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An extended multidimensional Hardy-Hilbert-type inequality with a general homogeneous kernel

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Abstract

In this paper, by the use of the weight coefficients, the transfer formula and the technique of real analysis, an extended multidimensional Hardy–Hilbert–type inequality with a general homogeneous kernel and a best possible constant factor is given. Moreover, the equivalent forms, the operator expressions and a few examples are considered.

Keywords: Hardy–Hilbert–type inequality, Weight coefficient, Equivalent form, Operator, Norm.
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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, a = \{a_m\}_{m=1}^{\infty} \in l^p, b = \{b_n\}_{n=1}^{\infty} \in l^q$, $||a||_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, $||b||_q > 0$, then we have the following Hardy–Hilbert's inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q \tag{1.1}$$

and the following Hilbert–type inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq||a||_p||b||_q, \tag{1.2}$$

with the best possible constant factor pq (cf. [1], Theorem 315, Theorem 341). Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [1], [2], [3]).

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Assuming that $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are positive sequences, with

$$U_m = \sum_{i=1}^{m} \mu_i, V_n = \sum_{j=1}^{n} \upsilon_j \ (m, n \in \mathbf{N} = \{1, 2, \ldots\}),$$

we have the following Hardy–Hilbert–type inequality (cf. [1], Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{1.3}$$

For $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, inequality (1.3) reduces to (1.1).

In 2014, Yang and Chen [4] gave the following multidimensional Hilbert–type inequality: For $i_0, j_0 \in \mathbf{N}, \alpha, \beta > 0$,

$$||x||_\alpha := \left(\sum_{k=1}^{i_0} |x^{(k)}|^\alpha\right)^{\frac{1}{\alpha}} \quad (x = (x^{(1)}, \ldots, x^{i_0}) \in \mathbf{R}^{i_0}),$$

$$||y||_\beta := \left(\sum_{k=1}^{j_0} |y^{(k)}|^\beta\right)^{\frac{1}{\beta}} \quad (y = (y^{(1)}, \ldots, y^{(j_0)}) \in \mathbf{R}^{j_0}),$$

$0 < \lambda_1 + \eta \leq i_0, 0 < \lambda_2 + \eta \leq j_0, \lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, we have

$$\begin{aligned} &\sum_n \sum_m \frac{(\min\{||m||_\alpha, ||n||_\beta\})^\eta}{(\max\{||m||_\alpha, ||n||_\beta\})^{\lambda+\eta}} a_m b_n \\ &< \ K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \left[\sum_m ||m||_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right]^{\frac{1}{p}} \left[\sum_n ||n||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{1.4}$$

where, $\sum_m = \sum_{m_{i_0}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty}$, $\sum_n = \sum_{n_{j_0}=1}^{\infty} \cdots \sum_{n_1=1}^{\infty}$, the series in the right hand side are the positive, and the best possible constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ is indicated by

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{\lambda + 2\eta}{(\lambda_1 + \eta)(\lambda_2 + \eta)}.$$

For $i_0 = j_0 = \lambda = 1, \eta = 0, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.4) reduces to (1.2). The other results on this type of inequalities were provided by [5]–[17].

In 2015, Shi and Yang [18] gave another extension of (1.2) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{\max\{U_m, V_n\}} < pq \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{1.5}$$

Some other results on Hardy–Hilbert–type inequalities were given by [19]–[25].

In this paper, by the use of the weight coefficients, the transfer formula and the technique of real analysis, an extended multidimensional Hardy–Hilbert–type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of (1.4) and (1.5). Moreover, the equivalent forms, the operator expressions and a few particular examples are considered.

2. Some lemmas

If $\mu_i^{(k)} > 0$ $(k = 1, \dots, i_0; i = 1, \dots, m), \upsilon_j^{(l)} > 0$ $(l = 1, \dots, j_0; j{=}1, \dots, n)$, then we set

$$U_m^{(k)} := \sum_{i=1}^{m} \mu_i^{(k)} \ (k = 1, \dots, i_0), V_n^{(l)} := \sum_{j=1}^{n} \upsilon_j^{(l)} \ (l = 1, \dots, j_0),$$
$$U_m = (U_m^{(1)}, \dots, U_m^{(i_0)}), V_n = (V_n^{(1)}, \dots, V_n^{(j_0)}) \ (m, n \in \mathbf{N}).$$

We also set functions $\mu_k(t) := \mu_m^{(k)}, t \in (m-1, m]$ $(m \in \mathbf{N})$; $\upsilon_l(t) := \upsilon_n^{(l)}$, $t \in (n-1, n]$ $(n \in \mathbf{N})$, and

$$U_k(x) := \int_0^x \mu_k(t)dt \ (k = 1, \dots, i_0),$$

$$V_l(y) := \int_0^y \upsilon_l(t)dt \ (l = 1, \dots, j_0),$$
$$U(x) := (U_1(x), \dots, U_{i_0}(x)), V(y) := (V_1(y), \dots, V_{j_0}(y)) \ (x, y \geq 0).$$

It follows that $U_k(m) = U_m^{(k)}$ $(k = 1, \dots, i_0; m \in \mathbf{N}), V_l(n) = V_n^{(l)}$ $(l = 1, \dots, j_0; n \in \mathbf{N})$, and for $x \in (m-1, m)$, $U_k'(x) = \mu_k(x) = \mu_m^{(k)}$ $(k = 1, \dots, i_0; m \in \mathbf{N})$; for $y \in (n-1, n), V_l'(y) = \upsilon_l(y) = \upsilon_n^{(l)}$ $(l = 1, \dots, j_0; n \in \mathbf{N})$.

Lemma 2.1. (Yang and Chen, [21]) Suppose that $g(t)(>0)$ is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0, \infty)$ $(n_0 \in \mathbf{N})$, satisfying $\int_0^\infty g(t)dt \in \mathbf{R}_+$. We have

$$\int_1^\infty g(t)dt < \sum_{n=1}^{\infty} g(n) < \int_0^\infty g(t)dt. \tag{2.1}$$

Lemma 2.2. If $i_0 \in \mathbf{N}, \alpha, M > 0, \Psi(u)$ is a non-negative measurable function in $(0, 1]$, and

$$D_M := \left\{ x \in \mathbf{R}_+^{i_0}; u = \sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha \leq 1 \right\},$$

then we have the following transfer formula (cf. [26]):

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_s = \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha}-1} du.$$

Lemma 2.3. *If* $i_0, j_0 \in \mathbf{N}$, $\alpha, \beta > 0$,

$$c_1 := \min_{1 \leq i \leq i_0} \{\mu_1^{(i)}\}, c_2 := \min_{1 \leq j \leq j_0} \{\upsilon_1^{(j)}\},$$

then for $\varepsilon > 0$, *we have*

$$\sum_m ||U_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c_1^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + O_1(1), \tag{2.2}$$
$$\sum_n ||V_n||_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \leq \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon c_2^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + O_2(1) \ (\varepsilon \to 0^+). \tag{2.3}$$

Proof . For $c_1 > 0, M > c_1 i_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, & 0 < u \le \frac{c_1^\alpha i_0}{M^\alpha}, \\ \frac{1}{(Mu^{1/\alpha})^{i_0+\varepsilon}}, & \frac{c_1^\alpha i_0}{M^\alpha} < u \le 1. \end{cases}$$

By (2.2), it follows that

$$\int_{\{x\in\mathbf{R}_+^{i_0};x_i\ge c\}} \frac{dx}{||x||_\alpha^{i_0+\varepsilon}} = \lim_{M\to\infty} \int\cdots\int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1\cdots dx_{i_0}$$

$$= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_{c_1^\alpha i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha}-1}}{(Mu^{1/\alpha})^{i_0+\varepsilon}} du = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c_1^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.$$

We have

$$\sum_m ||U_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \le H_0 + \sum_{i=1}^{i_0} H_i,$$

$$H_0 := \sum_{\{m\in\mathbf{N}^{i_0};m_i\ge 2\}} ||U_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)},$$

$$H_i := \sum_{\{m\in\mathbf{N}^{i_0};m_i=1,m_k\ge 1(k\ne i)\}} ||U_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}.$$

Then by (2.1) and the above result, we find

$$0 < H_0 = \sum_{\{m\in\mathbf{N}^{i_0};m_i\ge 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-1\le x_i<m_i\}} ||U(m)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx$$

$$< \sum_{\{m\in\mathbf{N}^{i_0};m_i\ge 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-1\le x_i<m_i\}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx$$

$$= \int_{\{x\in\mathbf{R}_+^{i_0};x_i\ge 1\}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx\,(v=U(x))$$

$$\le \int_{\{v\in\mathbf{R}_+^{i_0};v_i\ge c_1\}} ||v||_\alpha^{-i_0-\varepsilon}dv = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c_1^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.$$

Without losing generality, we estimate H_{i_0}. If $i_0 = 1$, then we find

$$0 < H_{i_0} = (\mu_1^{(1)})^{-1-\varepsilon}\mu_1^{(1)} = (\mu_1^{(1)})^{-\varepsilon} < \infty;$$

if $i_0 \ge 2$, then for $m_{i_0} = 1$, we find

$$H_{i_0} = \sum_{m\in\mathbf{N}^{i_0-1}} \int_{\{x\in\mathbf{R}_+^{i_0-1};m_i-1<x_i\le m_i\}} \frac{\mu_1^{(i_0)}\prod_{k=1}^{i_0-1}\mu_k(x)dx}{[(\mu_1^{(i_0)})^\alpha + \sum_{i=1}^{i_0-1}(U_m^{(i)})^\alpha]^{\frac{i_0+\varepsilon}{\alpha}}}$$

$$\le \mu_1^{(i_0)}\int_{\mathbf{R}_+^{i_0-1}} \frac{\prod_{k=1}^{i_0-1}\mu_k(x)}{[(\mu_1^{(i_0)})^\alpha + \sum_{i=1}^{i_0-1}(U_i(x))^\alpha]^{\frac{i_0+\varepsilon}{\alpha}}}dx.$$

Setting $v = (U_1(x), \ldots, U_{i_0-1}(x))$, by (2.2), we have

$$\begin{aligned}
0 < H_{i_0} &\le \mu_1^{(i_0)} \int_{\mathbf{R}_+^{i_0-1}} \frac{1}{[(\mu_1^{(i_0)})^\alpha + M^\alpha \sum_{i=1}^{i_0-1}(\frac{v_i}{M})^\alpha]^{\frac{i_0+\varepsilon}{\alpha}}} dv \\
&= \mu_1^{(i_0)} \lim_{M\to\infty} \frac{M^{i_0-1}\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} \int_0^1 \frac{u^{\frac{i_0-1}{\alpha}-1}}{[(\mu_1^{(i_0)})^\alpha + M^\alpha u]^{\frac{i_0+\varepsilon}{\alpha}}} du \\
&\overset{t=\frac{M^\alpha u}{(\mu_1^{(i_0)})^\alpha}}{=} (\mu_1^{(i_0)})^{-\varepsilon} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} \int_0^\infty \frac{t^{\frac{i_0-1}{\alpha}-1} dt}{(1+t)^{\frac{i_0+\varepsilon}{\alpha}}} \\
&= (\mu_1^{(i_0)})^{-\varepsilon} \frac{\Gamma^{i_0-1}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0-1}{\alpha})} B(\frac{i_0-1}{\alpha}, \frac{1+\varepsilon}{\alpha}) < \infty.
\end{aligned}$$

Hence, we have

$$\sum_m ||U_m||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \le \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon c_1^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \sum_{i=1}^{i_0} O_i(1)\,,$$

namely, (2.2) follows. In the same way, we have (2.3). □

Definition 2.4. If $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(vx, vy) = v^{-\lambda} k_\lambda(x,y)\,(v, x, y > 0)$, for any fixed $y > 0\,(x > 0), k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}$ $(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}})$ is decreasing with respect to $x \in \mathbf{R}_+$ $(y \in \mathbf{R}_+)$, and strict decreasing in an interval $(a_y, \infty)\,((b_x, \infty)) \subset (0, \infty)$,

$$k(\lambda_1) := \int_0^\infty k_\lambda(u, 1) u^{\lambda_1 - 1} du \in \mathbf{R}_+,$$

then for $i_0, j_0 \in \mathbf{N}$, $\alpha, \beta > 0$, we define two weight coefficients $w(\lambda_1, n)$ $(n \in \mathbf{N}^{j_0})$ and $W(\lambda_2, m)$ $(m \in \mathbf{N}^{i_0})$ as follows:

$$w(\lambda_1, n) := \sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta) \frac{||V_n||_\beta^{\lambda_2}}{||U_m||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}, \tag{2.4}$$

$$W(\lambda_2, m) := \sum_n k_\lambda(||U_m||_\alpha, ||V_n||_\beta) \frac{||U_m||_\alpha^{\lambda_1}}{||V_n||_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}. \tag{2.5}$$

Example 2.5. For $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda$, $0 < \lambda_1 + \eta \le i_0, 0 < \lambda_2 + \eta \le j_0$, we set

$$k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}\ (x, y > 0).$$

Then for any fixed $y > 0$,

$$k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}} = \begin{cases} \frac{1}{y^{\lambda+\eta} x^{i_0-\lambda_1-\eta}}, & 0 < x < y, \\ \frac{y^\eta}{x^{i_0+\lambda_2+\eta}}, & x \ge y, \end{cases}$$

is decreasing in $x \in \mathbf{R}_+$, and strictly decreasing in $([y]+1, \infty)$. In the same way, for fixed $x >$

$0, k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}$ is decreasing in $y \in \mathbf{R}_+$, and strictly decreasing in $([x]+1,\infty)$. We still have

$$\begin{aligned} k(\lambda_1) &= \int_0^\infty \frac{(\min\{u,1\})^\eta}{(\max\{u,1\})^{\lambda+\eta}}\frac{1}{u^{1-\lambda_1}}du \\ &= \int_0^1 \frac{u^\eta}{u^{1-\lambda_1}}du + \int_1^\infty \frac{1}{u^{\lambda+\eta}}\frac{1}{u^{1-\lambda_1}}du \\ &= \frac{\lambda+2\eta}{(\lambda_1+\eta)(\lambda_2+\eta)} \in \mathbf{R}_+. \end{aligned}$$

Note 1. For $b,\alpha>0$, we have

$$\frac{d}{dx}(b+x^\alpha)^{\frac{1}{\alpha}} = (b+x^\alpha)^{\frac{1}{\alpha}-1}x^{\alpha-1} > 0 \quad (x>0).$$

Hence, with regards to the assumptions of Definition 2.4, for $m_i-1<x_i<m_i$ $(i=1,\dots,i_0; m\in\mathbf{N}^{\mathbf{i_0}})$, we have $||U(m)||_\alpha > ||U(x)||_\alpha$ and

$$k_\lambda(||U_m||_\alpha,||V_n||_\beta)\frac{1}{||U(m)||_\alpha^{i_0-\lambda_1}} < k_\lambda(||U(x)||_\alpha,||V_n||_\beta)\frac{1}{||U(x)||_\alpha^{i_0-\lambda_1}};$$

for $m_i<x_i<m_i+1(i=1,\dots,i_0;m\in\mathbf{N}^{i_0})$, $n_j<y_j<n_j+1(j=1,\dots,j_0;n\in\mathbf{N}^{j_0})$, $\frac{\varepsilon}{p},\frac{\varepsilon}{q}>0$, we have $||U(m)||_\alpha<||U(x)||_\alpha$, $||V(n)||_\beta<||V(y)||_\beta$ and

$$\begin{aligned} &\frac{k_\lambda(||U_m||_\alpha,||V_n||_\beta)}{||U_m||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}||V_n||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \\ &= \frac{k_\lambda(||U(m)||_\alpha,||V(n)||_\beta)}{||U(m)||_\alpha^{i_0-\lambda_1}||V(n)||_\beta^{j_0-\lambda_2}}\frac{1}{||U(m)||_\alpha^{\frac{\varepsilon}{p}}||V(n)||_\beta^{\frac{\varepsilon}{q}}} \\ &> \frac{k_\lambda(||U(x)||_\alpha,||V(y)||_\beta)}{||U(x)||_\alpha^{i_0-\lambda_1}||V(y)||_\beta^{j_0-\lambda_2}}\frac{1}{||U(x)||_\alpha^{\frac{\varepsilon}{p}}||V(y)||_\beta^{\frac{\varepsilon}{q}}} \\ &= \frac{k_\lambda(||U(x)||_\alpha,||V(y)||_\beta)}{||U(x)||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}||V(y)||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}}. \end{aligned} \tag{2.6}$$

Lemma 2.6. *With regards to the assumptions of Definition* 2.4, *we have*

$$w(\lambda_1,n) < K_\alpha(\lambda_1)\ (n\in\mathbf{N}^{j_0}), \tag{2.7}$$

$$W(\lambda_2,m) < K_\beta(\lambda_1)\ (m\in\mathbf{N}^{i_0}), \tag{2.8}$$

where

$$K_\beta(\lambda_1) = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}k(\lambda_1), K_\alpha(\lambda_1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1);$$

Proof . By (2.1), (2.2) and Note 1, it follows that

$$\begin{aligned} w(\lambda_1,n) &= \sum_m \int_{\{x\in\mathbf{R}_+^{i_0};m_i-1<x_i\le m_i\}} \frac{k_\lambda(||U_m||_\alpha,||V_n||_\beta)||V_n||_\beta^{\lambda_2}}{||U(m)||_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_m^{(k)}dx \\ &< \sum_m \int_{\{x\in\mathbf{R}_+^{i_0};m_i-1<x_i\le m_i\}} \frac{k_\lambda(||U(x)||_\alpha,||V_n||_\beta)||V_n||_\beta^{\lambda_2}}{||U(x)||_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_k(x)dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) \frac{||V_n||_\beta^{\lambda_2}}{||U(x)||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_k(x) dx \\
&\overset{u=U(x)}{\leq} \int_{\mathbf{R}_+^{i_0}} k_\lambda(||u||_\alpha, ||V_n||_\beta) \frac{||V_n||_\beta^{\lambda_2}}{||u||_\alpha^{i_0-\lambda_1}} du \\
&= \lim_{M\to\infty} \int_{\mathbf{D}_M} k_\lambda(M[\sum_{i=1}^{i_0} (\frac{x_i}{M})^\alpha]^{1/\alpha}, ||V_n||_\beta) \frac{M^{\lambda_1 - i_0} ||V_n||_\beta^{\lambda_2} dx}{[\sum_{i=1}^{j_0} (\frac{x_i}{M})^\alpha]^{(i_0-\lambda_1)/\alpha}} \\
&= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 k_\lambda(Mu^{1/\alpha}, ||V_n||_\beta) \frac{||V_n||_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1} du}{M^{i_0-\lambda_1} u^{(i_0-\lambda_1)/\alpha}} \\
&= \lim_{M\to\infty} \frac{M^{\lambda_1}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 k_\lambda(Mu^{1/\alpha}, ||V_n||_\beta) ||V_n||_\beta^{\lambda_2} u^{\frac{\lambda_1}{\alpha}-1} du \\
&\overset{v=\frac{Mu^{1/\alpha}}{||V_n||_\beta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_0^\infty k_\lambda(v,1) v^{\lambda_1-1} dv = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k(\lambda_1) = K_\alpha(\lambda_1).
\end{aligned}
$$

Hence, we have (2.7). In the same way, we have (2.8). □

3. Main results

We set the functions

$$
\Phi(m) := \frac{||U_m||_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \; (m \in \mathbf{N}^{i_0}),
$$

$$
\Psi(n) := \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} \; (n \in \mathbf{N}^{j_0}),
$$

and the following normed spaces:

$$
l_{p,\Phi} = \left\{ a = \{a_m\}; ||a||_{p,\Phi} := (\sum_m \Phi(m)|a_m|^p)^{\frac{1}{p}} < \infty \right\},
$$

$$
l_{q,\Psi} = \left\{ b = \{b_n\}; ||b||_{q,\Psi} := (\sum_n \Psi(n)|b_n|^q)^{\frac{1}{q}} < \infty \right\},
$$

$$
l_{p,\Psi^{1-p}} := \left\{ c = \{c_n\}; ||c||_{p,\Psi^{1-p}} := (\sum_n \Psi^{1-p}(n)|c_n|^p)^{\frac{1}{p}} < \infty \right\}.
$$

Theorem 3.1. *With regards to the assumptions of Definition 2.4, if* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0, a = \{a_m\} \in l_{p,\Phi}, b = \{b_n\} \in l_{q,\Psi}, ||a||_{p,\Phi}, ||b||_{q,\Psi} > 0$, *then we have the following equivalent inequalities*

$$
I := \sum_n \sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta) a_m b_n < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\Phi} ||b||_{q,\Psi}, \tag{3.1}
$$

$$
\begin{aligned}
J &:= \left\{ \sum_n \frac{\prod_{k=1}^{j_0} \upsilon_n^{(k)}}{||V_n||_\beta^{j_0-p\lambda_2}} \left[\sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta) a_m \right]^p \right\}^{\frac{1}{p}} \\
&< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||a||_{p,\Phi}
\end{aligned} \tag{3.2}
$$

where

$$K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1).$$

Proof . By Hölder's inequality with weight (cf. [27]), we have

$$\begin{aligned} I &= \sum_n \sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta) \\ &\times \left[\frac{||U_m||_\alpha^{\frac{i_0-\lambda_1}{q}} (\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{\frac{1}{p}} a_m}{||V_n||_\beta^{\frac{j_0-\lambda_2}{p}} (\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}}}\right] \left[\frac{||V_n||_\beta^{\frac{j_0-\lambda_2}{p}} (\prod_{k=1}^{i_0} \mu_m^{(k)})^{\frac{1}{q}} b_n}{||U_m||_\alpha^{\frac{i_0-\lambda_1}{q}} (\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{\frac{1}{p}}}\right] \\ &\le \left[\sum_m W(\lambda_2, m) \frac{||U_m||_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}}\right]^{\frac{1}{p}} \left[\sum_n w(\lambda_1, n) \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})}\right]^{\frac{1}{q}}. \end{aligned}$$

Then by (2.7) and (2.8), we have (3.1). We set

$$b_n := \frac{\prod_{l=1}^{j_0} \upsilon_n^{(l)}}{||V_n||_\beta^{j_0-p\lambda_2}} \left[\sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta) a_m\right]^{p-1} \quad (n \in \mathbf{N}^{j_0}).$$

Then we have $J = ||b||_{q,\Psi}^{q-1}$. Since the right–hand side of (3.2) is finite, it follows that $J < \infty$. If $J = 0$, then (3.2) is trivially valid; if $J > 0$, then by (3.1), we have

$$\begin{aligned} ||b||_{q,\Psi}^q &= J^p = I < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi}||b||_{q,\Psi}, \\ ||b||_{q,\Psi}^{q-1} &= J < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi}, \end{aligned}$$

namely, (3.2) follows. On the other hand, assuming that (3.2) is valid, by Hölder's inequality (cf. [27]), we have

$$\begin{aligned} I &= \sum_n \left[\frac{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{1/p}}{||V_n||_\beta^{(j_0/p)-\lambda_2}} \sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta) a_m\right] \\ &\times \frac{||V_n||_\beta^{(j_0/p)-\lambda_2}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{1/p}} b_n \le J||b||_{q,\Psi}. \end{aligned} \tag{3.3}$$

Then by (3.2), we have (3.1), which is equivalent to (3.2). □

Theorem 3.2. *With regards to the assumptions of Theorem 3.1, if* $\mu_m^{(k)} \ge \mu_{m+1}^{(k)}$ $(m \in \mathbf{N}), \upsilon_n^{(l)} \ge \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N}), U_\infty^{(k)} = V_\infty^{(l)} = \infty$ $(k = 1, \ldots, i_0, l = 1, \ldots, j_0)$, *then the constant factor* $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ *in (3.1) and (3.2) is the best possible.*

Proof . For $\varepsilon > 0$, we set

$$\begin{aligned} \widetilde{a} &= \{\widetilde{a}_m\}, \widetilde{a}_m := ||U_m||_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}} \prod_{k=1}^{i_0} \mu_m^{(k)} \ (m \in \mathbf{N}^{i_0}), \\ \widetilde{b} &= \{\widetilde{b}_n\}, \widetilde{b}_n := ||V_n||_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}} \prod_{l=1}^{j_0} \upsilon_n^{(l)} \ (n \in \mathbf{N}^{j_0}). \end{aligned}$$

Then by (2.2) and (2.3), we obtain

$$||\widetilde{a}||_{p,\Phi}||\widetilde{b}||_{q,\Psi} = \left[\sum_m \frac{||U_m||_\alpha^{p(i_0-\lambda_1)-i_0}\widetilde{a}_m^p}{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{p-1}}\right]^{\frac{1}{p}} \left[\sum_n \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0}\widetilde{b}_n^q}{(\prod_{l=1}^{j_0}\upsilon_n^{(l)})^{q-1}}\right]^{\frac{1}{q}}$$

$$= \left(\sum_m ||U_m||_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}\right)^{\frac{1}{p}} \left(\sum_n ||V_n||_\beta^{-j_0-\varepsilon}\prod_{l=1}^{j_0}\upsilon_n^{(j)}\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\varepsilon}\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{c_1^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O_1(1)\right)^{\frac{1}{p}} \times\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{c_2^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon O_2(1)\right)^{\frac{1}{q}}.$$

By (2.6), since $\mu_{m_i}^{(k)} \geq \mu_{m_i+1}^{(k)} = \mu_k(x)(m_i < x_i < m_i+1; m \in \mathbf{N}^{i_0})$, $\upsilon_{n_j}^{(l)} \geq \upsilon_{n_j+1}^{(l)} = \upsilon_l(y)$ $(n_j < y_j < n_j+1; n \in \mathbf{N}^{j_0})$, we find

$$\begin{aligned}
\widetilde{I} \ : &= \sum_n\sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta)\widetilde{a}_m\widetilde{b}_n \\
&= \sum_n\sum_m \int_{\{y\in\mathbf{R}_+^{j_0}; n_j\le y_j<n_j+1\}}\int_{\{x\in\mathbf{R}_+^{i_0}; m_i\le x_i<m_i+1\}} \frac{k_\lambda(||U_m||_\alpha, ||V_n||_\beta)}{||U_m||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}||V_n||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \\
&\quad \times\prod_{k=1}^{i_0}\mu_m^{(k)}\prod_{l=1}^{j_0}\upsilon_n^{(l)}dxdy \\
&> \sum_n\sum_m \int_{\{y\in\mathbf{R}_+^{j_0}; n_j\le y_j<n_j+1\}}\int_{\{x\in\mathbf{R}_+^{i_0}; m_i\le x_i<m_i+1\}} \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)}{||U(x)||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}} \\
&\quad \times\frac{1}{||V(y)||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}}\prod_{k=1}^{i_0}\mu_k(x)\prod_{l=1}^{j_0}\upsilon_l(y)dxdy \\
&= \int_{[1,\infty)^{j_0}}\int_{[1,\infty)^{i_0}} \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)}{||U(x)||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}||V(y)||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}}\prod_{k=1}^{i_0}\mu_k(x)\prod_{l=1}^{j_0}\upsilon_l(y)dxdy\,.
\end{aligned}$$

Setting $u = U(x), v = V(y)$, since $U_\infty^{(k)} = V_\infty^{(l)} = \infty$, for

$$c := \max_{1\le k\le i_0, 1\le l\le j_0}\{\mu_1^{(k)}, \upsilon_1^{(l)}\}\,,$$

we have

$$\begin{aligned}
\widetilde{I} &> \int_{[c,\infty)^{j_0}}\int_{[c,\infty)^{i_0}} \frac{k_\lambda(||u||_\alpha, ||v||_\beta)}{||u||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}||v||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}}dudv \\
&= \int_{[c,\infty)^{j_0}}\int_{[c,\infty)^{i_0}} \frac{k_\lambda(M_1[\sum_{i=1}^{i_0}(\frac{x_i}{M_1})^\alpha]^{\frac{1}{\alpha}}, M_2[\sum_{j=1}^{j_0}(\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}})dxdy}{\{M_1[\sum_{i=1}^{i_0}(\frac{x_i}{M_1})^\alpha]^{\frac{1}{\alpha}}\}^{i_0-\lambda_1+\frac{\varepsilon}{p}}\{M_2[\sum_{j=1}^{j_0}(\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}}\}^{j_0-\lambda_2+\frac{\varepsilon}{q}}}.
\end{aligned}$$

For $M_1 > ci_0^{1/\alpha}, M_2 > cj_0^{1/\beta}$, we put

$$\Psi_1(u) = \begin{cases} 0, & 0 < u \leq \frac{c^\alpha i_0}{M_1^\alpha}, \\ k_\lambda(M_1 u^{1/\alpha}, M_2[\sum_{j=1}^{j_0}(\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}})\frac{1}{(M_1 u^{1/\alpha})^{i_0-\lambda_1}}, & \frac{c^\alpha i_0}{M_1^\alpha} < u \leq 1, \end{cases}$$

$$\Psi_2(v) = \begin{cases} 0, & 0 < v \leq \frac{c^\beta j_0}{M_2^\beta}, \\ k_\lambda(M_1 u^{1/\alpha}, M_2 v^{1/\beta})\frac{1}{(M_2 v^{1/\beta})^{j_0-\lambda_2}}, & \frac{c^\beta j_0}{M_2^\beta} < v \leq 1, \end{cases}$$

By (2.2) twice, it follows that

$$\widetilde{I} > \lim_{M_1\to\infty}\lim_{M_2\to\infty}\frac{M_1^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\frac{M_2^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0}\Gamma(\frac{j_0}{\beta})}\int_{c^\alpha i_0/M_1^\alpha}^1 u^{\frac{i_0}{\alpha}-1}$$
$$\times\left[\int_{c^\beta j_0/M_2^\beta}^1 \frac{k_\lambda(M_1 u^{\frac{1}{\alpha}}, M_2 v^{\frac{1}{\beta}})v^{\frac{j_0}{\beta}-1}}{(M_1 u^{\frac{1}{\alpha}})^{i_0-\lambda_1+\frac{\varepsilon}{p}}(M_2 v^{\frac{1}{\beta}})^{j_0-\lambda_2+\frac{\varepsilon}{q}}}dv\right]du.$$

Setting $x = M_1 u^{\frac{1}{\alpha}}, y = M_2 v^{\frac{1}{\beta}}$ in the above, we find

$$\begin{aligned}
\widetilde{I} &> \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\int_{ci_0^{1/\alpha}}^\infty x^{\lambda_1-\frac{\varepsilon}{p}-1}\left(\int_{cj_0^{1/\beta}}^\infty k_\lambda(x,y)y^{\lambda_2-\frac{\varepsilon}{q}-1}dy\right)dx \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\int_{ci_0^{1/\alpha}}^\infty x^{-\varepsilon-1}\left(\int_0^{x/cj_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\int_{ci_0^{1/\alpha}}^\infty x^{-\varepsilon-1}\left(\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx\right. \\
&\quad \left.+\int_{ci_0^{1/\alpha}}^\infty x^{-\varepsilon-1}\left(\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^{x/cj_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx\right] \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\frac{1}{\varepsilon(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right. \\
&\quad \left.+\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty\left(\int_{cj_0^{1/\beta}v}^\infty x^{-\varepsilon-1}dx\right)k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right] \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\frac{1}{(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right. \\
&\quad \left.+\frac{1}{(cj_0^{1/\beta})^\varepsilon}\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty k_\lambda(v,1)v^{\lambda_1-\frac{\varepsilon}{p}-1}dv\right].
\end{aligned}$$

If there exists a constant $K \leq K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$, such that (3.1) is valid when replacing $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ by K, then we have $\varepsilon\widetilde{I} < \varepsilon K||\widetilde{a}||_{p,\Phi}||\widetilde{b}||_{q,\Psi}$, namely,

$$\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\frac{1}{(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right.$$
$$\left.+\frac{1}{(cj_0^{1/\beta})^\varepsilon}\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty k_\lambda(v,1)v^{\lambda_1-\frac{\varepsilon}{p}-1}dv\right]$$

$$< K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{c_1^{\varepsilon} i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}+\varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{c_2^{\varepsilon} j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon O_2(1)\right)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of Fatou lemma (cf. [28]), we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})k(\lambda_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \le K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},$$

and then $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) \le K$. Hence, $K = K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (3.1). The constant factor in (3.2) is still the best possible. Otherwise, we would reach a contradiction by (3.3) that the constant factor in (3.1) is not the best possible. □

4. Operator expressions

With regards to the assumptions of Theorem 3.2, in view of

$$c_n := \frac{\prod_{k=1}^{j_0}\upsilon_n^{(k)}}{||V_n||_\beta^{j_0-p\lambda_2}}\left[\sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta)a_m\right]^{p-1} \quad (n \in \mathbf{N}^{j_0})$$

$$c = \{c_n\}, ||c||_{p,\Psi^{1-p}} = J < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi} < \infty,$$

we can set the following definition:

Definition 4.1. Define a multidimensional Hardy–Hilbert–type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: For any $a \in l_{p,\Phi}$, there exists a unique representation $Ta = c \in l_{p,\Psi^{1-p}}$, satisfying

$$Ta(n) := \sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta)a_m \ (n \in \mathbf{N}^{j_0}).$$

For $b \in l_{q,\Psi}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n\left[\sum_m k_\lambda(||U_m||_\alpha, ||V_n||_\beta)a_m\right]b_n.$$

Then by Theorem 3.1, we have the following equivalent inequalities:

$$(Ta, b) < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi}||b||_{q,\Psi}, \tag{4.1}$$

$$||Ta||_{p,\Psi^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||a||_{p,\Phi}. \tag{4.2}$$

It follows that T is bounded with

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\Phi}}\frac{||Ta||_{p,\Psi^{1-p}}}{||a||_{p,\Phi}} \le K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1).$$

By Theorem 3.2, the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (4.2) is the best possible, we have

$$||T|| = K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}k(\lambda_1). \tag{4.3}$$

Example 4.2. (i) In view of Example 2.5, by (4.3), for $k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}$, we have

$$||T|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{\lambda+2\eta}{(\lambda_1+\eta)(\lambda_2+\eta)}.$$

(ii) For $k_\lambda(x,y) = \frac{1}{x^\lambda+y^\lambda}$ $(0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0, \lambda_1+\lambda_2 = \lambda)$, we find $k(\lambda_1) = \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}$, and then by (4.3), we have

$$||T|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}.$$

(iii) For $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda-y^\lambda}$ $(0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0, \lambda_1+\lambda_2 = \lambda)$, we find $k(\lambda_1) = [\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}]^2$, and then by (4.3), we have

$$||T|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \left[\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}\right]^2.$$

Remark 4.3. (i) For $0 < \lambda_1+\eta \leq i_0, 0 < \lambda_2+\eta \leq j_0$,

$$k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}} \quad (x,y > 0),$$

(3.1) reduces to (23) in [29], which is an extension of (1.4) for $\mu_i^{(k)} = \upsilon_j^{(l)} = 1$ $(k = 1,\ldots,i_0;$ $i = 1,\ldots,m, l = 1,\ldots,j_0; j=1,\ldots,n)$.

(ii) For $\lambda = i_0 = j_0 = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}, k_1(x,y) = \frac{1}{x+y}$ $(\frac{1}{\max\{x,y\}})$, (3.1) reduces to (1.3)) ((1.5)).

5. Conclusion

In this paper, by the use of the weight coefficients, the transfer formula and the technique of real analysis, an extended multidimensional Hardy–Hilbert–type inequality with a general homogeneous kernel and a best possible constant factor is given in Theorem 3.1 and Theorem 3.2, which is an extension of (1.4) and (1.5). Moreover, the equivalent forms, the operator expressions and a few particular examples are considered. The lemmas and theorems provide an extensive account of this type of inequalities.

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