Weak and Strong Convergence Theorems for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Nonself-Mappings

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Abstract

In this paper, we introduce and study a new iterative scheme to approximate a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive nonself-mappings in Banach spaces. Several strong and weak convergence theorems of the proposed iteration are established. The main results obtained in this paper generalize and refine some known results in the current literature.

Keywords: Generalized Asymptotically Quasi-Nonexpansive Nonself-Mappings, Common Fixed Points, Weak and Strong Convergence.

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1. Introduction

Let $X$ be a real Banach space, $C$ a nonempty closed convex subset of $X$ and $T$ a self-mapping of $C$. The fixed point set of $T$ is denoted by $F(T)$.

**Definition 1.1.** The mapping $T$ is said to be:

(i) Nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(ii) Quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x, y \in C$ and $p \in F(T)$;

(iii) Asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} r_n = 0$ and $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n \geq 1$;

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Asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence \( \{r_n\} \) in \([0, \infty)\) with $\lim_{n \to \infty} r_n = 0$ and $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, for all $x, p \in F(T)$ and $n \geq 1$;

Generalized quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence \( \{s_n\} \) in \([0, \infty)\) with $s_n \to 0$ as $n \to \infty$ such that $\|T^n x - p\| \leq \|x - p\| + s_n$, for all $x \in C$ and $p \in F(T)$ and $n \geq 1$;

Asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exist two sequences \( \{r_n\} \) and \( \{s_n\} \) in \([0, \infty)\) with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

Uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n \geq 1$.

From the above definition (1.1), it follows that

(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive;

(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive;

(iii) an asymptotically nonexpansive mapping with nonempty fixed points set is a generalized asymptotically quasi-nonexpansive;

(iv) an asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive;

(v) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive.

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume et al. [1] as an important generalization of asymptotically nonexpansive mappings. Recently, Deng and Liu [2] generalized the concept of generalized asymptotically quasi-nonexpansive self-mappings defined by Shahzad and Zegeye [3] to the case of nonself-mappings. Those mappings are defined as follows:

**Definition 1.2.** (see [1], [3]) Let $X$ be a real Banach space and $C$ a nonempty closed convex subset of $X$ and let $P : X \to C$ be the nonexpansive retraction of $X$ onto $C$. A nonself-mapping $T : C \to X$ is said to be

(i) Asymptotically nonexpansive if there exists a sequence \( \{r_n\} \) in \([0, \infty)\) with $\lim_{n \to \infty} r_n = 0$ such that $\|T(PT)^{n-1} x - T(PT)^{n-1} y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in C$ and $n \geq 1$;

(ii) Asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence \( \{r_n\} \) in \([0, \infty)\) with $\lim_{n \to \infty} r_n = 0$ such that $\|T(PT)^{n-1} x - p\| \leq (1 + r_n)\|x - p\|$, for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(iii) Generalized asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exist two sequences \( \{r_n\} \) and \( \{s_n\} \) in \([0, \infty)\) with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that $\|T(PT)^{n-1} x - p\| \leq (1 + r_n)\|x - p\| + s_n$, for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(iv) Uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $\|T(PT)^{n-1} x - T(PT)^{n-1} y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n \geq 1$. 

If $T$ is self-mapping, then $P$ becomes the identity mapping, so that $(i) - (iii)$ of Definition 1.2 reduce to $(iii), (iv)$ and $(vi)$ of Definition 1.1, respectively.

In this paper, we introduced a new iteration process for a finite family $\{T_i : i = 1, 2, 3, ..., m\}$ of nonexpansive mappings in real Banach spaces.

Let $X$ be a real Banach space, $C$ a nonempty closed convex subset of $X$ and $P : X \rightarrow C$ a nonexpansive retraction of $X$ onto $C$, and let $T_i : C \rightarrow X$ ($i = 1, 2, 3, ..., m$) be nonexpansive mappings. Let $\{x_n\}$ be a sequence defined by

$$x_0 \in C, \ x_{n+1} = S_n x_n, \ \forall n \geq 1 \quad (1.1)$$

where $S_n = P(\alpha_{0n} I + \alpha_{1n} T_1(PT_1)^{n-1} + \alpha_{2n} T_2(PT_2)^{n-1} + \alpha_{3n} T_3(PT_3)^{n-1} + \ldots + \alpha_{mn} T_m(PT_m)^{n-1})$ with $\alpha_{in} \in [0, 1]$ for $i = 1, 2, 3, ..., m$ and $\sum_{i=0}^{m} \alpha_{in} = 1$.

The main purpose of this paper is to prove strong convergence theorems of the iterative scheme (1.1) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive nonself-mappings in real Banach spaces.

2. Preliminaries and lemmas

In this section, we give some definitions and lemmas used in the main results. A subset $C$ of $X$ is said to be retract of $X$ if there exists a continuous mappings $P : X \rightarrow C$ such that $P(x) = x$ for all $x \in C$.

A mappings $T : C \rightarrow X$ with $F(T) \neq \emptyset$ is said to be;

(i) *Demiclosed at 0* if for each sequence $\{x_n\}$ converging weakly to $x$ and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$;

(ii) *semi-compact* if for each sequence $\{x_n\}$ with $\lim_{n \to \infty} \| x_n - Tx_n \| = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$;

(iii) *completely continuous* if for every bounded sequence $\{x_n\} \subset C$, there is a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ is convergent.

A Banach space $X$ is said to satisfy *Opial’s property* (see [4]) if for each $x \in X$ and each sequence $\{x_n\}$ weakly converges to $x$, the following condition holds for all $x \neq y$:

$$\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \|.$$  

**Lemma 2.1.** [7] Let the sequences $\{a_n\}$, $\{\delta_n\}$ and $\{c_n\}$ of real numbers satisfy:

$$a_{n+1} \leq (1 + \delta_n) a_n + c_n, \ \text{where} \ a_n \geq 0, \delta_n \geq 0, \ c_n \geq 0 \ \text{for all} \ n = 1, 2, 3, \ldots$$

and $\sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$. Then

(i) $\lim_{n \to \infty} a_n$ exists;

(ii) if $\lim_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.2.** [8] Let $X$ be a Banach space which satisfy Opial’s property and let $\{x_n\}$ be a sequence in $X$. Let $x, y \in X$ be such that $\lim_{n \to \infty} \| x_n - x \|$ and $\lim_{n \to \infty} \| x_n - y \|$ exists. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to $x$ and $y$, then $x = y$. 

Lemma 2.3. Let $X$ be a uniformly convex Banach. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for each $m \in \mathbb{N}$ and $j \in \{1, 2, 3, \ldots, m\},$
\[
\| \sum_{i=1}^{m} \alpha_i x_i \|^2 \leq \sum_{i=1}^{m} \alpha_i \|x\|^2 - \frac{\alpha_i}{m-1} \left( \sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|) \right),
\]
for all $x_i \in B_\varepsilon(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, 3, \ldots, m$ with $\sum_{i=1}^{m} \alpha_i = 1.$

3. Main Results

The aim of this section is to establish weak and strong convergence of the iterative scheme (1.1) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive nonself-mappings in a Banach space under some appropriate conditions.

Lemma 3.1. Let $C$ be a nonempty closed convex subset of a real Banach space $X$, and $T_i : C \to X,$ $(i = 1, 2, 3, \ldots, m)$ be family of generalized asymptotically quasi-nonexpansive nonself-mappings with the sequence $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ and $p_i \in F(T_i), i = 1, 2, \ldots, m.$ Suppose that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset,$ and the iterative sequence $\{x_n\}$, is defined by (1.1). Assume that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty.$ Then we get

(i) there exists two sequences $\{\delta_n\}, \{c_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty,$ $\sum_{n=1}^{\infty} c_n < \infty$ and $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + c_n$ for all $p \in F(T)$ and $n \geq 1;$

(ii) there exist $L, D > 0$ such that $\|x_{n+k} - p\| \leq L\|x_n - p\| + D,$ for all $p \in F(T)$ and $n, k \in \mathbb{N}.$

Proof. (i) Let $p \in F$ and $r_n = \max_{1 \leq i \leq k} \{r_{in}\}, s_n = \max_{1 \leq i \leq k} \{s_{in}\}$ for all $n.$ Since $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$ for all $i = 1, 2, 3, \ldots, m,$ we obtain that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty.$

For $i = 1, 2, 3, \ldots, m,$ we have
\[
\|x_{n+1} - p\| = \|S_n x_n - p\| \\
= \|P(\alpha_{0n} L + \alpha_{1n} T_1(PT_1)^{n-1} + \alpha_{2n} T_2(PT_2)^{n-1} + \ldots + \alpha_{mn} T_m(PT_m)^{n-1}) x_n - P(p)\| \\
\leq \alpha_{0n}\|x_n - p\| + \alpha_{1n}\|T_1(PT_1)^{n-1} x_n - p\| + \alpha_{2n}\|T_2(PT_2)^{n-1} x_n - p\| + \ldots + \alpha_{mn}\|T_m(PT_m)^{n-1} x_n - p\| \\
\leq \alpha_{0n}\|x_n - p\| + \alpha_{1n}((1 + r_{1n})\|x_n - p\| + s_{1n}) + \ldots + \alpha_{2n}((1 + s_{2n})\|x_n - p\| + s_{2n}) + \alpha_{mn}((1 + r_{mn})\|x_n - p\| + s_{mn}) \\
\leq (\alpha_{0n} + \alpha_{1n}(1 + r_{1n}) + \alpha_{2n}(1 + r_{2n}) + \ldots + \alpha_{mn}(1 + r_{mn}))\|x_n - p\| + s_{1n} + s_{2n} + s_{3n} + \ldots + s_{mn} \\
\leq (1 + mr_n)\|x_n - p\| + ms_n \\
= (1 + \delta_n)\|x_n - p\| + c_n
\] (3.1)
where $\delta_n = mr_n$ and $c_n = ms_n.$
(ii) If $t \geq 0$ then $1 + t \leq e^t$. Thus, from part (i) and for $n, k \in \mathbb{N}$, we have
\[
\|x_{n+k} - p\| \leq (1 + \delta_{n+k-1})\|x_{n+k-1} - p\| + c_{n+k-1}
\leq \exp\{\delta_{n+k-1}\|x_{n+k-1} - p\| + c_{n+k-1}
\leq \exp\{\delta_{n+k-1}\{(1 + \delta_{n+k-2})\|x_{n+k-2} - p\| + c_{n+k-2}\} + c_{n+k-1}
\leq \exp\{\delta_{n+k-1}\} \exp\{\delta_{n+k-2}\} \|x_{n+k-2} - p\| + \exp\{\delta_{n+k-1}\} c_{n+k-2} + c_{n+k-1}
\vdots
\leq \exp\{\sum_{i=0}^{k-1} \delta_{n+i}\} \|x_{n} - p\| + \exp\{\sum_{i=0}^{k-1} \delta_{n+i}\} \sum_{i=0}^{k-1} \delta_{n+i}
\] (3.2)

Setting $L = \exp(\sum_{i=1}^{\infty} \delta_i)$ and $D = L \sum_{i=1}^{\infty} c_i$, we obtain $\|x_{n+k} - p\| \leq L\|x_n - p\| + D$.
Thus (ii) is satisfied. □

**Lemma 3.2.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$, and $T_i : C \to X$, $(i = 1, 2, 3, \ldots, m)$ be family of uniformly $L_i$-Lipschitzian and generalized asymptotically quasi-nonexpansive nonself-mappings with the sequence $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ and $p_i \in F(T_i), i = 1, 2, \ldots, m$. Suppose that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, and $\liminf_{n \to \infty} \alpha_0 \alpha_n s_{in} > 0$, $\forall i = 1, 2, 3, \ldots, m$ and $\{x_n\}$ is defined by (1.1) such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Then we get

(i) $\lim_{n \to \infty} \|x_n - p\|$ exists, $\forall p \in F(T)$;

(ii) $\lim_{n \to \infty} \|x_n - T_i x_n\| = 0$, for each $i = 1, 2, \ldots, m$.

**Proof.**

(i) By lemmas 2.1 and 3.1(i), we obtain that $\lim_{n \to \infty} \|x_n - p\|$ exists.

(ii) From (i), we have that $\{x_n\}$ is bounded. For each $i = 1, 2, 3, \ldots, m$, we have
\[
\|T_i(PT_i)^{n-1}x_n - p\| \leq (1 + r_{in})\|x_n - p\| + s_{in}
\leq (1 + r_n)\|x_n - p\| + s_{n}
\]

It follows that $\{T_i(PT_i)^{n-1}x_n - p\}$ is bounded $\forall i = 1, 2, 3, \ldots, m$.
Put $r = \sup\{\|T_i(PT_i)^{n-1}x_n - p\| : 1 \leq i \leq m, n \in \mathbb{N}\} + \sup\{\|x_n - p\| : n \in \mathbb{N}\}$. Let $1 \leq i \leq m$. By Lemma 2.3, there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that
\[
\|\sum_{i=1}^{m} \alpha_i x_i\|^2 \leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \frac{\alpha_i}{m-1} (\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|)),
\] (3.3)
for all $x_i \in B_{r}(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, 3, \ldots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$.
By (3.3), we have
\[
\|x_{n+1} - p\|^2 = \|\alpha_{0n}(x_n - p) + \alpha_{1n}(T_1(PT_1)^{n-1}x_n - p) + \ldots + \alpha_{mn}(T_m(PT_m)^{n-1}x_n - p)\|^2
\leq \alpha_{0n}\|x_n - p\|^2 + \alpha_{1n}(1 + r_{1n})\|x_n - p\| + s_{1n})^2 + \ldots
\]
\[ \alpha_0 n \| x_n - p \|^2 + \sum_{i=1}^{m} \alpha_i r_n \| x_n - p \|^2 + 2 \sum_{i=1}^{m} \alpha_i s_n \| x_n - p \| \]
\[ + \sum_{i=1}^{m} \alpha_i r_n^2 \| x_n - p \|^2 + 2 \sum_{i=1}^{m} \alpha_i s_n \| x_n - p \| \]
\[ - \frac{\alpha_0}{m} \sum_{i=1}^{m} \alpha_i g(\| x_n - T_i(PT_i)^{n-1} x_n \|). \]

It follows that
\[ \frac{\alpha_0}{m} \left( \sum_{i=1}^{m} \alpha_i g(\| x_n - T_i(PT_i)^{n-1} x_n \|) \right) \leq \left( \| x_{n+1} - p \|^2 - \| x_n - p \|^2 \right) + 2 \sum_{i=1}^{m} \alpha_i r_n \| x_n - p \|^2 \]
\[ + \sum_{i=1}^{m} \alpha_i r_n^2 \| x_n - p \|^2 + 2 \sum_{i=1}^{m} \alpha_i s_n \| x_n - p \| \]
\[ + \sum_{i=1}^{m} \alpha_i s_n^2. \]

Since \( \lim_{n \to \infty} \| x_n - p \| \) exists, \( \lim_{n \to \infty} r_n = 0 = \lim_{n \to \infty} s_n \) and \( \lim \inf_{n \to \infty} \alpha_0 n \alpha_i > 0 \) for each \( i = 1, 2, 3, \ldots, m \), it follows that \( \lim_{n \to \infty} g(\| x_n - T_i(PT_i)^{n-1} x_n \|) = 0. \)

Since \( g \) is continuous strictly increasing with \( g(0) = 0 \), we can conclude that
\[ \lim_{n \to \infty} \| x_n - T_i(PT_i)^{n-1} x_n \| = 0, \forall i = 1, 2, 3, \ldots, m. \] (3.4)

For each \( n \in \mathbb{N} \), we have
\[ \| x_{n+1} - T_1(PT_1)^{n-1} x_n \| = \| \alpha_0 n x_n + \alpha_1 n T_1(PT_1)^{n-1} x_n + \alpha_2 n T_2(PT_2)^{n-1} x_n + \ldots \]
\[ + \alpha_m n T_m(PT_m)^{n-1} x_n - T_1(PT_1)^{n-1} x_n \|
\[ \leq \alpha_0 n \| x_n - T_1(PT_1)^{n-1} x_n \| + \alpha_2 n \| T_2(PT_2)^{n-1} x_n - T_1(PT_1)^{n-1} x_n \| + \ldots + \alpha_m n \| T_m(PT_m)^{n-1} x_n - T_1(PT_1)^{n-1} x_n \|
\[ \leq \alpha_0 n \| x_n - T_1(PT_1)^{n-1} x_n \| + \alpha_2 n \| T_2(PT_2)^{n-1} x_n - x_n \|
\[ + \alpha_2 n \| x_n - T_1(PT_1)^{n-1} x_n \| + \ldots + \alpha_m n \| T_m(PT_m)^{n-1} x_n - x_n \|
\[ + \alpha_m n \| x_n - T_1(PT_1)^{n-1} x_n \|. \] (3.5)

From (3.4) and (3.5), we have
\[ \| x_{n+1} - T_1(PT_1)^{n-1} x_n \| \to 0 \text{ as } n \to \infty \] (3.6)
Theorem 3.5. Under the hypotheses of Lemma 3.2, assume that

\begin{align*}
\text{a convergent subsequence}
\end{align*}

Theorem 3.4. Under the hypotheses of Lemma 3.2, assume that one of

\begin{align*}
\text{the iterative sequence}
\end{align*}

Proof. Suppose that

\begin{align*}
\text{the iterative sequence}
\end{align*}

It follows from (3.4), (3.6) and (3.7) that

\begin{align*}
\|x_n - T_ix_n\| = 0
\end{align*}

This completes the proof. \(\square\)

Theorem 3.3. Under the hypotheses of Lemma 3.2, assume that one of \(T_i\) is completely continuous. Then the iterative sequence \(\{x_n\}\) defined by (1.1) converges strongly to a common fixed point of the family \(\{T_i : i = 1, 2, 3, \ldots, m\}\).

Proof. Suppose that \(T_{i_0}\) is completely continuous for some \(i_0 \in \{1, 2, \ldots, m\}\). Since \(\{x_n\}\) is bounded, \(\{x_n\}\) has a subsequence \(\{x_{n_k}\}\) such that \(T_{i_0}x_{n_k} \to p\). By Lemma 3.2 (ii), we have \(\lim_{n \to \infty} \|x_n - T_ix_n\| = 0\), \(\forall i = 1, 2, \ldots, m\). It follows that

\begin{align*}
\|x_{n_k} - p\| &\leq \|x_{n_k} - T_{i_0}x_{n_k}\| + \|T_{i_0}x_{n_k} - p\| \to 0.
\end{align*}

Thus \(x_{n_k} \to p\). By the continuity of \(T_i\), we have

\begin{align*}
\|p - T_ip\| = \lim_{k \to \infty} \|x_{n_k} - T_ix_{n_k}\| = 0, \forall i = 1, 2, \ldots, m.
\end{align*}

Hence \(p \in F\). By Lemma 3.2 (i), we have that \(\lim_{n \to \infty} \|x_n - p\|\) exists. This implies that \(\lim_{n \to \infty} \|x_n - p\| = 0\). \(\square\)

Theorem 3.4. Under the hypotheses of Lemma 3.2, assume that one of \(T_i\) is semi-compact. Then the iterative sequence \(\{x_n\}\) defined by (1.1) converges strongly to a common fixed point of the family \(\{T_i : i = 1, 2, 3, \ldots, m\}\).

Proof. Suppose that \(T_{i_0}\) is semi-compact for some \(i_0 \in \{1, 2, \ldots, m\}\). By Lemma 3.2 (ii), we have \(\lim_{n \to \infty} \|x_n - T_ix_n\| = 0\), \(\forall i = 1, 2, \ldots, m\). Since \(\{x_n\}\) is bounded and \(T_{i_0}\) is semi-compact, \(\{x_n\}\) has a convergent subsequence \(\{x_{n_k}\}\) such that \(x_{n_k} \to p\). By the continuity of \(T_i\), we have

\begin{align*}
\|p - T_ip\| = \lim_{k \to \infty} \|x_{n_k} - T_ix_{n_k}\| = 0, \forall i = 1, 2, \ldots, m.
\end{align*}

Hence \(p \in F\). By Lemma 3.2 (i), we have that \(\lim_{n \to \infty} \|x_n - p\|\) exists. This implies that \(\lim_{n \to \infty} \|x_n - p\| = 0\). \(\square\)

Theorem 3.5. Under the hypotheses of Lemma 3.2, assume that \((I - T_i)\) is demiclosed at 0, for each \(i = 1, 2, \ldots, m\). Then the iterative sequence \(\{x_n\}\) defined by (1.1) converges weakly to a common fixed point of the family \(\{T_i : i = 1, 2, 3, \ldots, m\}\).
Proof. Let $p \in F$. By Lemma 3.2(i), we have $\lim_{n \to \infty} \|x_n - p\|$ exists, and hence $\{x_n\}$ is bounded. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly $q_1 \in C$. By Lemma 3.2(ii), $\lim_{n \to \infty} \|x_n - T_i x_n\| = 0$. Since $(I - T_i)$ is demiclosed at 0, for each $i = 1, 2, \ldots, m$, we obtain that $T_i q_1 = q_1$. That is, $q_1 \in F$. Next, we show that $\{x_n\}$ converges weakly to $q_1$. Take another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $q_2 \in C$. Again, as above, we can conclude that $q_2 \in F$. By Lemma 3.2 we obtain that $q_1 = q_2$. This show that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$. □

Remark 3.6. If $\{T_i : C \to C\}_i^m$ is a finite family of self-mappings, then the mapping $S_n$ in (1.1) is reduced to $S_n = \alpha_0 n + \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \ldots + \alpha_m T_m$ by Cholamjiak and Suantai [9]. So results obtained in the paper generalized those in [9].

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