A Unique Common Fixed Point Theorem for Six Maps in G-metric Spaces

K. P. R. Rao\textsuperscript{a,*}, K. B. Lakshmi\textsuperscript{a}, Z. Mustafa\textsuperscript{b}

\textsuperscript{a}Department of Applied Mathematics, Acharya Nagarjuna University-Dr. M. R. Appa Row Campus, Nuzvid-521 201, Andhra Pradesh, India.
\textsuperscript{b}Department of Mathematics, The Hashemite University, P.O. 330127, Zarqa 13115, Jordan.

(Communicated by M. B. Ghaemi)

Abstract

In this paper we obtain a unique common fixed point theorem for six weakly compatible mappings in G-metric spaces.

Keywords: G-metric, Common Fixed Points, Compatible Mappings.

2010 MSC: 47H10, 54H25.

1. Introduction

Dhage \cite{1, 2, 3, 4} et al. introduced the concept of $D$-metric spaces as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims \cite{12} and Naidu et al. \cite{8, 9, 10} demonstrated that most of the claims concerning the fundamental topological structure of $D$-metric space are incorrect, alternatively, Mustafa and Sims introduced in \cite{13} more appropriate notion of generalized metric space which called G-metric spaces, and obtained some topological properties. Later Zeiad Mustafa, Hamed Obiedat and Fadi Awawdeh \cite{13}, Mustafa, Shatanawi and Bataineh \cite{15}, Mustafa and Sims \cite{16}, Shatanawi \cite{11} and Renu Chugh, Tamanna Kadian, Anju Rani and B.E. Rhoades \cite{7} et al. obtained some fixed point theorems for a single map in G-metric spaces. In this paper, we obtain a unique common fixed point theorem for six weakly compatible mappings in G-metric spaces and obtain some theorems of \cite{11} as corollaries to our theorem. First, we present some known definitions and propositions in G-metric spaces.

*Corresponding author

Email addresses: kprrao2004@yahoo.com (K. P. R. Rao), zmagablh@hu.edu.jo (Z. Mustafa)

Received: Mar 2011    Revised: May 2012
Definition 1.1. Let $X$ be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G₁): $G(x, y, z) = 0$ if $x = y = z$,
(G₂): $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G₃): $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G₄): $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, symmetry in all three variables,

Then the function $G$ is called a generalized metric or a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

Definition 1.2. Let $(X, G)$ be a $G$-metric space and $\{x_n\}$ be a sequence in $X$. A point $x \in X$ is said to be limit of $\{x_n\}$ iff $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$. In this case, the sequence $\{x_n\}$ is said to be $G$-convergent to $x$.

Definition 1.3. Let $(X, G)$ be a $G$-metric space and $\{x_n\}$ be a sequence in $X$. $\{x_n\}$ is called $G$-Cauchy iff $\lim_{n, m, l \to \infty} G(x_l, x_n, x_m) = 0$. $(X, G)$ is called $G$-complete if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $(X, G)$.

Proposition 1.4. In a $G$-metric space, $(X, G)$, the following are equivalent.

1. The sequence $\{x_n\}$ is $G$-Cauchy.
2. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.

Proposition 1.5. Let $(X, G)$ be a $G$-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proposition 1.6. Let $(X, G)$ be a $G$-metric space. Then for any $x, y, z, a \in X$, it follows that

(i) if $G(x, y, z) = 0$ then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, z) \leq 2G(x, x, y)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{3}{2}G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Proposition 1.7. Let $(X, G)$ be a $G$-metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent

(i) $\{x_n\}$ is $G$-convergent to $x$,
(ii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x, x) \to 0$ as $m, n \to \infty$.

Definition 1.8. Let $(X, G)$ and $(X', G')$ be two $G$-metric spaces, and let $f : (X, G) \to (X', G')$ be a function, then $f$ is said to be $G$-continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function $f$ is $G$-continuous at $X$ if and only if it is $G$-continuous at all $a \in X$.

Proposition 1.9. Let $(X, G)$, and $(X', G')$ be two $G$-metric spaces. Then a function $f : X \to X'$ is $G$-continuous at a point $x \in X$ if and only if it is $G$-sequentially continuous at $x$; that is, whenever $(x_n)$ is $G$-convergent to $x$ we have $(f(x_n))$ is $G$-convergent to $f(x)$.

Definition 1.10. A pair of self mappings is called weakly compatible if they commute at their coincidence points.
2. Main Results

Following to Matkowski [3], let \( \Phi \) denote the set of all continuous nondecreasing functions \( \phi : [0, \infty) \to [0, \infty) \) such that \( \phi^n(t) \to 0 \) as \( n \to \infty \) for all \( t > 0 \). It is clear that \( \phi(t) < t \) for all \( t > 0 \) and \( \phi(0) = 0 \).

**Theorem 2.1.** Let \((X, G)\) be a \( G \)-metric space and \(S, T, R, f, g, h : X \to X\) be satisfying

(i) \(S(X) \subseteq g(X)\), \(T(X) \subseteq h(X)\) and \(R(X) \subseteq f(X)\),
(ii) each one of \(f(X), g(X)\) and \(h(X)\) is a complete subspace of \(X\),
(iii) the pairs \((S, f), (T, g)\) and \((R, h)\) are weakly compatible, and

\[
(iv) \quad G(Sx, Ty, Rz) \\
\leq \phi \left( \max \left\{ \frac{1}{3} G(fx, Sx, Ty) + G(gy, Ty, Rz) + G(hz, Rz, Sx), \right. \right. \\
\left. \left. \frac{1}{3} G(fx, Ty, hz) + G(Sx, gy, hz) + G(fx, gy, Rz) \right\} \right)
\]

for all \(x, y, z \in X\), where \( \phi \in \Phi \).

Then either one of the pairs \((S, f), (T, g)\) and \((R, h)\) has a coincidence point or the maps \(S, T, R, f, g\) and \( h \) have a unique common fixed point in \(X\).

**Proof.** Choose \(x_0 \in X\). By (i), there exist \(x_1, x_2, x_3 \in X\) such that \(Sx_0 = gx_1 = y_0, \) say , \(Tx_1 = hx_2 = y_1, \) say and \(Rx_2 = fx_3 = y_2, \) say.

Inductively, there exist sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(y_{3n} = Sx_{3n} = gx_{3n+1}, y_{3n+1} = Tx_{3n+1} = hx_{3n+2}, \) and \(y_{3n+2} = Rx_{3n+2} = fx_{3n+3}, \) where \(n = 0, 1, 2, \ldots\)

If \(y_{3n} = y_{3n+1}\) then \(x_{3n+1}\) is a coincidence point of \(g\) and \(T\).
If \(y_{3n+1} = y_{3n+2}\) then \(x_{3n+2}\) is a coincidence point of \(h\) and \(R\).
If \(y_{3n+2} = y_{3n+3}\) then \(x_{3n+3}\) is a coincidence point of \(f\) and \(S\).

Now assume that \(y_n \neq y_{n+1}\) for all \(n\).

Denote \(d_n = G(y_n, y_{n+1}, y_{n+2})\).

Putting \(x = x_{3n}, y = x_{3n+1}, z = x_{3n+2}\) in (iv), we get

\[
d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}) = G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}) \\
\leq \phi \left( \max \left\{ \frac{1}{3} G(fx_{3n}, Sx_{3n}, Tx_{3n+1}), \right. \right. \\
\left. \left. \frac{1}{3} G(Sx_{3n}, fy_{3n+1}, Ty_{3n+1}) + \right. \right. \right.

\[
\leq \phi \left( \max \left\{ \frac{1}{3} G(fy_{3n+1}, Ty_{3n+1}, hz_{3n+2}), \right. \right. \\
\left. \left. \frac{1}{3} G(Ty_{3n+1}, hx_{3n+2}, Sx_{3n}), \right. \right. \right.

\[
\leq \phi \left( \max \left\{ \frac{1}{3} [d_{3n-1} + d_{3n}], \right. \right. \\
\left. \left. \frac{1}{3} [d_{3n-1} + d_{3n} + d_{3n+1}], \right. \right. \right.

If \(d_{3n} \geq d_{3n-1}\) then from (1), we have \(d_{3n} \leq \phi(d_{3n}) < d_{3n}\). It is a contradiction. Hence \(d_{3n} \leq d_{3n-1}\).

Now from (1), \(d_{3n} \leq \phi(d_{3n-1})\).

Similarly, by putting \(x = x_{3n+3}, y = x_{3n+1}, z = x_{3n+2}\) and \(x = x_{3n+3}, \)

\[
y = x_{3n+4}, z = x_{3n+2}\) in (iv), we get

\[
d_{3n+1} \leq \phi(d_{3n}) \quad \text{(2)}
\]

and

\[
d_{3n+2} \leq \phi(d_{3n+1}) \quad \text{(3)}
\]
respectively. Thus from (1), (2) and (3), we have
\[
G(y_n, y_{n+1}, y_{n+2}) \leq \phi(G(y_{n-1}, y_n, y_{n+1}) \\
\leq \phi^3(G(y_{n-2}, y_{n-1}, y_n) \\
\leq \phi^n(G(y_0, y_1, y_2))
\]
From (G3) and (4), we have
\[
G(y_n, y_{n+1}) \leq G(y_{n+1}, y_{n+1}, y_{n+2}) \leq \phi^n(G(y_0, y_1, y_2))
\]
Now for \( m > n \), from (G5) and (4), we have
\[
G(y_n, y_m) \leq G(y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots + G(y_{m-1}, y_{m-1}, y_m) \\
\leq \phi^n(G(y_0, y_1, y_2)) + \phi^{n+1}(G(y_0, y_1, y_2)) + \ldots + \phi^{m-1}(G(y_0, y_1, y_2)) \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \quad \text{since} \quad \phi^n(t) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad \text{for all} \quad t > 0
\]
Hence \( \{y_n\} \) is G-Cauchy. Suppose \( f(X) \) is G-complete. Then there exist \( p, t \in X \) such that \( y_{3n+2} \rightarrow p = ft \). Since \( \{y_n\} \) is G-Cauchy, it follows that \( y_{3n} \rightarrow p \) and \( y_{3n+1} \rightarrow p \) as \( n \rightarrow \infty \).
\[
G(St, Tx_{3n+1}, Rx_{3n+2}) \leq \phi \left( \max \left\{ G(fp, fx_{3n+1}, hx_{3n+2}), \frac{1}{3}[G(ft, St, Tx_{3n+1}) + G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, St)] \right\} \right)
\]
Letting \( n \rightarrow \infty \), we get
\[
G(St, p, p) \leq \phi \left( \max \left\{ 0, \frac{1}{3}[G(p, St, p) + 0 + G(p, p, St)], \frac{1}{4}[0 + G(St, p, p) + 0] \right\} \right).
\]
\[
G(St, p, p) \leq \phi(G(St, p, p)) \quad \text{since} \quad \phi \text{ is nondecreasing.}
\]
Hence \( St = p \). Thus \( p = ft = St \).
Since the pair \( (S, f) \) is weakly compatible, we have \( fp = Sp \).
Putting \( x = p, y = x_{3n+1}, z = x_{3n+2} \) in (iv), we get
\[
G(Sp, Tx_{3n+1}, Rx_{3n+2}) \leq \phi \left( \max \left\{ G(fp, gx_{3n+1}, hx_{3n+2}), \frac{1}{3}[G(fp, Sp, Tx_{3n+1}) + G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, Sp)] \right\} \right)
\]
Letting \( n \rightarrow \infty \), we have
\[
G(Sp, p, p) \leq \phi \left( \max \left\{ G(Sp, p, p), \frac{1}{3}[G(Sp, Sp, p) + 0 + G(p, p, Sp)], \frac{1}{4}[G(Sp, p, p) + G(Sp, p, p) + G(Sp, p, p)] \right\} \right)
\]
Since \( G(Sp, Sp, p) \leq 2G(Sp, p, p) \), we have \( G(Sp, p, p) \leq \phi(G(Sp, p, p)) \)
Thus \( Sp = p \). Hence \( fp = Sp = p \).
Since \( p = Sp \in g(X) \), there exists \( v \in X \) such that \( p = gv \). Putting \( x = p, y = v, z = x_{3n+2} \) in (iv), we get

\[
G( Sp, Tv, R x_{3n+2} ) \\
\leq \phi \left( \max \left\{ \frac{1}{3} [G(fp, Sp, Tp) + G(gp, Tp, R x_{3n+2}) + G(hx_{3n+2}, R x_{3n+2}, Sp)], \frac{1}{4} [G(fp, Tv, hx_{3n+2}) + G(Sp, gv, hx_{3n+2})] \\
+ G(fp, gv, R x_{3n+2}) \right\} \right)
\]

Letting \( n \to \infty \), we deduce that

\[
G(p, Tv, p) \leq \phi \left( \max \left\{ \frac{1}{3} [G(p, p, Tv) + G(p, Tv, p) + 0], \frac{1}{4} [G(p, Tv, p) + 0 + 0] \right\} \right)
\]

\[
\leq \phi(G(p, Tv, p)) , \text{ since } \phi \text{ is nondecreasing.}
\]

Thus \( Tv = p \), so that \( p = Tv = gv \).

Since the pair \((T, g)\) is weakly compatible, we have \(Tp = gp\).

\[
G( Sp, Tp, R x_{3n+2} ) \\
\leq \phi \left( \max \left\{ \frac{1}{3} [G(fp, gp, hx_{3n+2}) + G(gp, Tp, R x_{3n+2}) + G(hx_{3n+2}, R x_{3n+2}, Sp)], \frac{1}{4} [G(fp, Tp, hx_{3n+2}) + G(Sp, gp, hx_{3n+2})] \\
+ G(fp, gp, R x_{3n+2}) \right\} \right)
\]

Letting \( n \to \infty \), we have

\[
G(p, Tp, p) \leq \phi \left( \max \left\{ \frac{1}{3} [G(p, p, Tp) + G(Tp, Tp, p) + 0], \frac{1}{4} [G(p, Tp, p) + G(p, Tp, p) + G(p, Tp, p)] \right\} \right) , \text{ since } G(Tp, Tp, p) \leq 2G(Tp, p, p), \text{we have, } G(p, Tp, p) \leq \phi(G(p, Tp, p)).
\]

Thus \( Tp = p \). Hence \( gp = Tp = p \).

Since \( p = Tp \in h(X) \), there exists \( w \in X \) such that \( p = hw \).

Putting \( x = p, y = p, z = w \) in (iv), we get

\[
G( Sp, Tp, Rw ) \\
\leq \phi \left( \max \left\{ \frac{1}{3} [G(fp, gp, hw) + G(gp, Tp, Rw) + G(hw, Rw, Sp)], \frac{1}{4} [G(fp, Tp, hw) + G(Sp, gp, hw)] \\
+ G(fp, gp, Rw) \right\} \right)
\]

\[
G(p, p, Rw) \leq \phi \left( \max \left\{ 0, \frac{1}{3} [0 + G(p, p, Rw) + G(p, Rw, p)], \frac{1}{4} [0 + 0 + G(p, p, Rw)] \right\} \right)
\]

\[
\leq \phi(G(p, p, Rw)) , \text{ since } \phi \text{ is nondecreasing.}
\]

Thus \( Rw = p \), so that \( p = hw = Rw \).

Since the pair \((R, h)\) is weakly compatible, we have \( Rp = hp \).

Putting \( x = p, y = p, z = p \) in (iv), we get

\[
G(p, p, Rp) = G( Sp, Tp, Rp ) \\
\leq \phi \left( \max \left\{ \frac{1}{3} [0 + G(p, p, Rp) + G(Rp, Rp, p)], \frac{1}{4} [0 + 0 + G(p, p, Rp)] \\
+ G(p, p, Rp) \right\} \right)
\]
Since $G(Rp, Rp, p) \leq 2G(p, p, Rp)$, we have $G(p, p, Rp) \leq \phi(G(p, p, Rp))$.
Thus $Rp = p$ so that $Rp = hp = p$.  

From (5), (6) and (7), it follows that $p$ is a common fixed point of $S, T, R, f, g$ and $h$.

Uniqueness of common fixed point follows easily from (iv). Similarly, we can prove the theorem when $g(X)$ or $h(X)$ is a complete subspace of $X$. □

Corollary 2.2. Let $(X, G)$ be a $G$-metric space and $S, T, R, f, g, h : X \to X$ be satisfying
(i) $S(X) \subseteq g(X), T(X) \subseteq h(X)$ and $R(X) \subseteq f(X)$,
(ii) one of $f(X), g(X)$ and $h(X)$ is a complete subspace of $X$,
(iii) the pairs $(S, f), (T, g)$ and $(R, h)$ are weakly compatible and
(iv) $G(Sx, Ty, Rz) \leq \phi(G(fx, gy, hz))$
for all $x, y, z \in X$, where $\phi \in \Phi$.
Then the maps $S, T, R, f, g$ and $h$ have a unique common fixed point in $X$.

Corollary 2.3. Let $(X, G)$ be a complete $G$-metric space and $S, T, R : X \to X$ be satisfying
$G(Sx, Ty, Rz) \leq \phi(G(x, y, z))$ for all $x, y, z \in X$, where $\phi \in \Phi$.
Then the maps $S, T$ and $R$ have a unique common fixed point, say, $p \in X$ and $S, T$ and $R$ are $G$-continuous at $p$.

Proof. There exists $p \in X$ such that $p$ is the unique common fixed point of $S, T$ and $R$ as in Theorem 2.1.
Let $\{y_n\}$ be any sequence in $X$ which $G$-converges to $p$.
Then
$G(Sy_n, Sp, Sp) = G(Sy_n, Tp, Rp) \leq \phi(G(y_n, p, p)) \to 0$ as $n \to \infty$.
Hence $S$ is $G$-continuous at $p$.
Similarly, we can show that $T$ and $R$ are also $G$-continuous at $p$. □

Remark 2.4. Theorem 3.1, Corollaries 3.2 to 3.5 of [11] follows from Corollary 2.3 with $S = T = R$.

3. Acknowledgement

The authors would like to thank the referee for his valuable suggestions on the manuscript.

References


