Generalization of Titchmarsh’s Theorem for the Dunkl Transform


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(Communicated by M. B. Ghaemi)

Abstract

Using a generalized spherical mean operator, we obtain the generalization of Titchmarsh’s theorem for the Dunkl transform for functions satisfying the Lipschitz condition in $L^2(\mathbb{R}^d, w_k)$, where $w_k$ is a weight function invariant under the action of an associated reflection groups.

Keywords: Dunkl Operator, Dunkl Transform, Generalized Spherical Mean Operator.


1. Introduction

Titchmarsh ([9], Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. ([9]) Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:

(1) $\|f(t+h)−f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ as $h \to 0$

(2) $\int_{|\lambda| \geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \to \infty$

where $\mathcal{F}(f)$ stands for the Fourier transform of $f$.

In this paper, we prove the generalization of Theorem 1.1 in the Dunkl transform setting by means of the generalized spherical mean operator.

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Received: January 2012 Revised: September 2012
2. Preliminaries

In order to confirm the basic and standard notation, we briefly overview the theory of Dunkl operators and related harmonic analysis. Main references are [1, 2, 3, 4, 7, 8, 9].

Let $\mathbb{R}^d$ be the Euclidean space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and let $|x| = \sqrt{\langle x, x \rangle}$. For $\alpha$ in $\mathbb{R}^d \setminus \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$. A finite set $\mathbb{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $\mathbb{R} \cap \mathbb{R} \alpha = \{\alpha, -\alpha\}$ and $\sigma_{\mathbb{R}} = \mathbb{R}$ for all $\alpha \in \mathbb{R}$. For a given root system $\mathbb{R}$, reflections $\sigma_\alpha, \alpha \in \mathbb{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathbb{R}$. We fix a $\beta \in \mathbb{R}^d \cup \alpha \in \mathbb{R} \ H_\alpha$ and define a positive root system $\mathbb{R}^+ = \{\alpha \in \mathbb{R} / \langle \alpha, \beta \rangle > 0\}$.

A function $k: \mathbb{R} \rightarrow \mathbb{C}$ on $\mathbb{R}$ is called a multiplicity function if it is invariant under the action of $W$. Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$.

We consider the weight function $w_k(x) = \prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{2k(\alpha)}$, where $w_k$ is $W$-invariant and homogeneous of degree $2\gamma$ where

$$\gamma = \sum_{\alpha \in \mathbb{R}^+} k(\alpha).$$

We let $\eta$ be the normalized surface measure on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ and set

$$d\eta_k(y) = w_k(y) d\eta(y).$$

Then $\eta_k$ is a $W$-invariant measure on $S^{d-1}$, we let $d_k = \eta_k(S^{d-1})$.

The Dunkl operators $T_j, 1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group $W$ and the multiplicity function $k$ are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathbb{R}^+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^d).$$

where $\alpha_j = \langle \alpha, e_j \rangle$; $(e_1, ..., e_d)$ being the canonical basis of $\mathbb{R}^d$ and $C^1(\mathbb{R}^d)$ is the space of functions of class $C^1$ on $\mathbb{R}^d$.

The Dunkl kernel $E_k$ on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F. Dunkl in [2]. For $y \in \mathbb{R}^d$ the function $x \mapsto E_k(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases}
T_j u(x, y) = y_j u(x, y) & \text{for } 1 \leq j \leq d \\
u(0, y) = 1 & \text{for all } y \in \mathbb{R}^d
\end{cases}$$

This kernel has unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

M. Röler has proved in [6] the following integral representation for the Dunkl kernel

$$E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x(y), \quad x \in \mathbb{R}^d, \ z \in \mathbb{C}^d$$

where $\mu_x$ is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, |x|)$ of center 0 and radius $|x|$. 
Proposition 2.1. [4] Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$. Then

1. $E_k(z, 0) = 1$
2. $E_k(z, w) = E_k(w, z)$
3. $E_k(\lambda z, w) = E_k(z, \lambda w)$
4. For all $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}$, $x \in \mathbb{R}^d$, $z \in \mathbb{C}^d$, we have

$$|D^\nu z E_k(x, z)| \leq |x|^{|\nu|} \exp(|x| |\text{Re} z|)$$

where

$$D^\nu z = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \ldots \partial z_d^{\nu_d}}; \quad |\nu| = \nu_1 + \ldots + \nu_d.$$

In particular

$$|D^\nu z E_k(ix, z)| \leq |x|^{|\nu|}$$

for all $x, z \in \mathbb{R}^d$

We denote by $L^2_k(\mathbb{R}^d) = L^2(\mathbb{R}^d, w_k(x) dx)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{k, 2} = \left( \int_{\mathbb{R}^d} |f(x)|^2 w_k(x) dx \right)^{1/2}$$

The Dunkl transform is defined for $f \in L^1_k(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_k(x) dx)$ by

$$\hat{f}(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) w_k(x) dx.$$

where the constant $c_k$ is given by

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}} w_k(z) dz.$$

According to [3, 4] we have the following results:

1. When both $f$ and $\hat{f}$ are in $L_k^1(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) E_k(ix, \xi) w_k(\xi) d\xi, \quad x \in \mathbb{R}^d$$

2. (Plancherel’s theorem) The Dunkl transform on $S(\mathbb{R}^d)$, the space of Schwartz functions, extends uniquely to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

K. Trimèche has introduced [8] the Dunkl translation operators $\tau_x$, $x \in \mathbb{R}^d$. For $f \in L^2_k(\mathbb{R}^d)$ and we have

$$\hat{\tau_x(f)}(\xi) = E_k(ix, \xi) \hat{f}(\xi)$$

and

$$\tau_x(f)(y) = c_k^{-1} \int_{\mathbb{R}^d} \hat{\xi} E_k(ix, \xi) E_k(iy, \xi) w_k(\xi) d\xi.$$
Appealing to Parseval theorem and Proposition 2.1 we see that
\[ \|\tau_x f\|_{k,2} \leq \|f\|_{k,2} \ \forall x \in \mathbb{R}^d. \]

The generalized spherical mean operator for \( f \in L_k^2(\mathbb{R}^d) \) is defined by
\[ M_h f(x) = \frac{1}{d_k} \int_{S^{d-1}} \tau_x(hy)d\eta_k(y), \ x \in \mathbb{R}^d, h > 0 \]

From \[5\], we have \( M_h f \in L_k^2(\mathbb{R}^d) \) whenever \( f \in L_k^2(\mathbb{R}^d) \) and
\[ \|M_h f\|_{k,2} \leq \|f\|_{k,2} \]
for all \( h > 0 \).

For \( p \geq -\frac{1}{2} \), we introduce the normalized Bessel function \( j_p \) defined by
\[ j_p(z) = \frac{\Gamma(p + 1)}{\pi} \sum_{n=0}^{\infty} (-1)^n (z/2)^{2n} \frac{n!}{n!(n + p + 1)}, \ z \in \mathbb{C} \]  \hspace{1cm} (2.2)

where \( \Gamma \) is the gamma-function.

From \[3\] we have
\[ \frac{1}{d_k} \int_{S^{d-1}} E_k(iy, x)d\eta_k(y) = j_{\gamma + \frac{d}{2} - 1}(|x|) \] \hspace{1cm} (2.3)

for all \( x \in \mathbb{R}^d \). This shows in particular that \( x \rightarrow j_{\gamma + \frac{d}{2} - 1}(|x|) \) is a smooth bounded function with
\[ |j_{\gamma + \frac{d}{2} - 1}(|x|)| \leq 1 \] \hspace{1cm} (2.4)

From (2.2) we obtain
\[ \lim_{z \rightarrow 0} j_{\gamma + \frac{d}{2} - 1}(z) - 1 - z^2 \neq 0 \]

by consequence, there exist \( c > 0 \) and \( \eta > 0 \) such that
\[ |z| \leq \eta \implies |j_{\gamma + \frac{d}{2} - 1}(z) - 1| \geq c|z|^2 \] \hspace{1cm} (2.5)

The integral representation of the Dunkl kernel (2.1) and (2.3) yield
\[ |j_{\gamma + \frac{d}{2} - 1}(|x|)| \leq |x|. \] \hspace{1cm} (2.6)

**Proposition 2.2.** Let \( f \in L_k^2(\mathbb{R}^d) \). Then
\[ M_h f(\xi) = j_{\gamma + \frac{d}{2} - 1}(h|\xi|) \hat{f}(\xi) \]

**Proof.** (See [5]). \( \square \)
3. Generalization of Titchmarsh’s Theorem

In this section we give the main result of this paper. We need first to define the $\psi$-Dunkl Lipschitz class.

**Definition 3.1.** A function $f \in L^2_k(\mathbb{R}^d)$ is said to be in the $\psi$-Dunkl Lipschitz class, denoted by $\text{Lip}(\psi, 2, k)$; if:

$$\|M_h f(\cdot) - f(\cdot)\|_{k, 2} = O(\psi(h))$$

as $h \to 0$.

where $\psi(t)$ is a continuous increasing function on $[0, \infty)$, $\psi(0) = 0$ and $\psi(t) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$ and this function verify $\int_{0}^{1/h} \psi(s^{-2})ds = O(\frac{1}{h^2}\psi(h^2))$ as $h \to 0$

**Theorem 3.2.** Let $f \in L^2_k(\mathbb{R}^d)$. Then the following are equivalents

1. $f \in \text{Lip}(\psi, 2, k)$
2. $\int_{|x| \geq r} |\hat{f}(x)|^2 w_k(x)dx = O(\psi(r^{-2}))$ as $r \to \infty$

**Proof.** 1) $\implies$ 2) Suppose that $f \in \text{Lip}(\psi, 2, k)$. Then we have

$$\|M_h f - f\|_{k, 2} = O(\psi(h)) \text{ as } h \to 0.$$

Parseval Theorem and Proposition 2.2, we obtain

$$\|M_h f - f\|_{k, 2}^2 = \int_{\mathbb{R}^d} |1 - j_{\gamma + \frac{d}{2} - 1}(h|x|)|^2 |\hat{f}(x)|^2 w_k(x)dx$$

Formula (2.5) gives

$$\int_{\frac{\eta r}{2} \leq |x| \leq \frac{\eta}{2}} |1 - j_{\gamma + \frac{d}{2} - 1}(h|x|)|^2 |\hat{f}(x)|^2 w_k(x)dx \geq \frac{\eta^4}{16} \int_{\frac{\eta r}{2} \leq |x| \leq \frac{\eta}{2}} |\hat{f}(x)|^2 w_k(x)dx$$

There exists then a positive constant $C$ such that

$$\int_{\frac{\eta r}{2} \leq |x| \leq \frac{\eta}{2}} |\hat{f}(x)|^2 w_k(x)dx \leq C \int_{\mathbb{R}^d} |1 - j_{\gamma + \frac{d}{2} - 1}(h|x|)|^2 |\hat{f}(x)|^2 w_k(x)dx$$

$$\leq C \psi(h^2).$$

For all $h > 0$, we obtain

$$\int_{r \leq |x| \leq 2r} |\hat{f}(x)|^2 w_k(x)dx \leq C \psi(2^{-2} \eta r^{-2})$$

Thus there exists $K > 0$ such that

$$\int_{r \leq |x| \leq 2r} |\hat{f}(x)|^2 w_k(x)dx \leq K \psi(r^{-2})$$

So that
\[\int_{|x| \geq r} |\hat{f}(x)|^2w_k(x)dx = \left[ \int_{r \leq |x| \leq 2r} + \int_{2r \leq |x| \leq 4r} + \int_{4r \leq |x| \leq 8r} \ldots \right] |\hat{f}(x)|^2w_k(x)dx = O(\psi(r^{-2}) + \psi(2^{-2}r^{-2})\ldots) \]

This proves that
\[\int_{|x| \geq r} |\hat{f}(x)|^2w_k(x)dx = O(\psi(r^{-2}))\]

2) \implies 1) Suppose now that
\[\int_{|x| \geq r} |\hat{f}(x)|^2w_k(x)dx = O(\psi(r^{-2})) \text{ as } r \to \infty.\]

We have to show that
\[\int_0^\infty x^{2\gamma+d-1}|1 - j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx = O(\psi(h^2)) \text{ as } h \to 0,\]
where we have set
\[\varphi(x) = \int_{\mathbb{S}^{d-1}} |\hat{f}(xy)|^2w_k(y)dy\]

we write
\[\int_0^\infty x^{2\gamma+d-1}|1 - j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx = I_1 + I_2\]
where
\[I_1 = \int_0^{\frac{1}{h}} x^{2\gamma+d-1}|1 - j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx.\]

and
\[I_2 = \int_{\frac{1}{h}}^\infty x^{2\gamma+d-1}|1 - j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx.\]

From [2.4], we have
\[I_2 \leq 4 \int_{\frac{1}{h}}^\infty x^{2\gamma+d-1}\varphi(x)dx = O(\psi(h^2)) \text{ as } h \to 0\]

Set
\[g(s) = \int_s^\infty x^{2\gamma+d-1}\varphi(x)dx\]
From (2.6), with an integration by parts yields

\[ I_1 \leq -h^2 \int_0^1 s^2 g'(s) ds \]
\[ \leq -g\left(\frac{1}{h}\right) + 2h^2 \int_0^1 sg(s) ds \]
\[ \leq Ch^2 \int_0^1 s\psi(s^{-2}) ds \]
\[ \leq Ch^2 \frac{1}{h^2} \psi(h^2) \]
\[ \leq C\psi(h^2). \]

and this ends the proof. □

References