–valued mappings, Integral type.
2010 MSC: Primary 47H10; Secondary 54H25.

1. Introduction

In the line of research of multi–valued mappings, one of the initial results was given by Nadler in [19]. He extended the Banach contraction principle to mappings that was associating a nonempty closed bounded set to any point of a metric space. After that, many authors have studied the

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existence and uniqueness of fixed point for multi–valued mappings in metric spaces (see, for instance, [1, 3, 13, 14, 15, 16, 19, 23] and references therein).

In 2009, Ilić and Rakočević [11] proved that quasi–contraction maps on normal cone metric spaces have a unique fixed point. Then, Kadelburg *et al.* [12] generalized their results by considering an additional assumption. Also, they proved that quasi–contraction maps on cone metric spaces have the property (P) whenever the contractivity constant α belongs to the interval $(0, \frac{1}{2})$. Later, Rezapour *et al.* [22] proved similar results without the additional assumption and for $\alpha \in (0, 1)$ by providing a new technical proof. In 2011, Wardowski [24] published a paper where he tried to test fixed point results for set–valued contractions on normal cone metric spaces. In 2011, Amini–Harandi [4] proved a result on the existence of fixed points of set–valued quasi–contraction maps in metric spaces by using the Rezapour *et al.*'s technique given in [22]. But, like Kadelburg *et al.* [12], he could only prove it for $\alpha \in (0, \frac{1}{2})$. Then, he proposed this question: Does any quasicontraction have a fixed point for $\alpha \in (0, 1)$? In 2012, Rezapour *et al.* [10] introduced quasi–contraction type multi–valued mappings and showed that the main result of Amini–Harandi also holds for quasi–contraction type multi–valued mappings. Then, they arised this question: Is there any quasi–contraction which is not a quasi–contraction type? In 2013, Mohammadi *et al.* [17] gave an affirmative answer to this question. So, the problem of Amini–Harandi has remained open up to this time. In this paper, we introduce multi–valued Ćirić–generalized weak quasi–contraction mappings of integral type and prove the existence of unique strict fixed point for such mappings by adding a necessary and sufficient condition, called "approximate strict fixed point property". Our result gives a partial positive answer to the problem of Amini–Harandi.

2. Preliminaries

Let (X, d) be a metric space and $\mathcal{CB}(X)$ the set of all nonempty closed bounded subsets of X. Assume that $\mathcal{H}$ be the Hausdorff metric on $\mathcal{CB}(X)$. Let $T : X \to 2^X$ be a multi–valued mapping. An element $x \in X$ is said to be an strict fixed point of T whenever $Tx = \{x\}$. Also it is said that T has the approximate strict fixed point property whenever

$$\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0,$$

or equivalently

$$\inf_{x\in X} \mathcal{H}(\{x\}, Tx) = 0,$$

see [2] for more details. In 2010, Amini–Harandi [2] proved that some multi–valued mappings $T : X \to \mathcal{CB}(X)$ have unique endpoint (strict fixed point) if and only if have the approximate endpoint (strict fixed point) property. After that, Moradi and Khojasteh [18] generalized Amini–Harandi's result by considering the same properties. A mapping $T : X \to X$ is said to be a *weak contraction* if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \le \alpha\, N(x, y) \quad \text{for all } x, y \in X,$$

where

$$N(x, y) = \max\left\{ d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{d(x, Ty) + d(y, Tx)}{2} \right\}. \tag{2.1}$$

Similarly, a multi–valued mapping $T : X \to \mathcal{CB}(X)$ is said to be *weak contraction* if there exists $\alpha \in [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \le \alpha\, N(x, y) \quad \text{for all } x, y \in X,$$

where $N(x, y)$ is also given by (2.1). Daffer and Kaneko [8] proved the following fixed point theorem for multi–valued weak contraction mappings.

Theorem 2.1. (Daffer and Kaneko, [8]) Let (X, d) be a complete metric space. Suppose that $T : X \to \mathcal{CB}(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,

$$\mathcal{H}(Tx, Ty) \le \alpha N(x, y)$$

for all $x, y \in X$ (i.e., it is a weak contraction). If the function $x \mapsto d(x, Tx)$ is lower semi–continuous, then there exists a point $x_0 \in X$ such that $x_0 \in Tx_0$.

An extension of the previous contractions can be considered as follows. A multi–valued mapping $T : X \to \mathcal{CB}(X)$ is said to be a *multi–valued quasi–contraction* whenever there exists $\alpha \in (0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \le \alpha\, M(x, y) \quad \text{for all} \;\; x, y \in X,$$

where

$$M(x, y) = \max\{\, d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\,\}.$$

In 2011, Amini–Harandi proved in [4] the following result about the existence of fixed points of multi–valued quasi–contractions in metric spaces by using the technique given in [22].

Theorem 2.2. (Amini–Harandi, [4]) Let (X, d) be a complete metric space and let $T : X \to \mathcal{CB}(X)$ be a multi–valued quasi-contraction for some $\alpha \in [0, \frac{1}{2})$. Then T has a fixed point.

Immediately he proposed the following question.
Question (A). Does the conclusion of Theorem 2.2 remains true for any $\alpha \in [\frac{1}{2}, 1)$?

On the other hand, in 2001, Branciari [5] generalized the Banach contraction principle to integral type mappings by using a Lebesgue integrable mapping $\varphi : [0, +\infty) \to [0, +\infty)$ which is summable on each compact subset of $[0, +\infty)$ such that $\int_0^\epsilon \varphi(t)dt > 0$, for any $\epsilon > 0$. Denote by Φ and Ψ, the set of all Lebesgue integrable mapping $\varphi : [0, +\infty) \to [0, +\infty)$ which is summable on each compact subset of $[0, +\infty)$ such that $\int_0^\epsilon \varphi(t)dt > 0$ for any $\epsilon > 0$ and upper bounded on $[0, +\infty)$ and the family of all upper semicontinuous (u.s.c) mappings $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$ for all $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$, respectively.

Definition 2.1. We say that $T : X \to \mathcal{CB}(X)$ is a Ćirić–generalized weak quasicontractive multi–valued mapping of integral type whenever there exist two mappings $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\int_0^{\mathcal{H}(Tx,Ty)} \varphi(t)dt \le \psi\Big(\int_0^{M(x,y)} \varphi(t)dt\Big) \quad \text{for all } x, y \in X. \tag{2.2}$$

3. Main results

The following theorem is the main result of this study.

Theorem 3.1. *Let (X, d) be a complete metric space, and let $T : X \to \mathcal{CB}(X)$ be a Ćirić–generalized weak quasicontractive multi–valued mapping of integral type. Then, T has a unique strict fixed point if and only if T has the approximate strict fixed point property.*

Proof . If T has an strict fixed point, obviously, it has the approximate strict fixed point property. Conversely, let T has the approximate strict fixed point property. Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}\mathcal{H}(\{x_n\}, Tx_n) = 0$. Now for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned} M(x_n, x_m) &= \max\{d(x_n, x_m), d(x_n, Tx_n), d(x_m, Tx_m), d(x_n, Tx_m), d(x_m, Tx_n)\} \\ &\le d(x_n, x_m) + \mathcal{H}(\{x_n\}, Tx_n) + \mathcal{H}(\{x_m\}, Tx_m) \\ &\le \mathcal{H}(Tx_n, Tx_m) + 2\mathcal{H}(\{x_n\}, Tx_n) + 2\mathcal{H}(\{x_m\}, Tx_m). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^{M(x_n,x_m)} \varphi(t)dt &\le \int_0^{\mathcal{H}(Tx_n,Tx_m)+2\mathcal{H}(\{x_n\},Tx_n)+2\mathcal{H}(\{x_m\},Tx_m)} \varphi(t)dt \\ &= \int_0^{\mathcal{H}(Tx_n,Tx_m)} \varphi(t)dt + \int_{\mathcal{H}(Tx_n,Tx_m)}^{\mathcal{H}(Tx_n,Tx_m)+2\mathcal{H}(\{x_n\},Tx_n)+2\mathcal{H}(\{x_m\},Tx_m)} \varphi(t)dt \\ &\le \int_0^{\mathcal{H}(Tx_n,Tx_m)} \varphi(t)dt + M(2\mathcal{H}(\{x_n\}, Tx_n) + 2\mathcal{H}(\{x_m\}, Tx_m)) \\ &\le \psi(\int_0^{M(x_n,x_m)} \varphi(t)dt) + M(2\mathcal{H}(\{x_n\}, Tx_n) + 2\mathcal{H}(\{x_m\}, Tx_m)), \end{aligned} \tag{3.1}$$

where M is a positive number which $\varphi(t) \le M$ for all $t \ge 0$. Thus, from (3.1), we have

$$\begin{aligned} \limsup_{n,m\to\infty} \int_0^{M(x_n,x_m)} \varphi(t)dt &\le \limsup_{n,m\to\infty} \psi(\int_0^{M(x_n,x_m)} \varphi(t)dt) \\ &\le \psi(\limsup_{n,m\to\infty} \int_0^{M(x_n,x_m)} \varphi(t)dt). \end{aligned} \tag{3.2}$$

From (3.2), one can conclude that

$$\limsup_{n,m\to\infty} \int_0^{M(x_n,x_m)} \varphi(t)dt = 0$$

and so $\limsup_{n,m\to\infty} M(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. We claim that x is an strict fixed point of T. To see this, we have

$$\begin{aligned} \int_0^{\mathcal{H}(\{x\},Tx)} \varphi(t)dt &\le \int_0^{d(x,x_n)+\mathcal{H}(\{x_n\},Tx_n)+\mathcal{H}(Tx_n,Tx)} \varphi(t)dt \\ &= \int_0^{\mathcal{H}(Tx_n,Tx)} \varphi(t)dt + \int_{\mathcal{H}(Tx_n,Tx)}^{\mathcal{H}(Tx_n,Tx)+d(x,x_n)+\mathcal{H}(\{x_n\},Tx_n)} \varphi(t)dt \\ &\le \int_0^{\mathcal{H}(Tx_n,Tx)} \varphi(t)dt + M(d(x, x_n) + \mathcal{H}(\{x_n\}, Tx_n)) \\ &\le \psi(\int_0^{M(x_n,x)} \varphi(t)dt) + M(d(x, x_n) + \mathcal{H}(\{x_n\}, Tx_n)). \end{aligned} \tag{3.3}$$

Taking the limit on both sides of (3.3) shows that

$$\begin{aligned} \int_0^{\mathcal{H}(\{x\},Tx)} \varphi(t)dt &\le \limsup_{n\to\infty} \psi(\int_0^{M(x_n,x)} \varphi(t)dt) \\ &\le \psi(\limsup_{n\to\infty} \int_0^{M(x_n,x)} \varphi(t)dt). \end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned} M(x_n, x) &= \max\{d(x_n, x), d(x_n, Tx_n), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\} \\ &\le d(x_n, x) + \mathcal{H}(\{x_n\}, Tx_n) + \mathcal{H}(\{x\}, Tx), \end{aligned}$$

which implies

$$\limsup_{n\to\infty} M(x_n, x) \leq \mathcal{H}(\{x\}, Tx).$$

Consequently, (3.4) yields that

$$\int_0^{\mathcal{H}(\{x\},Tx)} \varphi(t)dt \leq \psi(\int_0^{\mathcal{H}(\{x\},Tx)} \varphi(t)dt)$$

and so $\mathcal{H}(\{x\}, Tx) = 0$. This means that $Tx = \{x\}$. The uniqueness of strict fixed point is concluded from (2.2). □

The following Corollary is a generalization of Moradi and Khojasteh [18].

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multi–valued mapping such that there exists $\psi \in \Psi$ satisfying*

$$\mathcal{H}(Tx, Ty) \leq \psi(M(x, y)) \quad \textit{for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then, T has unique strict fixed point if and only if T has the approximate strict fixed point property.

Proof . Define $\varphi(t) = 1$ for all $t \geq 0$ and apply Theorem 3.1. □

Denote by $Fix(T)$ and $Sfix(T)$ the set of all fixed points and strict fixed points of T, respectively. The following corollary is a partial positive answer to Question (A).

Corollary 3.3. *Let (X, d) be a complete metric space and let $T : X \to \mathcal{CB}(X)$ be a multi–valued mapping such that*

$$\mathcal{H}(Tx, Ty) \leq \alpha\, M(x, y)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then, T has a unique strict fixed point in X if and only if T has the approximate strict fixed point property. In such a case, $Sfix(T) = Fix(T)$.

Proof . Using Corollary 3.2 for $\psi(t) = \alpha\, t$, we will have T has a unique strict fixed point in X if and only if T has the approximate strict fixed point property. If this is the case, then we shall show that $Sfix(T) = Fix(T)$. Obviously, $Sfix(T) \subseteq Fix(T)$. It is sufficient to show that $Fix(T) \subseteq Sfix(T)$. Let $y \in Fix(T)$ and x be the unique strict fixed point of T. Then,

$$\begin{aligned} d(x, y) &\leq \mathcal{H}(Tx, Ty) \\ &\leq \alpha\, M(x, y) \\ &= \alpha \max\{\, d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\,\} \\ &\leq \alpha\, d(x, y), \end{aligned}$$

which implies $d(x, y) = 0$ and so $y = x \in Sfix(T)$. □

The following example shows that Corollary 3.2 is a real generalization of Moradi and Khojasteh [18]. Also, this example supports Corollary 3.3 as a real generalization of Daffer and Kaneko [8] and a partial positive answer to Question (A).

Example 3.1. Let

$$M_1 = \{\frac{m}{n} : m = 0, 1, 3, 9, ..., n = 1, 4, ..., 3k + 1, ...\},$$

$$M_2 = \{\frac{m}{n} : m = 1, 3, 9, ..., n = 2, 5, ..., 3k + 2, ...\}$$

and $M = M_1 \cup M_2$ with the usual metric. Define $T : M \to \mathcal{CB}(\mathcal{M})$ by

$$Tx = \begin{cases} \left[0, \frac{3x}{5}\right] \cap M, & x \in M_1, \\ \left[0, \frac{x}{8}\right] \cap M, & x \in M_2. \end{cases}$$

If $x, y \in M_1$, then

$$H(Tx, Ty) = |\frac{3x}{5} - \frac{3y}{5}| = \frac{3}{5}|x - y| \leq \frac{3}{5}M(x, y).$$

If $x, y \in M_2$, then

$$H(Tx, Ty) = |\frac{x}{8} - \frac{y}{8}| = \frac{1}{8}|x - y| \leq \frac{3}{5}M(x, y).$$

If x be, for example, in M_1 and y in M_2, then

$$H(Tx, Ty) = |\frac{3x}{5} - \frac{y}{8}| = \frac{3}{5}|x - \frac{5}{24}y|.$$

Now if $x > \frac{5}{24}y$, then

$$H(Tx, Ty) = \frac{3}{5}(x - \frac{5}{24}y) \leq \frac{3}{5}(x - \frac{1}{8}y) = \frac{3}{5}d(x, Ty) \leq \frac{3}{5}M(x, y)$$

and if $x < \frac{5}{24}y$, then

$$H(Tx, Ty) = \frac{3}{5}(\frac{5}{24}y - x) \leq \frac{3}{5}(y - x) \leq \frac{3}{5}M(x, y).$$

We see that $H(Tx, Ty) \leq \frac{3}{5}M(x, y)$ for all $x, y \in M$. To check that T satisfies approximate strict fixed point property, let $x_n = \frac{1}{3n+1}$, for all $n \in \mathcal{N}$. Then,

$$H(\{x_n\}, Tx_n) = H\left(\{\frac{1}{3n + 1}\}, [0, \frac{3}{5(3n + 1)}]\right) = \frac{2}{5(3n + 1)} \to 0$$

as $n \to \infty$. Hence by Corollary 3.3, T has a unique strict fixed point. Here $T0 = \{0\}$ and $x = 0$ is the unique strict fixed point. Also note that $Sfix(T) = Fix(T) = \{0\}$. Now let $x = 1, y = \frac{1}{2}$, then $H(Tx, Ty) = |\frac{3}{5} - \frac{1}{16}| = \frac{43}{80}$ and

$$N(x, y) = \max\left\{|1 - \frac{1}{2}|, |1 - \frac{3}{5}|, |\frac{1}{2} - \frac{1}{16}|, \frac{1}{2}(|1 - \frac{1}{16}| + 0)\right\} = \frac{1}{2}.$$

We see that $H(Tx, Ty) = \frac{43}{80} > \frac{1}{2} = N(x, y) > \psi(N(x, y))$. So, we can't applly Daffer and Kaneko [8] and Moradi and Khojasteh [18] in this example. Also for $x = 1, y = \frac{1}{2}$, we have

$$M(x, y) = \max\left\{|1 - \frac{1}{2}|, |1 - \frac{3}{5}|, |\frac{1}{2} - \frac{1}{16}|, |1 - \frac{1}{16}|, 0\right\} = \frac{15}{16}.$$

So $\frac{H(Tx,Ty)}{M(x,y)} = \frac{43/80}{15/16} = \frac{43}{75} > \frac{1}{2}$. Hence T is not a quasi–contraction with constant contraction in $[0, \frac{1}{2})$ and so we can't apply Amini–Harandi [4] in this example.

After Corollary 3.3, it is natural to arise the following question.
Question (B). Under what condition for a multi–valued quasi–contraction one may have $Fix(T) \setminus Sfix(T) \neq \varnothing$? In the result below we give a fixed point result for integral type self mappings.

Corollary 3.4. *Let (X, d) be a complete metric space, and let $f : X \to X$ be a self–mapping such that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \psi(\int_0^{M(x,y)} \varphi(t)dt) \quad \text{for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Then f has a unique fixed point if and only if f has the approximate fixed point property.

Proof . Define $T : X \to \mathcal{CB}(X)$ by $Tx = \{fx\}$ and apply Theorem 3.1. □

In the remain of this work, we show that the condition (2.2) is sufficient for f to satisfy the approximate fixed point property. We say that the self–mapping $T : X \to X$ is a Ćirić–generalized weak quasi–contraction of integral type whenever there exist $\varphi \in \Phi$ and $0 \leq q < 1$ such that

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq q \int_0^{M(x,y)} \varphi(t)dt \quad \text{for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

For each $A \subseteq X$ and $x \in X$, consider the following notions:

- $\delta(A) = \sup\{d(x, y)|x, y \in A\}$,
- $\mathcal{O}(x, n) = \{x, Tx, \ldots, T^n x\} \quad (n = 1, 2, \ldots)$,
- $\mathcal{O}(x, \infty) = \{x, Tx, \ldots\}$.

The following lemmas play the crucial role throughout the paper:

Lemma 3.5. *Let (X, d) be a metric space, and let $T : X \to X$ be a Ćirić–generalized weak quasi–contraction of integral type. Then for any $x \in X$ and positive integers i and j, $i, j \in \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, we have*

$$\int_0^{d(T^i x,T^j x)} \varphi(t)dt \leq q \int_0^{\delta(\mathcal{O}(x,n))} \varphi(t)dt. \tag{3.5}$$

Proof . We have

$$\int_0^{d(T^i x,T^j x)} \varphi(t)dt = \int_0^{d(TT^{i-1}x,TT^{j-1}x)} \varphi(t)dt \leq q \int_0^{M(T^{i-1}x,T^{j-1}x)} \varphi(t)dt. \tag{3.6}$$

But we know that

$$\begin{aligned} M(T^{i-1}x, T^{j-1}x) &= \max\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), d(T^{j-1}x, T^j x) \\ &\quad , d(T^{i-1}x, T^j x), d(T^{j-1}x, T^i x)\} \\ &\leq \delta(\mathcal{O}(x, n)). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), one can obtain (3.5) easily. □

Remark 3.1. From Lemma 3.5, considering $T : X \to X$ as a Ćirić–generalized weak quasi–contraction of integral type and n as positive integer, yields that, for any $x \in X$ there exists a positive integer $k \le n$ such that $\delta(\mathcal{O}(x,n)) = d(x, T^k x)$.

Lemma 3.6. *Let (X, d) be a metric space and $T : X \to X$ be a Ćirić–generalized weak quasi–contraction of integral type. Then, for any $x \in X$,*

$$\int_0^{\delta(\mathcal{O}(x,\infty))} \varphi(t)dt \le \frac{1}{1-q} M d(x, Tx),$$

where M is the upper bound of φ.

Proof . Let x be an arbitrary element of X. Since $\mathcal{O}(x,1) \subseteq \mathcal{O}(x,2) \subseteq \ldots$, we have

$$\delta(\mathcal{O}(x,\infty)) = \sup_{n\in\mathbb{N}} \delta(\mathcal{O}(x,n)).$$

Hence it is sufficient to prove that

$$\int_0^{\delta(\mathcal{O}(x,n))} \varphi(t)dt \le \frac{1}{1-q} M d(x, Tx) \quad \text{for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ be arbitrary. From the Remark 3.1, there exists a positive integer $k \le n$ such that $\delta(\mathcal{O}(x,n)) = d(x, T^k x)$. Now we have

$$\begin{array}{rcl} \int_0^{\delta(\mathcal{O}(x,n))} \varphi(t)dt &=& \int_0^{d(x,T^kx)} \varphi(t)dt \le \int_0^{d(x,Tx)+d(Tx,T^kx)} \varphi(t)dt \\ &=& \int_0^{d(Tx,T^kx)} \varphi(t)dt + \int_{d(Tx,T^kx)}^{d(Tx,T^kx)+d(x,Tx)} \varphi(t)dt \\ &\le& q\int_0^{\delta(\mathcal{O}(x,n))} \varphi(t)dt + Md(x,Tx) \end{array} \tag{3.8}$$

and the desired result is obtained from (3.8). □

The following theorem shows that the approximate fixed point property holds automatically for Ćirić–generalized weak quasi–contraction self mappings of integral type.

Theorem 3.7. *Let (X, d) be a metric space and $T : X \to X$ be a Ćirić–generalized weak quasi–contraction of integral type. Then T has the approximate fixed point property.*

Proof . Choose a fixed element $x \in X$. Define a sequence $\{x_n\}_{n\ge0}$ with $x_n = T^n x$, for all $n \ge 0$. We claim that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. To see this, we have

$$\int_0^{d(x_n,Tx_n)} \varphi(t)dt = \int_0^{d(TT^{n-1}x,T^2T^{n-1}x)} \varphi(t)dt \le q\int_0^{\delta(\mathcal{O}(T^{n-1}x,2))} \varphi(t)dt. \tag{3.9}$$

From the Remark 3.1, there exists a positive integer $k_1 \le 2$, such that

$$\delta(\mathcal{O}(T^{n-1}x, 2)) = d(T^{n-1}x, T^{k_1}T^{n-1}x).$$

Hence

$$\begin{array}{rcl} \int_0^{\delta(\mathcal{O}(T^{n-1}x,2))} \varphi(t)dt &=& \int_0^{d(T^{n-1}x,T^{k_1}T^{n-1}x)} \varphi(t)dt \\ &=& \int_0^{d(TT^{n-2}x,T^{k_1+1}T^{n-2}x)} \varphi(t)dt \le q\int_0^{\delta(\mathcal{O}(T^{n-2}x,3))} \varphi(t)dt. \end{array} \tag{3.10}$$

Combining (3.9) and (3.10) shows that

$$\int_0^{d(x_n,Tx_n)} \varphi(t)dt \le q^2 \int_0^{\delta(\mathcal{O}(T^{n-2}x,3))} \varphi(t)dt.$$

By continuing this process, we obtain

$$\begin{aligned}\int_0^{d(x_n,Tx_n)} \varphi(t)dt &\le q^n \int_0^{\delta(\mathcal{O}(x,n+1))} \varphi(t)dt \\ &\le \tfrac{q^n}{1-q} M d(x,Tx).\end{aligned}$$

Since $q^n \to 0$ as $n \to \infty$, so we have $d(x_n, Tx_n) \to 0$ as $n \to \infty$. □

As an application of Corollary 3.4 and Theorem 3.7, we obtain the following fixed point result which extends the Ćirić's theorem ([7], Theorem 1) and Moradi and Khojasteh's result ([18], Corollary 2.7).

Corollary 3.8. Let (X, d) be a complete metric space, and let $T : X \to X$ be a Ćirić–generalized weak quasi-contraction of integral type. Then T has a unique fixed point in X.

4. Strict fixed points of α–ψ–quasicontractive multi–valued mappings of integral type

Let $\alpha : X \times X \to [0, \infty)$ is a mapping and $\psi \in \Psi$. We say that $T : X \to \mathcal{CB}(X)$ is a Ćirić-generalized α–ψ–quasicontractive multi–valued mapping of integral type whenever there exist $\alpha : X\times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) \int_0^{\mathcal{H}(Tx,Ty)} \varphi(t)dt \le \psi(\int_0^{M(x,y)} \varphi(t)dt)$$

for all $x, y \in X$. The following theorem guarantees existence of at least one strict fixed point for α–ψ–quasicontractive multi–valued mappings of integral type.

Theorem 4.1. *Let (X, d) be a complete metric space and let $\alpha : X\times X \to [0, \infty)$ be a mapping. Also suppose that $T : X \to \mathcal{CB}(X)$ is a Ćirić–generalized α–ψ–quasicontractive multi–valued mapping of integral type satisfying the following conditions:*

(i) there exists a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_m) \ge 1$ for all $m, n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Tx_n) = 0,$$

(ii) for any sequence $\{x_n\}$ in X which $\alpha(x_n, x_m) \ge 1$ for all $m, n \in \mathbb{N}$ and $x_n \to x$, we have $\alpha(x_n, x) \ge 1$ for all n.

Then, T has a strict fixed point in X.

Proof . As in the argument in Theorem 3.1, for any $m, n \ge N$, we have

$$\begin{aligned}\int_0^{M(x_n,x_m)} \varphi(t)dt &\le \int_0^{\mathcal{H}(Tx_n,Tx_m)+2\mathcal{H}(\{x_n\},Tx_n)+2\mathcal{H}(\{x_m\},Tx_m)} \varphi(t)dt \\ &= \int_0^{\mathcal{H}(Tx_n,Tx_m)} \varphi(t)dt + \int_{\mathcal{H}(Tx_n,Tx_m)}^{\mathcal{H}(Tx_n,Tx_m)+2\mathcal{H}(\{x_n\},Tx_n)+2\mathcal{H}(\{x_m\},Tx_m)} \varphi(t)dt \\ &\le \alpha(x_n, x_m) \int_0^{\mathcal{H}(Tx_n,Tx_m)} \varphi(t)dt + M(2\mathcal{H}(\{x_n\}, Tx_n) + 2\mathcal{H}(\{x_m\}, Tx_m)) \\ &\le \psi(\int_0^{M(x_n,x_m)} \varphi(t)dt) + M(2\mathcal{H}(\{x_n\}, Tx_n) + 2\mathcal{H}(\{x_m\}, Tx_m)), \qquad (4.1)\end{aligned}$$

where M is a positive number which $\varphi(t) \leq M$ for all $t \geq 0$. Thus from (4.1) and Theorem 3.1, we conclude that $\limsup_{n,m\to\infty} M(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} d(x_n, x) = 0$. Then by assumption (ii), we will have $\alpha(x_n, x) \geq 1$ for all $n \geq N$. So for any $n \geq N$, we have

$$\begin{aligned}\int_0^{H(\{x\},Tx)} \varphi(t)dt &\leq \int_0^{d(x,x_n)+\mathcal{H}(\{x_n\},Tx_n)+\mathcal{H}(Tx_n,Tx)} \varphi(t)dt \\ &\leq \int_0^{\mathcal{H}(Tx_n,Tx)} \varphi(t)dt + \int_{\mathcal{H}(Tx_n,Tx)}^{\mathcal{H}(Tx_n,Tx)+d(x,x_n)+\mathcal{H}(\{x_n\},Tx_n)} \varphi(t)dt \\ &\leq \alpha(x_n, x) \int_0^{\mathcal{H}(Tx_n,Tx)} \varphi(t)dt + M(d(x, x_n) + \mathcal{H}(\{x_n\}, Tx_n)) \\ &\leq \psi(\int_0^{M(x_n,x)} \varphi(t)dt) + M(d(x, x_n) + \mathcal{H}(\{x_n\}, Tx_n)).\end{aligned}$$

Letting $n \to \infty$ and applying Theorem 3.1, we obtain

$$\int_0^{\mathcal{H}(\{x\},Tx)} \varphi(t)dt \leq \psi(\int_0^{\mathcal{H}(\{x\},Tx)} \varphi(t)dt)$$

and so $\mathcal{H}(\{x\}, Tx) = 0$. This means that $Tx = \{x\}$. □

Note that adding the following property to the conditions of Theorem 4.1 we can obtain uniqueness of the strict fixed point:

(U) $\alpha(x, y) \geq 1$ for any strict fixed points $x, y \in X$.

The following example supports the integral type version of α–ψ–quasicontractive multi–valued mappings.

Example 4.1. Let $X = [0, 1]$ be endowed with the usual metric $d(x, y) = |x - y|$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = t^{\frac{1}{t}}[\frac{1-\ln t}{t^2}]$ for $0 < t < e$ and 0 otherwise. Then, for any $0 \leq \tau < e$, we have $\int_0^\tau \varphi(t)dt = \tau^{\frac{1}{\tau}}$. Also, define $T : X \to \mathcal{CB}(X)$ by $Tx = [0, \frac{x}{x+1}]$ and $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ if $x, y \in \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$ and 0 otherwise.

If $x = 0$ and $y = \frac{1}{n}$, then

$$\begin{aligned}\alpha(x, y) \textstyle\int_0^{\mathcal{H}(Tx,Ty)} \varphi(t)dt &= \textstyle\int_0^{\frac{1}{n+1}} \varphi(t)dt = (\frac{1}{n+1})^{n+1} = \frac{1}{n+1}(\frac{1}{n+1})^n \leq \frac{1}{2}(\frac{1}{n})^n \\ &= \psi(\textstyle\int_0^{d(x,y)} \varphi(t)dt) \\ &\leq \psi(\textstyle\int_0^{M(x,y)} \varphi(t)dt),\end{aligned}$$

where $\psi(t) = \frac{1}{2}t$.

If $x, y \in \{\frac{1}{n} | n \in \mathbb{N}\}$, then assume $x = \frac{1}{n}$ and $y = \frac{1}{m}$. In this case, we have

$$\begin{aligned}\alpha(x, y) \textstyle\int_0^{\mathcal{H}(Tx,Ty)} \varphi(t)dt &= \textstyle\int_0^{|\frac{1}{n+1}-\frac{1}{m+1}|} \varphi(t)dt \\ &= (\frac{|m-n|}{(n+1)(m+1)})^{\frac{(n+1)(m+1)}{|m-n|}} \\ &= (\frac{|m-n|}{(n+1)(m+1)})^{\frac{n+m+1}{|m-n|}} (\frac{|m-n|}{nm})^{\frac{nm}{|m-n|}} (\frac{mn}{(n+1)(m+1)})^{\frac{nm}{|m-n|}} \\ &\leq \frac{1}{2}(1)(\frac{|m-n|}{nm})^{\frac{nm}{|m-n|}} = \psi(\textstyle\int_0^{d(x,y)} \varphi(t)dt) \leq \psi(\textstyle\int_0^{M(x,y)} \varphi(t)dt).\end{aligned}$$

We see that

$$\alpha(x,y)\int_0^{H(Tx,Ty)}\varphi(t)dt \le \psi(\int_0^{M(x,y)}\varphi(t)dt)$$

for all $x, y \in X$. Define a sequence $\{x_n\}$ by $x_n = \frac{1}{n}$. Then, we conclude that $\alpha(x_n, x_m) = 1$ for all $m, n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Tx_n) = \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, the mapping T which is defined in above satisfies conditions of Theorem 4.1 and so T has a unique strict fixed point in X. Note that $T0 = \{0\}$.

The following result is a generalization of Mohammadi et al. [17].

Corollary 4.2. *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multi–valued mapping such that there exist $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)\mathcal{H}(Tx,Ty) \le \psi(M(x,y)), \quad \text{for all } x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Moreover, let the following conditions hold:

(i) there exists a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_m) \ge 1$ for all $m, n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Tx_n) = 0,$$

(ii) for any sequence $\{x_n\}$ in X which $\alpha(x_n, x_m) \ge 1$ for all $m, n \in \mathbb{N}$ and $x_n \to x$, we have $\alpha(x_n, x) \ge 1$ for all n.

Then, T has a strict fixed point in X.

Proof . Define $\varphi(t) = 1$ for all $t \ge 0$ and apply Theorem 4.1. □

Example 4.2. Let M_1 and M_2 be as in Example 3.1, $M_3 = \{2k : k \in \mathcal{N}\}$ and $M = M_1 \cup M_2 \cup M_3$ with the usual metric. Define $T : M \to \mathcal{CB}(\mathcal{M})$ by

$$Tx = \begin{cases} \left[0, \frac{3x}{5}\right] \cap (M_1 \cup M_2), & x \in M_1, \\ \left[0, \frac{x}{8}\right] \cap (M_1 \cup M_2), & x \in M_2, \\ \{2x\}, & x \in M_3 \end{cases}$$

and $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in M_1 \cup M_2, \\ 0, & otherwise. \end{cases}$$

Then, if $x, y \in M_1 \cup M_2$, as we saw in Example 3.1, $\alpha(x, y)\mathcal{H}(Tx, Ty) = \mathcal{H}(Tx, Ty) \le \frac{3}{5}M(x, y)$. So, we have $\alpha(x, y)\mathcal{H}(Tx, Ty) \le \psi(M(x, y))$ for all $x, y \in M$, where $\psi(t) = \frac{3}{5}t$ for all $t \ge 0$. Put $x_n = \frac{1}{3n+1}$ for all $n \in \mathcal{N}$. Then, since $x_n \in M_1$ for all $n \in \mathcal{N}$, we have $\alpha(x_n, x_m) = 1$ for all $m, n \in \mathcal{N}$. Also in Example 3.1 we saw that $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Tx_n) = 0$. It is easy to check that the condition (ii) in Theorem 4.2 holds. So, by Corollary 4.2, T has a strict fixed point. Here $T0 = \{0\}$. Now let $x = 1, y = 2$, we have $H(Tx, Ty) = |4 - \frac{3}{5}| = \frac{17}{5}$ and $M(x, y) =$

$\max\{|1-2|, |1-\frac{3}{5}|, |2-4|, |1-4|, |2-\frac{3}{5}|\} = 3$. So $H(Tx,Ty) = \frac{17}{5} > 3 = M(x,y) > \psi(M(x,y))$. Hence, we can't apply Theorem 3.1 in this example. Also , for $x = 1, y = \frac{1}{2}$, we have $H(Tx,Ty) = \frac{43}{80}$ and $N(x,y) = \frac{1}{2}$. We see that

$$\alpha(x,y)H(Tx,Ty) = H(Tx,Ty) = \frac{43}{80} > \frac{1}{2} = N(x,y) > \psi(N(x,y)).$$

Therefore, the main result of Mohammadi *et al.* [17] is not applicable in this example.

Finally, in the following corollary we give a result for quasicontractive multi–valued mappings of integral type in ordered metric spaces.

Corollary 4.3. *Let (X,d) be a complete metric space and $\preceq$ be an order on X, $\psi \in \Psi$ and $\varphi \in \Phi$. Let $T : X \to \mathcal{CB}(X)$ be a multi–valued mapping such that for any $x, y \in X$ which x, y are comparable,*

$$\int_0^{\mathcal{H}(Tx,Ty)} \varphi(t)dt \leq \psi(\int_0^{M(x,y)} \varphi(t)dt).$$

Moreover, let the following conditions hold:

(i) there exists a sequence $\{x_n\}$ in X such that x_n, x_m are comparable, for all $m, n \in N$ and $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Tx_n) = 0$,

(ii) for any sequence $\{x_n\}$ in X which x_n, x_m are comparable for all $m, n \in \mathbb{N}$ and $x_n \to x$, we have x_n, x are comparable, for all n.

Then, T has a strict fixed point in X.

Proof . Define the mapping $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = 1$ whenever x, y are comparable and $\alpha(x,y) = 0$ otherwise. Then apply Theorem 4.1. □

5. Conclusion

Motivated by Amini–Harandi's question [4], we introduced Ćirić–generalized weak quasicontractive multi–valued mappings of integral type and proved existence of strict fixed point for such mappings which is a partiall positive answer for the question. Also we provided an example to support our main result. This example also shows that our main result is a real generalization of Moradi and Khojasteh [18]. Also, we introduced α–ψ–quasicontractive multi–valued mappings of integral type and presented an example which shows the usability of integral type multi–valued mappings. Then we gave an example to demonstrate that our result is a real generalization of Mohammadi *et al.* [17].

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