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Generalization of Titchmarsh's Theorem for the Dunkl Transform

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Abstract

Using a generalized spherical mean operator, we obtain the generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the Lipschitz condition in $L^2(\mathbb{R}^d, w_k)$, where w_k is a weight function invariant under the action of an associated reflection groups.

Keywords: Dunkl Operator, Dunkl Transform, Generalized Spherical Mean Operator.
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1. Introduction

Titchmarsh ([9], Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. *[9] Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:*

(1) $\|f(t+h) - f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$

(2) $\int_{|\lambda|\geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow \infty$

where $\mathcal{F}(f)$ stands for the Fourier transform of f.

In this paper, we prove the generalization of Theorem 1.1 in the Dunkl transform setting by means of the generalized spherical mean operator.

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2. Preliminaries

In order to confirm the basic and standard notation, we briefly overview the theory of Dunkl operators and related harmonic analysis. Main references are [1, 2, 3, 4, 7, 8, 9].

Let $\mathbb{R}^d$ be the Euclidean space equipped with a scalar product $\langle , \rangle$ and let $|x| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $\mathrm{H}_\alpha \subset \mathbb{R}^d$ orthogonal to α. A finite set $\mathrm{R} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $\mathrm{R} \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ and $\sigma\mathrm{R} = \mathrm{R}$ for all $\alpha \in \mathrm{R}$. For a given root system R, reflections σ_α, $\alpha \in \mathrm{R}$, generate a finite group $\mathrm{W} \subset O(d)$, called the reflection group associated with R. We fix a $\beta \in \mathbb{R}^d \backslash \cup_{\alpha\in\mathrm{R}} \mathrm{H}_\alpha$ and define a positive root system $\mathrm{R}_+ = \{\alpha \in \mathrm{R}/\langle\alpha, \beta\rangle > 0\}$.
A function $k : \mathrm{R} \longrightarrow \mathbb{C}$ on R is called a multiplicity function if it is invariant under the action of W. Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \mathrm{R}$.

We consider the weight function

$$w_k(x) = \prod_{\alpha\in\mathrm{R}_+} |\langle\alpha, x\rangle|^{2k(\alpha)},$$

where w_k is W-invariant and homogeneous of degree 2γ where

$$\gamma = \sum_{\alpha\in\mathrm{R}_+} k(\alpha).$$

We let η be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set

$$d\eta_k(y) = w_k(y)d\eta(y).$$

Then η_k is a W-invariant measure on $\mathbb{S}^{d-1}$, we let $d_k = \eta_k(\mathbb{S}^{d-1})$.

The Dunkl operators T_j, $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k are the first-order differential-difference operators given by

$$\mathrm{T}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in\mathrm{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle}, \quad f \in \mathrm{C}^1(\mathbb{R}^d).$$

where $\alpha_j = \langle\alpha, e_j\rangle$; $(e_1, ..., e_d)$ being the canonical basis of $\mathbb{R}^d$ and $\mathrm{C}^1(\mathbb{R}^d)$ is the space of functions of class C^1 on $\mathbb{R}^d$.

The Dunkl kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F. Dunkl in [2]. For $y \in \mathbb{R}^d$ the function $x \longmapsto \mathrm{E}_k(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases} \mathrm{T}_j u(x, y) = y_j u(x, y) & \text{for } \quad 1 \leq j \leq d \\ u(0, y) = 1 & \text{for all } \quad y \in \mathbb{R}^d \end{cases}$$

This kernel has unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

M. Röler has proved in [6] the following integral representation for the Dunkl kernel

$$\mathrm{E}_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \; x \in \mathbb{R}^d, \; z \in \mathbb{C}^d \tag{2.1}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $\mathrm{B}(0, |x|)$ of center 0 and raduis $|x|$.

Proposition 2.1. *[4] Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$. Then*

1. $\mathrm{E}_k(z, 0) = 1$
2. $\mathrm{E}_k(z, w) = \mathrm{E}_k(w, z)$
3. $\mathrm{E}_k(\lambda z, w) = \mathrm{E}_k(z, \lambda w)$
4. *For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}$, $x \in \mathbb{R}^d$, $z \in \mathbb{C}^d$, we have*

$$|\mathrm{D}_z^\nu \mathrm{E}_k(x, z)| \leq |x|^{|\nu|} exp(|x||Rez|)$$

where

$$\mathrm{D}_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} .....\partial z_d^{\nu_d}}; \quad |\nu| = \nu_1 + ... + \nu_d.$$

In particulier

$$|\mathrm{D}_z^\nu \mathrm{E}_k(ix, z)| \leq |x|^{|\nu|}$$

for all $x, z \in \mathbb{R}^d$

We denote by $\mathrm{L}_k^2(\mathbb{R}^d) = \mathrm{L}^2(\mathbb{R}^d, w_k(x)dx)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{k,2} = \left(\int_{\mathbb{R}^d} |f(x)|^2 w_k(x) dx \right)^{1/2}$$

The Dunkl transform is defined for $f \in \mathrm{L}_k^1(\mathbb{R}^d) = \mathrm{L}^1(\mathbb{R}^d, w_k(x)dx)$ by

$$\widehat{f}(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) \mathrm{E}_k(-i\xi, x) w_k(x) dx.$$

where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}} w_k(z) dz.$$

According to [3, 4] we have the following results:

1. When both f and $\widehat{f}$ are in $\mathrm{L}_k^1(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) \mathrm{E}_k(ix, \xi) w_k(\xi) d\xi, \quad x \in \mathbb{R}^d$$

2. (Plancherel's theorem) The Dunkl transform on $\mathrm{S}(\mathbb{R}^d)$, the space of Schwartz functions, extends uniquely to an isometric isomorphism on $\mathrm{L}_k^2(\mathbb{R}^d)$.

K. Trimèche has introduced [8] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$. For $f \in \mathrm{L}_k^2(\mathbb{R}^d)$ and we have

$$\widehat{\tau_x(f)}(\xi) = \mathrm{E}_k(ix, \xi) \widehat{f}(\xi)$$

and

$$\tau_x(f)(y) = c_k^{-1} \int_{\mathbb{R}^d} \widehat{f}(\xi) \mathrm{E}_k(ix, \xi) \mathrm{E}_k(iy, \xi) w_k(\xi) d\xi.$$

Applealing to Parseval theorem and Proposition 2.1 we see that

$$\|\tau_x f\|_{k,2} \leq \|f\|_{k,2} \quad \forall x \in \mathbb{R}^d.$$

The generalized spherical mean operator for $f \in \mathrm{L}_k^2(\mathbb{R}^d)$ is defined by

$$\mathrm{M}_h f(x) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \tau_x(hy) d\eta_k(y), \quad x \in \mathbb{R}^d, h > 0$$

From [5], we have $\mathrm{M}_h f \in \mathrm{L}_k^2(\mathbb{R}^d)$ whenever $f \in \mathrm{L}_k^2(\mathbb{R}^d)$ and

$$\|\mathrm{M}_h f\|_{k,2} \leq \|f\|_{k,2}$$

for all $h > 0$.

For $p \geq -\frac{1}{2}$, we introduce the normalized Bessel functuion j_p defined by

$$j_p(z) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+p+1)}, \; z \in \mathbb{C} \tag{2.2}$$

where Γ is the gamma-function.

From [3] we have

$$\frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \mathrm{E}_k(iy, x) d\eta_k(y) = j_{\gamma+\frac{d}{2}-1}(|x|) \tag{2.3}$$

for all $x \in \mathbb{R}^d$. This shows in particular that $x \longrightarrow j_{\gamma+\frac{d}{2}-1}(|x|)$ is a smooth bounded function with

$$|j_{\gamma+\frac{d}{2}-1}(|x|)| \leq 1 \tag{2.4}$$

From (2.2) we obtain

$$\lim_{z \longrightarrow 0} \frac{j_{\gamma+\frac{d}{2}-1}(z) - 1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ such that

$$|z| \leq \eta \Longrightarrow |j_{\gamma+\frac{d}{2}-1}(z) - 1| \geq c|z|^2 \tag{2.5}$$

The integral representation of the Dunkl kernel (2.1) and (2.3) yield

$$|j_{\gamma+\frac{d}{2}-1}(|x|)| \leq |x|. \tag{2.6}$$

Proposition 2.2. *Let $f \in \mathrm{L}_k^2(\mathbb{R}^d)$. Then*

$$\widehat{\mathrm{M}_h f}(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi)$$

Proof .(See [5]).□

3. Generalization of Titchmarsh's Theorem

In this section we give the main result of this paper. We need first to define the ψ-Dunkl Lipschitz class.

Definition 3.1. *A function $f \in \mathrm{L}^2_k(\mathbb{R}^d)$ is said to be in the ψ-Dunkl Lipschitz class, denoted by $Lip(\psi, 2, k)$; if:*

$$\|\mathrm{M}_h f(.) - f(.)\|_{k,2} = O(\psi(h))$$

as $h \longrightarrow 0$.
where $\psi(t)$ is a continuous increasing function on $[0, \infty)$, $\psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$ and this function verify $\int_0^{1/h} s\psi(s^{-2})ds = O(\frac{1}{h^2}\psi(h^2))$ as $h \longrightarrow 0$

Theorem 3.2. *Let $f \in \mathrm{L}^2_k(\mathbb{R}^d)$. Then the following are equivalents*

1. $f \in Lip(\psi, 2, k)$
2. $\int_{|x|\geq r} |\widehat{f}(x)|^2 w_k(x)dx = O(\psi(r^{-2}))$ *as* $r \longrightarrow \infty$

Proof . 1) $\Longrightarrow$ 2) Suppose that $f \in Lip(\psi, 2, k)$. Then we have

$$\|\mathrm{M}_h f - f\|_{k,2} = O(\psi(h)) \text{ as } h \longrightarrow 0.$$

Parseval Theorem and Proposition 2.2, we obtain

$$\|\mathrm{M}_h f - f\|^2_{k,2} = \int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|x|)|^2 |\widehat{f}(x)|^2 w_k(x)dx$$

Formula (2.5) gives

$$\int_{\frac{\eta}{2h}\leq|x|\leq\frac{\eta}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|x|)|^2 |\widehat{f}(x)|^2 w_k(x)dx \geq \frac{c^2\eta^4}{16} \int_{\frac{\eta}{2h}\leq|x|\leq\frac{\eta}{h}} |\widehat{f}(x)|^2 w_k(x)dx$$

There exists then a positive constant C such that

$$\begin{aligned} \int_{\frac{\eta}{2h}\leq|x|\leq\frac{\eta}{h}} |\widehat{f}(x)|^2 w_k(x)dx &\leq C \int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|x|)|^2 |\widehat{f}(x)|^2 w_k(x)dx \\ &\leq C\psi(h^2). \end{aligned}$$

For all $h > 0$, we obtain

$$\int_{r\leq|x|\leq 2r} |\widehat{f}(x)|^2 w_k(x)dx \leq C\psi(2^{-2}\eta r^{-2})$$

Thus there exists $K > 0$ such that

$$\int_{r\leq|x|\leq 2r} |\widehat{f}(x)|^2 w_k(x)dx \leq K\psi(r^{-2})$$

So that

$$\begin{aligned}\int_{|x|\geq r}|\widehat{f}(x)|^2w_k(x)dx &= \left[\int_{r\leq|x|\leq 2r}+\int_{2r\leq|x|\leq 4r}+\int_{4r\leq|x|\leq 8r}....\right]|\widehat{f}(x)|^2w_k(x)dx\\ &= O(\psi(r^{-2})+\psi(2^{-2}r^{-2})......)\\ &= O(\psi(r^{-2})+\psi(r^{-2})+.......)\\ &= O(\psi(r^{-2})).\end{aligned}$$

This proves that

$$\int_{|x|\geq r}|\widehat{f}(x)|^2w_k(x)dx = O(\psi(r^{-2}))$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{|x|\geq r}|\widehat{f}(x)|^2w_k(x)dx = O(\psi(r^{-2})) \text{ as } r\longrightarrow\infty.$$

We have to show that

$$\int_0^{\infty}x^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx = O(\psi(h^2)) \text{ as } h\longrightarrow 0,$$

where we have set

$$\varphi(x)=\int_{\mathbb{S}^{d-1}}|\widehat{f}(xy)|^2w_k(y)dy$$

we write

$$\int_0^{\infty}x^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx = \mathrm{I}_1+\mathrm{I}_2$$

where

$$\mathrm{I}_1=\int_0^{\frac{1}{h}}x^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx.$$

and

$$\mathrm{I}_2=\int_{\frac{1}{h}}^{\infty}x^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hx)|^2\varphi(x)dx.$$

From (2.4), we have

$$\mathrm{I}_2\leq 4\int_{\frac{1}{h}}^{\infty}x^{2\gamma+d-1}\varphi(x)dx = O(\psi(h^2)) \text{ as } h\longrightarrow 0$$

Set

$$g(s)=\int_s^{\infty}x^{2\gamma+d-1}\varphi(x)dx$$

From (2.6) , with an integration by parts yields

$$\begin{aligned}
\mathrm{I}_1 &\leq -h^2\int_0^{\frac{1}{h}} s^2 g'(s)ds \\
&\leq -g(\frac{1}{h}) + 2h^2\int_0^{\frac{1}{h}} sg(s)ds \\
&\leq Ch^2\int_0^{\frac{1}{h}} s\psi(s^{-2})ds \\
&\leq Ch^2\frac{1}{h^2}\psi(h^2) \\
&\leq C\psi(h^2).
\end{aligned}$$

and this ends the proof. □

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