Int. J. Nonlinear Anal. Appl. 3 (2012) No. 1, 9-16
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Weak and Strong Convergence Theorems for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Nonself-Mappings

P. Yatakoat, S. Suantai*

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.

(Communicated by M. Eshaghi Gordji)

Abstract

In this paper, we introduce and study a new iterative scheme to approximate a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive nonself-mappings in Banach spaces. Several strong and weak convergence theorems of the proposed iteration are established. The main results obtained in this paper generalize and refine some known results in the current literature.

Keywords: Generalized Asymptotically Quasi-Nonexpansive Nonself-Mappings, Common Fixed Points, Weak and Strong Convergence.
2010 MSC: 47H09, 47H10.

1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X and T a self-mapping of C. The fixed point set of T is denote by $F(T)$.

Definition 1.1. *The mapping T is said to be;*

(i) Nonexpansive *if* $\|Tx - Ty\| \leq \|x - y\|$ *for all* $x, y \in C$;

(ii) Quasi-nonexpansive *if* $F(T) \neq \emptyset$ *and* $\|Tx - p\| \leq \|x - p\|$ *for all* $x, y \in C$ *and* $p \in F(T)$;

(iii) Asymptotically nonexpansive *if there exists a sequence* $\{r_n\}$ *in* $[0, \infty)$ *with* $\lim_{n\to\infty} r_n = 0$ *and* $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, *for all* $x, y \in C$ *and* $n \geq 1$;

*Corresponding author
Email addresses: p_yatakoat@npu.ac.th (P. Yatakoat), scmti005@chiangmai.ac.th (S. Suantai)

Received: January 2011 *Revised:* June 2012

(iv) Asymptotically quasi-nonexpansive *if* $F(T) \neq \emptyset$ *and there exists a sequence* $\{r_n\}$ *in* $[0, \infty)$ *with* $\lim_{n\to\infty} r_n = 0$ *and* $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, *for all* $x \in C, p \in F(T)$ *and* $n \geq 1$*;*

(v) Aeneralized quasi-nonexpansive *if* $F(T) \neq \emptyset$ *and there exists a sequences* $\{s_n\}$ *in* $[0, \infty)$ *with* $s_n \to 0$ *as* $n \to \infty$ *such that* $\|T^n x - p\| \leq \|x - p\| + s_n$, *for all* $x \in C$ *and* $p \in F(T)$ *and* $n \geq 1$*;*

(iv) Aeneralized asymptotically quasi-nonexpansive *if* $F(T) \neq \emptyset$ *and there exist two sequences* $\{r_n\}$ *and* $\{s_n\}$ *in* $[0, \infty)$ *with* $r_n \to 0$ *and* $s_n \to 0$ *as* $n \to \infty$ *such that* $\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n$, *for all* $x \in C$, $p \in F(T)$ *and* $n \geq 1$.

(vii) Uniformly L-Lipschitzian *if there exists a constant* $L > 0$ *such that* $\|T^n x - T^n y\| \leq L\|x - y\|$, *for all* $x, y \in C$ *and* $n \geq 1$.

From the above definition (1.1) it follows that

(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive;

(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive;

(iii) an asymptotically nonexpansive mapping with nonempty fixed points set is a generalized asymptotically quasi-nonexpansive;

(iv) an asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive;

(v) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive.

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume et al. [1] as an important generalization of asymptotically nonexpansive mappings.

Recently, Deng and Liu [2] generalized the concept of generalized asymptotically quasi-nonexpansive self-mappings defined by Shahzad and Zegeye [3] to the case of nonself-mappings. Those mappings are defined as follows ;

Definition 1.2. *(see [1],[4]) Let* X *be a real Banach space and* C *a nonempty closed convex subset of* X *and let* $P : X \to C$ *be the nonexpansive retraction of* X *onto* C*. A nonself-mappins* $T : C \to X$ *is said to be*

(i) Asymptotically nonexpansive *if there exists a sequence* $\{r_n\}$ *in* $[0, \infty)$ *with* $\lim_{n\to\infty} r_n = 0$ *such that* $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + r_n)\|x - y\|$, *for all* $x, y \in C$ *and* $n \geq 1$*;*

(i) Asymptotically quasi-nonexpansive *if* $F(T) \neq \emptyset$ *and there exists a sequence* $\{r_n\}$ *in* $[0, \infty)$ *with* $\lim_{n\to\infty} r_n = 0$ *such that* $\|T(PT)^{n-1}x - p\| \leq (1 + r_n)\|x - p\|$, *for all* $x \in C, p \in F(T)$ *and* $n \geq 1$*;*

(iii) Generalized asymptotically quasi-nonexpansive *[5] if* $F(T) \neq \emptyset$ *and there exist two sequences* $\{r_n\}$ *and* $\{s_n\}$ *in* $[0, \infty)$ *with* $r_n \to 0$ *and* $s_n \to 0$ *as* $n \to \infty$ *such that* $\|T(PT)^{n-1}x - p\| \leq (1 + r_n)\|x - p\| + s_n$, *for all* $x \in C$, $p \in F(T)$ *and* $n \geq 1$.

(iv) Uniformly L-Lipschitzian *if there exists a constant* $L > 0$ *such that* $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$, *for all* $x, y \in C$ *and* $n \geq 1$.

If T is self-mapping, then P becomes the identity mapping, so that $(i)-(iii)$ of Definition 1.2 reduce to $(iii), (iv)$ and (vi) of Definition 1.1, respectively.

In this paper, we introduced a new iteration process for a finite family $\{T_i : i = 1, 2, 3, ..., m\}$ of generalized asymptotically quasi-nonexpansive nonself-mappings as follows:

Let X be a real Banach space, C a nonempty closed convex subset of X and $P : X \to C$ a nonexpansive retraction of X onto C, and let $T_i : C \to X$ $(i = 1, 2, 3, ..., m)$ be nonself-mappings. Let $\{x_n\}$ be a sequence defined by

$$x_0 \in C, \quad x_{n+1} = S_n x_n, \quad \forall n \geq 1 \tag{1.1}$$

where $S_n = P(\alpha_{0n} I + \alpha_{1n} T_1 (PT_1)^{n-1} + \alpha_{2n} T_2 (PT_2)^{n-1} + \alpha_{3n} T_3 (PT_3)^{n-1} + \ldots + \alpha_{mn} T_m (PT_m)^{n-1})$ with $\alpha_{in} \in [0, 1]$ for $i = 1, 2, 3, ..., m$ and $\sum_{i=0}^{m} \alpha_{in} = 1$.

The main purpose of this paper is to prove strong convergence theorems of the iterative scheme (1.1) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive nonself-mappings in real Banach spaces.

2. Preliminaries and lemmas

In this section, we give some definitions and lemmas used in the main results. A subset C of X is said to be *retract* of X if there exists a continuous mappings $P : X \to C$ such that $P(x) = x$ for all $x \in C$.

A mappings $T : C \to X$ with $F(T) \neq \emptyset$ is said to be;

(i) *Demiclosed at 0* if for each sequence $\{x_n\}$ converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$;

(ii) *semi-compact* if for each sequence $\{x_n\}$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$;

(iii) *completely continuous* if for every bounded sequence $\{x_n\} \subset C$, there is a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ is convergent.

A Banach space X is said to satisfy *Opial′s property* (see [6])if for each $x \in X$ and each sequence $\{x_n\}$ weakly converges to x, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Lemma 2.1. *[7] Let the sequences $\{a_n\}$, $\{\delta_n\}$ and $\{c_n\}$ of real numbers satisfy:*

$$a_{n+1} \leq (1 + \delta_n) a_n + c_n, \text{ where } a_n \geq 0, \delta_n \geq 0, \ c_n \geq 0 \text{ for all } n = 1, 2, 3, \ldots$$

and $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$. Then

(i) $\lim_{n\to\infty} a_n$ exists;

(ii) if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. *[8] Let X be a Banach space which satisfy Opial′s property and let $\{x_n\}$ be a sequence in X. Let $x, y \in X$ be such that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exists. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to x and y, then $x = y$.*

Lemma 2.3. *[9] Let X be a uniformly convex Banach. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for each $m \in \mathbb{N}$ and $j \in \{1, 2, 3, ..., m\}$,*

$$\|\sum_{i=1}^{m} \alpha_i x_i\|^2 \leq \sum_{i=1}^{m} \alpha_i \|x\|^2 - \frac{\alpha_i}{m-1}(\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|)),$$

for all $x_i \in B_r(0)$ and $\alpha_i \in [0,1]$ for all $i = 1, 2, 3, ..., m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

3. Main Results

The aim of this section is to establish weak and strong convergence of the iterative scheme (1.1) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive nonself-mappings in a Banach space under some appropriate conditions.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Banach space X, and $T_i : C \to X$, $(i = 1, 2, 3, \ldots, m)$ be family of generalized asymptotically quasi-nonexpansive nonself-mappings with the sequence $\{r_{in}\}, \{s_{in}\} \subset [0,\infty)$ and $p_i \in F(T_i), i = 1, 2, \ldots, m$. Suppose that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, and the iterative sequence $\{x_n\}$, is defined by (1.1). Assume that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Then we get*

(i) there exists two sequences $\{\delta_n\}, \{c_n\}$ in $[0,\infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n$ for all $p \in F(T)$ and $n \geq 1$;

(ii) there exist $L, D > 0$ such that $\|x_{n+k} - p\| \leq L\|x_n - p\| + D$, for all $p \in F(T)$ and $n, k \in \mathbb{N}$.

Proof . (i) Let $p \in F$ and $r_n = max_{1\leq i\leq k}\{r_{in}\}, s_n = max_{1\leq i\leq k}\{s_{in}\}$ for all n.
Since $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$ for all $i = 1, 2, 3, \ldots, m$, we obtain that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$.
For $i = 1, 2, 3, \ldots, m$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|S_n x_n - p\| \\
&= \|P(\alpha_{0n} I + \alpha_{1n} T_1 (PT_1)^{n-1} + \alpha_{2n} T_2 (PT_2)^{n-1} + \ldots \\
&\quad + \alpha_{mn} T_m (PT_m)^{n-1}) x_n - P(p)\| \\
&\leq \alpha_{0n}\|x_n - p\| + \alpha_{1n}\|T_1(PT_1)^{n-1}x_n - p\| + \alpha_{2n}\|T_2(PT_2)^{n-1}x_n - p\| + \ldots \\
&\quad + \alpha_{mn}\|T_m(PT_m)^{n-1}x_n - p\| \\
&\leq \alpha_{0n}\|x_n - p\| + \alpha_{1n}((1 + r_{1n})\|x_n - p\| + s_{1n}) + \ldots \\
&\quad + \alpha_{2n}((1 + r_{2n})\|x_n - p\| + s_{2n}) + \alpha_{mn}((1 + r_{mn})\|x_n - p\| + s_{mn}) \\
&\leq (\alpha_{0n} + \alpha_{1n}(1 + r_n) + \alpha_{2n}(1 + r_n) + \ldots + \alpha_{mn}(1 + r_n))\|x_n - p\| \\
&\quad + s_{1n} + s_{2n} + s_{3n} + \ldots + s_{mn} \\
&\leq (1 + m r_n)\|x_n - p\| + m s_n \\
&= (1 + \delta_n)\|x_n - p\| + c_n \qquad (3.1)
\end{aligned}$$

where $\delta_n = m r_n$ and $c_n = m s_n$.

(ii) If $t \geq 0$ then $1 + t \leq e^t$. Thus, from part (i) and for $n, k \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{n+k} - p\| &\leq (1+\delta_{n+k-1})\|x_{n+k-1} - p\| + c_{n+k-1} \\
&\leq exp\{\delta_{n+k-1}\}\|x_{n+k-1} - p\| + c_{n+k-1} \\
&\leq exp\{\delta_{n+k-1}\}[(1+\delta_{n+k-2})\|x_{n+k-2} - p\| + c_{n+k-2}] + c_{n+k-1} \\
&\leq exp\{\delta_{n+k-1}\}exp\{\delta_{n+k-2}\}\|x_{n+k-2} - p\| + exp\{\delta_{n+k-1}\}c_{n+k-2} + c_{n+k-1} \\
&\vdots \\
&\leq exp\{\sum_{i=0}^{k-1}\delta_{n+i}\}\|x_n - p\| + exp\{\sum_{i=0}^{k-1}\delta_{n+i}\}\sum_{i=0}^{k-1}\delta_{n+i} \qquad (3.2)
\end{aligned}$$

Setting $L = exp\{\sum_{i=1}^{\infty}\delta_i\}$ and $D = L\sum_{i=1}^{\infty}c_i$, we obtain $\|x_{n+k} - p\| \leq L\|x_n - p\| + D$.
Thus (ii) is satisfied. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $T_i : C \to X$, $(i = 1, 2, 3, \ldots, m)$ be family of uniformly L_i-Lipschitzian and generalized asymptotically quasi-nonexpansive nonself-mappings with the sequence $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ and $p_i \in F(T_i), i = 1, 2, \ldots, m$. Suppose that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, and $\liminf_{n\to\infty}\alpha_{0n}\alpha_{in} > 0$, $\forall i = 1, 2, 3, \ldots, m$ and $\{x_n\}$ is defined by (1.1) such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Then we get*

(i) $lim_{n\to\infty}\|x_n - p\|$ exists, $\forall p \in F(T)$;

(ii) $lim_{n\to\infty}\|x_n - T_ix_n\| = 0$, for each $i = 1, 2, \ldots, m$.

Proof .

(i) By lemmas 2.1 and 3.1(i), we obtain that $lim_{n\to\infty}\|x_n - p\|$ exists.

(ii) From (i), we have that $\{x_n\}$ is bounded. For each $i = 1, 2, 3, \ldots, m$, we have

$$\begin{aligned}
\|T_i(PT_i)^{n-1}x_n - p\| &\leq (1 + r_{in})\|x_n - p\| + s_{in} \\
&\leq (1 + r_n)\|x_n - p\| + s_n
\end{aligned}$$

It follows that $\{T_i(PT_i)^{n-1}x_n - p\}$ is bounded $\forall i = 1, 2, 3, \ldots, m$.
Put $r = sup\{\|T_i(PT_i)^{n-1}x_n - p\| : 1 \leq i \leq m, n \in \mathbb{N}\} + sup\{\|x_n - p\| : n \in \mathbb{N}\}$.
Let $1 \leq i \leq m$. By Lemma 2.3, there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\sum_{i=1}^{m}\alpha_i x_i\|^2 \leq \sum_{i=1}^{m}\alpha_i\|x\|^2 - \frac{\alpha_i}{m-1}(\sum_{i=1}^{m}\alpha_i g(\|x_j - x_i\|)), \qquad (3.3)$$

for all $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, 3, ..., m$ with $\sum_{i=1}^{m}\alpha_i = 1$.
By (3.3), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_{0n}(x_n - p) + \alpha_{1n}(T_1(PT_1)^{n-1}x_n - p) + \ldots + \alpha_{mn}(T_m(PT_m)^{n-1}x_n - p)\|^2 \\
&\leq \alpha_{0n}\|x_n - p\|^2 + \alpha_{1n}((1 + r_{1n})\|x_n - p\| + s_{1n})^2 + \ldots
\end{aligned}$$

$$
\begin{aligned}
&\quad +\alpha_{mn}((1+r_{mn})\|x_n-p\|+s_{mn})^2-\frac{\alpha_0}{m}(\sum_{i=1}^m\alpha_i g(\|x_n-T_i(PT_i)^{n-1}x_n\|))\\
&\leq \alpha_{0n}\|x_n-p\|^2+\sum_{i=1}^m\alpha_{in}\|x_n-p\|^2+2\sum_{i=1}^m\alpha_{in}r_n\|x_n-p\|^2\\
&\quad +\sum_{i=1}^m\alpha_{in}r_n^2\|x_n-p\|^2+2\sum_{i=1}^m\alpha_{in}(1+r_n)s_n\|x_n-p\|\\
&\quad +\sum_{i=1}^m\alpha_{in}s_n^2-\frac{\alpha_0}{m}(\sum_{i=1}^m\alpha_i g(\|x_n-T_i(PT_i)^{n-1}x_n\|))\\
&\leq \|x_n-p\|^2+2\sum_{i=1}^m\alpha_{in}r_n\|x_n-p\|^2+\sum_{i=1}^m\alpha_{in}r_n^2\|x_n-p\|^2\\
&\quad +2\sum_{i=1}^m\alpha_{in}(1+r_n)s_n\|x_n-p\|+\sum_{i=1}^m\alpha_{in}s_n^2\\
&\quad -\frac{\alpha_0}{m}(\sum_{i=1}^m\alpha_i g(\|x_n-T_i(PT_i)^{n-1}x_n\|)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{\alpha_0}{m}(\sum_{i=1}^m\alpha_i g(\|x_n-T_i(PT_i)^{n-1}x_n\|)) \leq\ & (\|x_{n+1}-p\|^2-\|x_n-p\|^2)+2\sum_{i=1}^m\alpha_{in}r_n\|x_n-p\|^2\\
&+\sum_{i=1}^m\alpha_{in}r_n^2\|x_n-p\|^2+2\sum_{i=1}^m\alpha_{in}(1+r_n)s_n\|x_n-p\|\\
&+\sum_{i=1}^m\alpha_{in}s_n^2.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exists,$\lim_{n\to\infty} r_n=0=\lim_{n\to\infty}s_n$ and $\liminf_{n\to\infty}\alpha_{0n}\alpha_{in}>0$ for each $i=1,2,3,\ldots,m$, it follows that $\lim_{n\to\infty}g(\|x_n-T_i(PT_i)^{n-1}x_n\|)=0$.
Since g is continuous strictly increasing with $g(0)=0$, we can conclude that

$$\lim_{n\to\infty}\|x_n-T_i(PT_i)^{n-1}x_n\|=0,\ \forall i=1,2,3,\ldots,m. \tag{3.4}$$

For each $n\in\mathbb{N}$, we have

$$
\begin{aligned}
\|x_{n+1}-T_1(PT_1)^{n-1}x_n\| &= \|\alpha_{0n}x_n+\alpha_{1n}T_1(PT_1)^{n-1}x_n+\alpha_{2n}T_2(PT_2)^{n-1}x_n+\ldots\\
&\quad +\alpha_{mn}T_m(PT_m)^{n-1}x_n-T_1(PT_1)^{n-1}x_n\|\\
&\leq \alpha_{0n}\|x_n-T_1(PT_1)^{n-1}x_n\|+\alpha_{2n}\|T_2(PT_2)^{n-1}x_n-T_1(PT_1)^{n-1}x_n\|+\\
&\quad \ldots+\alpha_{mn}\|T_m(PT_m)^{n-1}x_n-T_1(PT_1)^{n-1}x_n\|\\
&\leq \alpha_{0n}\|x_n-T_1(PT_1)^{n-1}x_n\|+\alpha_{2n}\|T_2(PT_2)^{n-1}x_n-x_n\|\\
&\quad +\alpha_{2n}\|x_n-T_1(PT_1)^{n-1}x_n\|+\ldots\\
&\quad +\alpha_{mn}\|T_m(PT_m)^{n-1}x_n-x_n\|+\alpha_{mn}\|x_n-T_1(PT_1)^{n-1}x_n\|. \qquad (3.5)
\end{aligned}
$$

From (3.4) and (3.5), we have

$$\|x_{n+1}-T_1(PT_1)^{n-1}x_n\|\to o\ \ as\ \ n\to\infty \tag{3.6}$$

From (3.4) and (3.6), where $n \to \infty$

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \to 0. \tag{3.7}$$

Since T_i is uniformly L_i-Lipschitzian, we have

$$\begin{aligned}
\|x_n - T_ix_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_i(PT_i)^{n-1}x_{n+1}\| + \\
&\quad \|T_i(PT_i)^{n-1}x_{n+1} - T_i(PT_i)^{n-1}x_n\| + \|T_i(PT_i)^{n-1}x_n - T_ix_n\| \\
&\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_i(PT_i)^{n-1}x_{n+1}\| + \\
&\quad L\|x_{n+1} - x_n\| + L\|(PT_i)^{n-1}x_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_i(PT_i)^{n-1}x_{n+1}\| + \\
&\quad L\|x_{n+1} - x_n\| + L\|T_i(PT_i)^{n-1}x_n - x_n\|.
\end{aligned} \tag{3.8}$$

It follows from (3.4), (3.6) and (3.7) that

$$\|x_n - T_ix_n\| = 0$$

This completes the proof. □

Theorem 3.3. *Under the hypotheses of Lemma 3.2, assume that one of T_i is completely continuous. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \dots, m\}$.*

Proof . Suppose that T_{i_0} is completely continuous for some $i_0 \in \{1, 2, \dots, m\}$. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $T_{i_0}x_{n_k} \to p$. By Lemma 3.2 (ii), we have $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0,\ \forall i = 1, 2, \dots, m$. It follows that

$$\|x_{n_k} - p\| \le \|x_{n_k} - T_{i_0}x_{n_k}\| + \|T_{i_0}x_{n_k} - p\| \to 0.$$

Thus $x_{n_k} \to p$. By the continuity of T_i, we have

$$\|p - T_ip\| = \lim_{k\to\infty} \|x_{n_k} - T_ix_{n_k}\| = 0,\ \forall i = 1, 2, \dots, m.$$

Hence $p \in F$. By Lemma 3.2 (i), we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. □

Theorem 3.4. *Under the hypotheses of Lemma 3.2, assume that one of T_i is semi-compact. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \dots, m\}$.*

Proof . Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \dots, m\}$. By Lemma 3.2 (ii), we have $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0,\ \forall i = 1, 2, \dots, m$. Since $\{x_n\}$ is bounded and T_{i_0} is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to p$. By the continuity of T_i, we have

$$\|p - T_ip\| = \lim_{k\to\infty} \|x_{n_k} - T_ix_{n_k}\| = 0,\ \forall i = 1, 2, \dots, m.$$

Hence $p \in F$. By Lemma 3.2 (i), we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. □

Theorem 3.5. *Under the hypotheses of Lemma 3.2, assume that $(I - T_i)$ is demiclosed at 0, for each $i = 1, 2, \dots, m$. Then the iterative sequence $\{x_n\}$ defined by (1.1) converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \dots, m\}$.*

Proof . Let $p \in F$. By Lemma 3.2 (i), we have $\lim_{n\to\infty} \|x_n - p\|$ exists, and hence $\{x_n\}$ is bounded. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly $q_1 \in C$. By Lemma 3.2 (ii), $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$. Since $(I - T_i)$ is demiclosed at 0, for each $i = 1, 2, \dots, m$, we obtain that $T_i q_1 = q_1$. That is, $q_1 \in F$. Next, we show that $\{x_n\}$ converges weakly to q_1. Take another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $q_2 \in C$. Again, as above, we can conclude that $q_2 \in F$. By Lemma 3.2, we obtain that $q_1 = q_2$. This show that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, m\}$. □

Remark 3.6. *If $\{T_i : C \to C\}_i^m$ is a finite family of self-mappings, then the mapping S_n in (1.1) is reduced to $S_n = \alpha_{0n} I + \alpha_{1n} T_1 + \alpha_{2n} T_2 + \alpha_{3n} T_3 + \ldots + \alpha_{mn} T_m$ by Cholamjiak and Suantai [9]. So results obtained in the paper generalized those in [9].*

4. Acknowledgments

This research was supported by grant from under the program Strategic Scholarships for Frontier Research Network for the Ph.D. Program Thai Doctoral degree from the Office of the Higher Education Commission, the Graduate School of Chiang Mai University and the Thailand Research Fund.

References

[1] C. E. Chidume, E. U. Ofoedu and H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., 280 (2003) 364–374.

[2] L. Deng and Q. Liu, *Scheme for nonself generalized asymptotically quasi-nonexpansive mappings*, Appl. Math. Comput., 205 (2008) 317–324.

[3] N. Shahzad and A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpanisve mappings in Banach spaces*, Fixed Point Theory Appl., article ID 18909, 10 pages, 2006.

[4] C. Wang, J. Zhu and L. An, *New iterative schemes dor asymptotically quasi-nonexpansive nonself-mappings in Banach spaces*, Comput. Math. Appl., 233 (2010) 2948–2955.

[5] N. Shahzad and H. Zegeye, *Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps*, Appl. Math. Comput., 189 (2007) 1058–1065.

[6] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., 73 (1967) 591–597.

[7] Z. H. Sun, *Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl., 286 (2003) 351–358.

[8] S. Suantai, *Weak and Strong convergence criteria of noor iterations for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., 311 (2005) 506–517.

[9] W. Cholamjiak and S. Suantai, *Weak and Strong convergence theorems for a finite family og generalized asymptotically quasi-nonexpansive mappings*, Comput. Math. Appl., 60 (2010) 1917–1923.

[10] S. Thianwan, *Common fixed point of new iterations for two asymptotically nonexpansive nonself-mappings in a Banach spaces*, Comput. Math. Appl., 224 (2009) 688–695.

[11] A. R. Khan, A. A. Domlo and H. Fukhar-ud-din, *Common fixed point Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., 341 (2008) 1–11.