Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 153-164
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2016.456

Algebras defined by homomorphisms

Feysal Hassani

Department of Mathematics, Payame Noor University, Tehran, Iran

(Communicated by M. Eshaghi)

Abstract

Let $\mathcal{R}$ be a commutative ring with identity, let A and B be two $\mathcal{R}$-algebras and $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism. We introduce a new algebra $A \times_\varphi B$, and give some basic properties of this algebra. Generalized 2-cocycle derivations on $A \times_\varphi B$ are studied. Accordingly, $A \times_\varphi B$ is considered from the perspective of Banach algebras.

Keywords: algebra; cocycle; generalized derivation; Banach algebra.
2010 MSC: 16W25.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity and let A be an $\mathcal{R}$-bimodule (algebra). An algebra A is called *prime* algebra if for $a, b \in A$, $aAb = 0$ implies that either $a = 0$ or $b = 0$, and it is called *semiprime* if for $a \in A$, $aAa = 0$ implies that $a = 0$. An $\mathcal{R}$-linear map $\delta : A \longrightarrow A$ is said to be a *derivation* if $d(xy) = d(x)y + xd(y)$ for every $x, y \in A$.

An $\mathcal{R}$-bilinear map $\gamma : A \times A \longrightarrow A$ is called 2-*cocycle* if it satisfies the following equation

$$a\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b)c = 0,$$

for every $a, b, c \in A$.

Let $\mathcal{R}$ be a ring and A and B algebras over $\mathcal{R}$ ($\mathcal{R}$-algebras). Let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism, i.e.

$$\varphi(rb + b') = r\varphi(b + b'), \quad \text{and} \quad \varphi(bb') = \varphi(b)\varphi(b'),$$

for every $b, b' \in B$. Let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism, an $\mathcal{R}$-linear map $\tau : B \longrightarrow A$ is called φ-derivation if

$$\tau(bb') = \varphi(b)\tau(b') + \tau(b)\varphi(b'),$$

Email address: hassany@pnu.ac.ir (Feysal Hassani)

Received: September 2015 *Revised:* June 2016

for all $b, b' \in B$. If $\varphi = \mathrm{id}_B$ (identity map on B) and A is a B-bimodule, then id_B-derivation is a derivation that we defined already.

An additive mapping $D : A \longrightarrow A$ is called generalization derivation if there exists a derivation $d : A \longrightarrow A$ such that $D(xy) = D(x)y + xd(y)$ for all $x, y \in A$ and we say D is a d-derivation [1]. These mappings studied by many authors on various rings and algebras (see [2, 7]). A new type of generalized derivation introduced by Nakajima in [6], that he called it generalized 2-cocycle derivation. An additive mapping $\delta : A \longrightarrow A$ is called generalized 2-cocycle derivation associate with 2-cocycle $\gamma : A \times A \longrightarrow A$, if

$$\delta(xy) = x\delta(y) + \delta(x)y + \gamma(x, y), \tag{1.1}$$

for all $x, y \in A$. This version of generalized derivations considered in [3, 4, 5] for various versions of algebras such as von-Neumann and triangular algebras.

In this paper, at first, we introduce a new algebra that defined with homomorphism product $\times_\varphi$, where φ is an $\mathcal{R}$-additive homomorphism between algebras over $\mathcal{R}$ ($\mathcal{R}$-bimodules). After that, we study generalized 2-cocycle derivation on these algebras.

2. An algebra with homomorphism product

In this section we introduce an algebra as a generalization of Cartesian product of two algebras. Let $\mathcal{R}$ be a commutative ring with identity, A and B algebras over $\mathcal{R}$ ($\mathcal{R}$-algebras) and let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism. We define an algebra $A \times_\varphi B$ over $\mathcal{R}$ with the following operations

$$(a, b) + (a', b') = (a + a', b + b') \quad \text{and} \quad (a, b)(a', b') = (aa' + a\varphi(b') + \varphi(b)a', bb'), \tag{2.1}$$

for every $(a, b), (a', b') \in A \times_\varphi B$.

We summarize some properties of $A \times_\varphi B$ as follows:

Proposition 2.1. *Let $\mathcal{R}$ be a commutative ring with identity, A and B algebras over $\mathcal{R}$ and let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism. Then*

(i) $A \times_\varphi B$ is commutative if and only if A and B are commutative.

(ii) if $A \times_\varphi B$ is prime, then A and B are prime. If A and B are prime and $a + \varphi(b) \neq 0$ for all $(a, b) \in A \times_\varphi B$, then $A \times_\varphi B$ is prime.

(iii) $A \times_\varphi B$ is semiprime if and only if then A and B are semiprime.

(iv) If $A \times_\varphi B$ is unital then A and B are unital. If A and B are unital with units e_A and e_B such that $\varphi(e_B) = e_A$, then $A \times_\varphi B$ is unital with unit $(0, e_B)$.

(v) $A \times_\varphi B$ is projective if and only if then A and B are projective.

(vi) $A \times_\varphi B$ is injective if and only if then A and B are injective.

Proof . (i) Let $A \times_\varphi B$ be commutative. For every $a_1, a_2 \in A$ we have $(a_1, 0), (a_2, 0) \in A \times_\varphi B$. Then

$$(a_1a_2, 0) = (a_1, 0)(a_2, 0) = (a_2, 0), (a_1, 0) = (a_2a_1, 0).$$

This implies that A is commutative. Similarly, B is commutative.

Conversely, let A and B be commutative. Then for every $(a_1, b_1), (a_2, b_2) \in A \times_\varphi B$ we have

$$\begin{aligned} (a_1, b_1)(a_2, b_2) &= (a_1a_2 + a_1\varphi(b_2) + \varphi(b_1)a_2, b_1b_2) \\ &= (a_2a_1 + \varphi(b_2)a_1 + a_2\varphi(b_1), b_2b_1) \\ &= (a_2, b_2)(a_1, b_1). \end{aligned}$$

(ii) Let $A \times_\varphi B$ be prime. For every $a_1, a_2 \in A$ we have $(a_1, 0), (a_2, 0) \in A \times_\varphi B$. If $a_1Aa_2 = 0$, then for every $a' \in A$ we have $a_1a'a_2 = 0$. Thus

$$(a_1a'a_2, 0) = (a_1, 0)(a', 0)(a_2, 0) = (0, 0).$$

This implies that $(a_1, 0) = (0, 0)$ or $(a_2, 0) = (0, 0)$. Hence, A is prime. Similarly we can show that B is prime.

Now, let A and B be prime. We shall show that if

$$(a_1, b_1)A \times_\varphi B(a_2, b_2) = (0, 0),$$

then $(a_1, b_1) = (0, 0)$ or $(a_2, b_2) = (0, 0)$. If $(a_1, b_1)A \times_\varphi B(a_2, b_2) = (0, 0)$, for every $(a', b') \in A \times_\varphi B$, we have $(a_1, b_1)(a', b')(a_2, b_2) = (0, 0)$. Then

$$\begin{aligned} (0, 0) &= (a_1, b_1)(a', b')(a_2, b_2) \\ &= (a_1a'a_2 + \varphi(b_1)a'a_2 + a_1\varphi(b')a_2 + \varphi(b_1b')a_2 + a_1a'\varphi(b_2) \\ &\quad \varphi(b_1)a'\varphi(b_2) + a_1\varphi(b')\varphi(b_2), b_1b'b_2). \end{aligned}$$

Therefore $b_1b'b_2 = 0$. Since B is prime, $b_1 = 0$ or $b_2 = 0$. If $b_1 = 0$, then

$$\begin{aligned} 0 &= a_1a'a_2 + a_1\varphi(b')a_2 + a_1a'\varphi(b_2) + a_1\varphi(b')\varphi(b_2) \\ &= a_1(a' + \varphi(b'))(a_2 + \varphi(b_2)). \end{aligned}$$

This implies that $a_1 = 0$ or $a_2 + \varphi(b_2) = 0$. But $a_2 + \varphi(b_2) = 0$ can not happen for every $(a_2, b_2) \in A \times_\varphi B$. This means that $a_1 = 0$. Hence, $(a_1, b_1) = (0, 0)$. If $b_2 = 0$, then

$$\begin{aligned} 0 &= a_1a'a_2 + \varphi(b_1)a'a_2 + a_1\varphi(b')a_2 + \varphi(b_1b')a_2 \\ &= (a_1 + \varphi(b_1))(a' + \varphi(b'))a_2. \end{aligned}$$

Consequently, $a_1 + \varphi(b_1) = 0$ or $a_2 = 0$. Since $a_1 + \varphi(b_1) = 0$ can not hold for every $(a_2, b_2) \in A \times_\varphi B$, $a_2 = 0$. Therefore $(a_2, b_2) = (0, 0)$. Thus, $A \times_\varphi B$ is prime.

(iii) Let $A \times_\varphi B$ be semiprime. For every $a \in A$ we have $(a, 0) \in A \times_\varphi B$. If $aAa = 0$, then for every $a' \in A$ we have $aa'a = 0$. Thus

$$(aa'a, 0) = (a, 0)(a', 0)(a, 0) = (0, 0).$$

This implies that $(a, 0) = (0, 0)$. Hence, A is semiprime. Similarly we can show that B is semiprime.

Now, let A and B be semiprime. We shall show that if $(a, b)(A \times_\varphi B)(a, b) = (0, 0)$, then $(a, b) = (0, 0)$. If $(a, b)(A \times_\varphi B)(a, b) = (0, 0)$, thus, for every $(a', b') \in A \times_\varphi B$, we have $(a, b)(a', b')(a, b) = (0, 0)$. Then

$$\begin{aligned} (0, 0) &= (a, b)(a', b')(a, b) \\ &= (aa'a + \varphi(b)a'a + a\varphi(b')a + \varphi(bb')a + aa'\varphi(b) + \varphi(b)a'\varphi(b) + a\varphi(b')\varphi(b), bb'b). \end{aligned}$$

Therefore $bb'b = 0$. Since B is semiprime, $b = 0$. This follows that $\varphi(b) = 0$, and consequently, $aa'a = 0$. Since A is semiprime, $a = 0$. Hence, $(a, b) = (0, 0)$ and $A \times_\varphi B$ is semiprime.

(iv) Assume $A \times_\varphi B$ is unital. Let (α, β) be the unit of $A \times_\varphi B$. Then for every $(a, b) \in A \times_\varphi B$,

$$\begin{aligned}(a,b) &= (a,b)(\alpha,\beta) = (a\alpha + a\varphi(\beta) + \varphi(b)\alpha, \beta b)\\ &= (a(\alpha+\varphi(\beta)) + \varphi(b)\alpha, \beta b)\\ &= (\alpha,\beta)(a,b) = ((\alpha+\varphi(\beta))a + \alpha\varphi(b), b\beta).\end{aligned}$$

Then by taking $a = 0$, we conclude that $\alpha\varphi(b) = 0$ for every $b \in B$. This follows that β is the unit of B and $\alpha + \varphi(\beta)$ is the unit of A.

If A and B are unital with units e_A and e_B such that $\varphi(e_B) = e_A$, then clearly $(0, e_B)$ is the unit of $A \times_\varphi B$.

The cases (v) and (vi) are easy. □

In the above results we have stated some properties of $A \times_\varphi B$, A and B that they have similar properties. An $\mathcal{R}$-algebra A is called *factorable*, if for every $a \in A$ there are $b, c \in A$ such that $a = bc$. Now; let A and B are factorable $\mathcal{R}$-algebras, then $A \times_\varphi B$ is not factorable.

Assume that $\alpha : A \longrightarrow \mathcal{R}$ is an $\mathcal{R}$-linear map. We denote the set of these maps by $Hom_{\mathcal{R}}(A, \mathcal{R})$ (in some litterateurs say that duality of A and denote by A^*).

Theorem 2.2. *Let $\mathcal{R}$ be a commutative ring with identity, A and B algebras over $\mathcal{R}$ and let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism. Then*

$$Hom_{\mathcal{R}}(A \times_\varphi B, \mathcal{R}) \cong Hom_{\mathcal{R}}(A, \mathcal{R}) \times Hom_{\mathcal{R}}(B, \mathcal{R}).$$

Proof . Define $T : Hom_{\mathcal{R}}(A \times_\varphi B, \mathcal{R}) \longrightarrow Hom_{\mathcal{R}}(A, \mathcal{R}) \times Hom_{\mathcal{R}}(B, \mathcal{R})$ by $T(\alpha)(a, b) = \alpha_A(a) + \alpha_B(b)$ for all $a \in A$, $b \in B$, where $\alpha|_A = \alpha_A$ and $\alpha|_B = \alpha_B$. It is easy to see that T is an $\mathcal{R}$-linear and injective map. For every $\alpha_A \in Hom_{\mathcal{R}}(A, \mathcal{R})$ and $\alpha_B \in Hom_{\mathcal{R}}(B, \mathcal{R})$, $(\alpha_A, \alpha_B) \in Hom_{\mathcal{R}}(A \times_\varphi B, \mathcal{R})$. This shows that T is surjective and proof is complete. □

Let $\mathcal{R}$ be a commutative ring with identity and let A be an $\mathcal{R}$-algebra. Let $\theta \in Hom_{\mathcal{R}}(A, \mathcal{R})$ such that $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$. We denote a subset of $Hom_{\mathcal{R}}(A, \mathcal{R})$ that consists all elements such as θ by $\mathbb{HOM}_{\mathcal{R}}(A, \mathcal{R})$.

Theorem 2.3. *Let $\mathcal{R}$ be a commutative ring with identity, A and B algebras over $\mathcal{R}$ and let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism. Then*

$$\mathbb{HOM}_{\mathcal{R}}(A \times_\varphi B, \mathcal{R}) = \{(\theta, \theta \circ \varphi) : \theta \in \mathbb{HOM}_{\mathcal{R}}(A, \mathcal{R})\} \cup \{(0, \psi) : \psi \in \mathbb{HOM}_{\mathcal{R}}(B, \mathcal{R})\}.$$

Proof . Set $\mathbb{K} = \{(\theta, \theta \circ \varphi) : \theta \in \mathbb{HOM}_{\mathcal{R}}(A, \mathcal{R})\} \cup \{(0, \psi) : \psi \in \mathbb{HOM}_{\mathcal{R}}(B, \mathcal{R})\}$. Clearly $(\theta, \theta \circ \varphi)$ and $(0, \psi)$ are in $Hom_{\mathcal{R}}(A \times_\varphi B, \mathcal{R})$. For every $(a, b), (a', b') \in A \times_\varphi B$ we have

$$\begin{aligned}(\theta,\theta\circ\varphi)((a,b)(a',b')) &= (\theta,\theta\circ\varphi)(aa' + a\varphi(b') + \varphi(b)a', bb')\\ &= \theta(a)\theta(a') + \theta(a)\theta\circ\varphi(b') + \theta\circ\varphi(b)\theta(a') + \theta\circ\varphi(b)\theta\circ\varphi(b')\\ &= (\theta(a) + \theta\circ\varphi(b))(\theta(a') + \theta\circ\varphi(b'))\\ &= (\theta,\theta\circ\varphi)(a,b)(\theta,\theta\circ\varphi)(a',b').\end{aligned} \tag{2.2}$$

Similarly for every $(a, b), (a', b') \in A \times_\varphi B$ we have

$$(0,\psi)((a,b)(a',b')) = \psi(b)\psi(b') = (0,\psi)(a,b)(0,\psi)(a',b'). \tag{2.3}$$

Thus, by (2.2) and (2.3), we have

$$\mathbb{K} \subseteq \mathbb{HOM}_{\mathcal{R}}(A \times_{\varphi} B, \mathcal{R}). \tag{2.4}$$

Let $\alpha \in \mathbb{HOM}_{\mathcal{R}}(A \times_{\varphi} B, \mathcal{R})$. By Theorem 2.2, $\mathbb{HOM}_{\mathcal{R}}(A \times_{\varphi} B, \mathcal{R}) \subseteq Hom_{\mathcal{R}}(A \times_{\varphi} B, \mathcal{R})$, so there are $\theta \in Hom_{\mathcal{R}}(A, \mathcal{R})$ and $\psi \in Hom_{\mathcal{R}}(B, \mathcal{R})$ such that $\alpha = (\theta, \psi)$. Then for every $(a, b), (a', b') \in A \times_{\varphi} B$

$$\begin{aligned} \alpha((a,b)(a',b')) &= \alpha(a,b)\alpha(a',b') = (\theta,\psi)(a,b)(\theta,\psi)(a',b') \\ &= (\theta(a) + \psi(b))(\theta(a') + \psi(b')). \end{aligned} \tag{2.5}$$

Also, for every $(a, b), (a', b') \in A \times_{\varphi} B$,

$$\begin{aligned} \alpha((a,b)(a',b')) &= (\theta,\psi)(aa' + a\varphi(b') + \varphi(b)a', bb') \\ &= \theta(aa') + \theta(a\varphi(b')) + \theta(\varphi(b)a') + \psi(bb'). \end{aligned} \tag{2.6}$$

By setting $b = b' = 0$ and above relations we have $\theta(aa') = \theta(a)\theta(a')$ for all $a, a' \in A$. This shows that $\theta \in \mathbb{HOM}_{\mathcal{R}}(A, \mathcal{R})$. Similarly, if we take $a = a' = 0$, we conclude that $\psi(bb') = \psi(b)\psi(b')$ for all $b, b' \in B$. Now, we shall show that if $\theta \neq 0$ then ψ of form $\theta \circ \varphi$. Assume that $\theta \neq 0$. Then by taking $a = a'$ and $b = b'$ in (2.5) and (2.6) we have

$$\theta(a)\psi(b) + \psi(b)\theta(a) = \theta(a)\theta \circ \varphi(b) + \theta \circ \varphi(b)\theta(a).$$

Since $\mathcal{R}$ is commutative, $\psi(b) = \theta \circ \varphi(b)$ for all $b \in B$. This means that $\psi = \theta \circ \varphi$. Thus

$$\mathbb{HOM}_{\mathcal{R}}(A \times_{\varphi} B, \mathcal{R}) \subseteq \mathbb{K}. \tag{2.7}$$

□

3. Generalized 2-cocycle derivations

In the whole of this section $\mathcal{R}$ is a commutative ring, A and B are unital $\mathcal{R}$-algebras with units e_A and e_B, respectively, and $\varphi : B \longrightarrow A$ is $\mathcal{R}$-linear algebras homomorphism such that $\varphi(e_B) = e_A$. In this section we study generalized 2-cocycle derivations on $A \times_{\varphi} B$.

Let $\gamma : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow A \times_{\varphi} B$ be a $\mathcal{R}$-bilinear map. Let $\gamma_1 : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow A$ and $\gamma_2 : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow B$ be the coordinate mapping associated to γ that is

$$\gamma((a_1, b_1), (a_2, b_2)) = (\gamma_1((a_1, b_1), (a_2, b_2)), \gamma_2((a_1, b_1), (a_2, b_2))),$$

for all $(a_1, b_1), (a_2, b_2) \in A \times_{\varphi} B$. Let $\gamma : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow A \times_{\varphi} B$ be a 2-cocycle, the coordinate mapping $\gamma_1 : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow A$ is said to correspond to a 2-cocycle on A if there exists a 2-cocycle $\gamma_A : A \times A \longrightarrow A$ such that $\gamma_1((a_1, b_1), (a_2, b_2)) = \gamma_A(a_1, a_2)$, for all $(a_1, b_1), (a_2, b_2) \in A \times_{\varphi} B$. Similarly, $\gamma_2 : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow B$ is said to correspond to a 2-cocycle on B if there exists a 2-cocycle $\gamma_B : B \times B \longrightarrow B$ such that $\gamma_2((a_1, b_1), (a_2, b_2)) = \gamma_B(b_1, b_2)$, for all $(a_1, b_1), (a_2, b_2) \in A \times_{\varphi} B$. Also, we assume that $\gamma_2((a_1, 0), (a_2, 0)) = 0$ and $\gamma_1((0, b_1), (0, b_2)) = 0$, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Lemma 3.1. *If $\gamma : (A \times_{\varphi} B) \times (A \times_{\varphi} B) \longrightarrow A \times_{\varphi} B$ is a 2-cocycle, then there are corresponding 2-cocycles $\gamma_A : A \times A \longrightarrow A$ and $\gamma_B : B \times B \longrightarrow B$.*

Proof . Define $\gamma_A : A \times A \longrightarrow A$ as follows

$$\gamma_A(a_1, a_2) = (e_A, 0)\ \gamma((a_1, 0), (a_2, 0))\ (e_A, 0), \tag{3.1}$$

for all $a_1, a_2 \in A$. Now, let $a_1, a_2, a_3 \in A$. Then

$$\begin{aligned} a_1\gamma_A(a_2, a_3) - \gamma_A(a_1a_2, a_3) + \gamma_A(a_1, a_2a_3) - \gamma_A(a_1, a_2)a_3 \\ = \ & (a_1, 0)\gamma((a_2, 0), (a_3, 0)) - \gamma((a_1a_2, 0), (a_3, 0)) + \gamma((a_1, 0), (a_2a_3, 0)) \\ & -\gamma((a_1, 0), (a_2, 0))(a_3, 0) \\ = \ & 0. \end{aligned}$$

This shows that γ_A is a 2-cocycle on A. Similarly, consider $\gamma_B : B \times B \longrightarrow B$ as follows

$$\gamma_B(b_1, b_2) = (0, e_B)\ \gamma((0, b_1), (0, b_2))\ (0, e_B), \tag{3.2}$$

for all $b_1, b_2 \in B$. □

Theorem 3.2. *Let $\delta : A \times_\varphi B \longrightarrow A \times_\varphi B$ be a generalized 2-cocycle derivation associate with $\gamma : (A \times_\varphi B) \times (A \times_\varphi B) \longrightarrow A \times_\varphi B$. Then there are corresponding 2-cocycles $\gamma_A : A \times A \longrightarrow A$, $\gamma_B : B \times B \longrightarrow B$, generalized 2-cocycle derivations $\delta_A : A \longrightarrow A$, $\delta_B : B \longrightarrow B$ associate with γ_A and γ_B, respectively, and a φ-derivation $\tau : B \longrightarrow A$.*

Proof . According to Lemma 3.1, there are corresponding 2-cocycles γ_A and γ_B. We prove the rest of the proof in some steps as follows:

Step 1. Let $\delta((0, e_B)) = (m, n)$. Then

$$\begin{aligned} (m, n) = \delta((0, e_B)(0, e_B)) &= \delta((0, e_B)) \\ &= \delta((0, e_B))\ (0, e_B) + (0, e_B)\ \delta((0, e_B)) + \alpha((0, e_B), (0, e_B)) \\ &= (m, n)\ (0, e_B) + (0, e_B)\ (m, n) + (0, \gamma_B((e_B, e_B))) \\ &= (m, n) + (m, n) + (0, \gamma_B((e_B, e_B))) \\ &= (2m, 2n + \gamma_B((e_B, e_B))). \end{aligned} \tag{3.3}$$

Thus, $m = 0$ and $n = -\gamma_B((e_B, e_B)))$.

Step 2. Assume $b \in B$ and $\delta((0, b)) = (m, n)$. Then by Step 1, we have

$$\begin{aligned} (m, n) = \delta((0, b)) &= \delta((0, e_B)(0, b)) \\ &= \delta((0, e_B))\ (0, b) + (0, e_B)\ \delta((0, b)) + \alpha((0, e_B), (0, b)) \\ &= (0, -\gamma_B((e_B, e_B)))\ (0, b) + (0, e_B)\ (m, n) + (0, \gamma_B((e_B, b))) \\ &= (0, -\gamma_B((e_B, e_B))b) + (m, n) + (0, \gamma_B((e_B, b))). \end{aligned} \tag{3.4}$$

This implies that $\gamma_B((e_B, b)) = \gamma_B((e_B, e_B))b$. On the other hand,

$$\begin{aligned} (m, n) = \delta((0, b)) &= \delta((0, b)(0, e_B)) \\ &= \delta((0, b))\ (0, e_B) + (0, b)\ \delta((0, e_B)) + \alpha((0, b), (0, e_B)) \\ &= (m, n)\ (0, e_B) + (0, b)\ (0, -\gamma_B((e_B, e_B))) + (0, \gamma_B((b, e_B))) \\ &= (0, -b\gamma_B((e_B, e_B))) + (m, n) + (0, \gamma_B((b, e_B))). \end{aligned} \tag{3.5}$$

Therefore $b\gamma_B((e_B, e_B)) = \gamma_B((b, e_B))$. Set $\delta((0, b)) = (\tau(b), \delta_B(b))$.

Step 3. Suppose that $\delta((e_A, 0)) = (m, n)$. Then

$$\begin{aligned}(m,n) &= \delta((e_A,0)) = \delta((e_A,0)(e_A,0)) \\ &= \delta((e_A,0))\ (e_A,0) + (e_A,0)\ \delta((e_A,0)) + \alpha((e_A,0),(e_A,0)) \\ &= (m,n)\ (e_A,0) + (e_A,0)\ (m,n) + (\gamma_A((e_A,e_A)),0) \\ &= (m+\varphi(n),0) + (m+\varphi(n),0) + (\gamma_A((e_A,e_A)),0) \\ &= (2m + 2\varphi(n) + \gamma_A((e_A,e_A)),0).\end{aligned} \tag{3.6}$$

Above relations means that $n = 0$ and consequently, $m = -\gamma_A((e_A, e_A))$.

Step 4. Let $\delta((a, 0)) = (m, n)$ for $a \in A$. Then Step 3, implies

$$\begin{aligned}(m,n) &= \delta((a,0)) = \delta((a,0)(e_A,0)) \\ &= \delta((a,0))\ (e_A,0) + (a,0)\ \delta((e_A,0)) + \alpha((a,0),(e_A,0)) \\ &= (m,n)\ (e_A,0) + (a,0)\ (-\gamma_A((e_A,e_A)),0) + (\gamma_A((a,e_A)),0) \\ &= (m+\varphi(n),0) + (-a\gamma_A((e_A,e_A)),0) + (\gamma_A((a,e_A)),0) \\ &= (m+\varphi(n) - a\gamma_A((e_A,e_A)) + \gamma_A((a,e_A)),0).\end{aligned} \tag{3.7}$$

Hence, $n = 0$ and $a\gamma_A((e_A, e_A)) = \gamma_A((a, e_A))$. On the other hand, by 3.7 we have

$$\begin{aligned}(m,n) &= \delta((a,0)) = \delta((e_A,0)(a,0)) \\ &= \delta((e_A,0))\ (a,0) + (e_A,0)\ \delta((a,0)) + \alpha((e_A,0),(a,0)) \\ &= (-\gamma_A((e_A,e_A)),0)\ (a,0) + (e_A,0)\ (m,0) + (\gamma_A((e_A,a)),0) \\ &= (-\gamma_A((e_A,e_A))a,0) + (m,0) + (\gamma_A((e_A,a)),0) \\ &= (-\gamma_A((e_A,e_A))a + m + \gamma_A((e_A,a)),0).\end{aligned} \tag{3.8}$$

Then $\gamma_A((e_A, a)) = \gamma_A((e_A, e_A))a$.

Step 5. By Steps 1 and 3 we have

$$\begin{aligned}&(-\gamma_A((e_A,e_A)),0) = \delta((e_A,0)) = \delta((e_A,0)(0,e_B)) \\ &= \delta((e_A,0))\ (0,e_B) + (e_A,0)\ \delta((0,e_B)) + \alpha((e_A,0),(0,e_B)) \\ &= (-\gamma_A((e_A,e_A)),0)\ (0,e_B) + (e_A,0)\ (0,-\gamma_B((e_B,e_B))) \\ &\quad + (\gamma_1((e_A,0),(0,e_B)),\gamma_2((e_A,0),(0,e_B))) \\ &= (-\gamma_A((e_A,e_A)) - \varphi(\gamma_B((e_B,e_B))),0) + (\gamma_1((e_A,0),(0,e_B)),\gamma_2((e_A,0),(0,e_B))).\end{aligned} \tag{3.9}$$

Above relation follows that $\varphi(\gamma_B((e_B, e_B))) = \gamma_1((e_A, 0), (0, e_B))$ and $\gamma_2((e_A, 0), (0, e_B)) = 0$. On the other hand

$$\begin{aligned}&(-\gamma_A((e_A,e_A)),0) = \delta((e_A,0)) = \delta((0,e_B)(e_A,0)) \\ &= (-\gamma_A((e_A,e_A)) - \varphi(\gamma_B((e_B,e_B))),0) + (\gamma_1((0,e_B),(e_A,0)),\gamma_2((0,e_B),(e_A,0))).\end{aligned} \tag{3.10}$$

Thus, $\gamma_2((0, e_B), (e_A, 0)) = 0$ and $\gamma_1((0, e_B), (e_A, 0) = \varphi(\gamma_B((e_B, e_B)))$. This means that

$$\gamma_1((e_A,0),(0,e_B)) = \gamma_1((0,e_B),(e_A,0))$$

.

Step 6. From Step 4 we have $\delta((a,0)) = (m,0)$ for every $a \in A$. Now, replace m with $\delta_A(a)$. Then

$$\begin{aligned}(\delta_A(a),0) &= \delta((a,0)) = \delta((a,0)(0,e_B))\\ &= \delta((a,0))\ (0,e_B) + (a,0)\ \delta((0,e_B)) + \alpha((a,0),(0,e_B))\\ &= (\delta_A(a),0)\ (0,e_B) + (a,0)\ (0,-\gamma_B((e_B,e_B)))\\ &\quad + (\gamma_1((a,0)),(0,e_B)),\gamma_2((a,0),(0,e_B)))\\ &= (\delta_A(a) - a\varphi(\gamma_B((e_B,e_B))),0) + (\gamma_1((a,0)),(0,e_B)),\gamma_2((a,0),(0,e_B))).\end{aligned} \tag{3.11}$$

This means that $\gamma_1((a,0)),(0,e_B)) = a\varphi(\gamma_B((e_B,e_B)))$ and $\gamma_2((a,0),(0,e_B)) = 0$. On the other hand,

$$\begin{aligned}&(\delta_A(a),0) = \delta((a,0)) = \delta((0,e_B)(a,0))\\ &= \delta((0,e_B))\ (a,0) + (0,e_B)\ \delta((a,0)) + \alpha((0,e_B),(a,0))\\ &= (0,-\gamma_B((e_B,e_B)))\ (a,0) + (0,e_B)\ (\delta_A(a),0) + (\gamma_1((0,e_B),(a,0))),\gamma_2((0,e_B),(a,0)))\\ &= (\delta_A(a) - \varphi(\gamma_B((e_B,e_B)))a,0) + (\gamma_1((0,e_B),(a,0)),\gamma_2((0,e_B),(a,0))).\end{aligned} \tag{3.12}$$

Then, $\gamma_1((0,e_B),(a,0)) = \varphi(\gamma_B((e_B,e_B)))a$ and $\gamma_2((0,e_B),(a,0)) = 0$.

Step 7. By Steps 2 and 3 and taking $\delta((0,b)) = \delta((0,b)) = (\tau(b),\delta_B(b))$, we have

$$\begin{aligned}&(\delta_A(\varphi(b)),0) = \delta((\varphi(b),0)) = \delta((e_A,0)(0,b))\\ &= \delta((e_A,0))\ (0,b) + (e_A,0)\ \delta((0,b)) + \alpha((e_A,0),(0,b))\\ &= (-\gamma_A((e_A,e_A)),0)\ (0,b) + (e_A,0)\ (\tau(b),\delta_B(b)) + (\gamma_1((e_A,0),(0,b)),\gamma_2((e_A,0),(0,b)))\\ &= (-\gamma_A((e_A,e_A))\varphi(b) + \tau(b) + \varphi(\delta_B(b)),0) + (\gamma_1((e_A,0),(0,b)),\gamma_2((e_A,0),(0,b))).\end{aligned} \tag{3.13}$$

Hence, (3.13), implies that $\gamma_2((e_A,0),(0,b)) = 0$ and

$$\delta_A(\varphi(b)) = -\gamma_A((e_A,e_A))\varphi(b) + \tau(b) + \varphi(\delta_B(b)) + \gamma_1((e_A,0),(0,b)). \tag{3.14}$$

As well as,

$$\begin{aligned}&(\delta_A(\varphi(b)),0) = \delta((\varphi(b),0)) = \delta((0,b)(e_A,0))\\ &= \delta((0,b))\ (e_A,0) + (0,b)\ \delta((e_A,0)) + \alpha((0,b),(e_A,0))\\ &= (\tau(b),\delta_B(b))\ (e_A,0) + (0,b)(-\gamma_A((e_A,e_A)),0) + (\gamma_1((0,b),(e_A,0)),\gamma_2((0,b),(e_A,0)))\\ &= (\tau(b) + \varphi(\delta_B(b)) - \varphi(b)\gamma_A((e_A,e_A)),0) + (\gamma_1((0,b),(e_A,0)),\gamma_2((0,b),(e_A,0))).\end{aligned} \tag{3.15}$$

Therefore $\gamma_2((0,b),(e_A,0)) = 0$ and

$$\delta_A(\varphi(b)) = \tau(b) + \varphi(\delta_B(b)) - \varphi(b)\gamma_A((e_A,e_A)) + \gamma_1((0,b),(e_A,0)). \tag{3.16}$$

By comparing (3.15) and (3.16) we have

$$\gamma_1((e_A,0),(0,b)) - \gamma_A((e_A,e_A))\varphi(b) = \gamma_1((0,b),(e_A,0)) - \varphi(b)\gamma_A((e_A,e_A)). \tag{3.17}$$

Step 8. From Step 6, we have

$$\begin{aligned}&(\delta_A(a_1a_2),0) = \delta((a_1a_2,0)) = \delta((a_1,0)(a_2,0))\\ &= \delta((a_1,0))\ (a_2,0) + (a_1,0)\ \delta((a_2,0)) + \alpha((a_1,0),(a_2,0))\\ &= (\delta_A(a_1),0)\ (a_2,0) + (a_1,0)\ (\delta(a_2),0) + (\gamma_1((a_1,0),(a_2,0)),\gamma_2((a_1,0),(a_2,0)))\\ &= (\delta_A(a_1)a_2 + a_1\delta_A(a_2) + \gamma_1((a_1,0),(a_2,0)),\gamma_2((a_1,0),(a_2,0)))\\ &= (\delta_A(a_1)a_2 + a_1\delta_A(a_2) + \gamma_A((a_1,a_2)),\gamma_2((a_1,0),(a_2,0))),\end{aligned} \tag{3.18}$$

for every $a_1, a_2 \in A$. This shows that δ_A is a generalized 2-cocycle derivation associated with γ_A.

Step 9. By Step 2, we have

$$\begin{aligned}&(\tau(b_1b_2), \delta_B(b_1b_2)) = \delta((0, b_1b_2)) = \delta((0, b_1)(0, b_2))\\&= \delta((0, b_1))\ (0, b_2) + (0, b_1)\ \delta((0, b_2)) + \alpha((0, b_1), (0, b_2))\\&= (\tau(b_1), \delta_B(b_1))\ (0, b_2) + (0, b_1)\ (\tau(b_2), \delta_B(b_2)) + (\gamma_1((0, b_1), (0, b_2)), \gamma_2((0, b_1), (0, b_2)))\\&= (\tau(b_1)\varphi(b_2) + \varphi(b_1)\tau(b_2), \delta_B(b_1)b_2 + b_1\delta_B(b_2) + \gamma_B((b_1, b_2))),\end{aligned} \tag{3.19}$$

for every $b_1, b_2 \in B$. Thus, τ is an φ-derivation from A into A and δ_B is a generalized 2-cocycle derivation associate with γ_B. □

4. Banach algebra point of view

Let A and $\mathcal{R}$ be Banach algebras such that A is a Banach $\mathcal{R}$-algebra with compatible actions

$$\alpha \cdot (ab) = (\alpha \cdot a)b,\ \ (ab) \cdot \alpha = a(b \cdot \alpha)$$

for all $a, b \in A, \alpha \in \mathcal{R}$.

Let $\mathcal{R}$ be a commutative Banach algebra with identity, let A and B be Banach algebras that are Banach $\mathcal{R}$-bimodule with compatible actions and let $\varphi : B \longrightarrow A$ be an $\mathcal{R}$-additive algebra homomorphism with $\|\varphi\| \leq 1$. Clearly, φ is not linear homomorphism. Then $A \times_\varphi B$ is a Banach algebra and a Banach $\mathcal{R}$-bimodule with the following norm:

$$\|(a, b)\| = \|a\|_A + \|b\|_B, \qquad\qquad (a \in A,\ b \in B).$$

According to Theorem 2.3, we have

$$\mathbb{HOM}_{\mathcal{R}}(A \times_\varphi B, \mathcal{R}) \cong \{(\theta, \theta \circ \varphi) : \theta \in \mathbb{HOM}_{\mathcal{R}}(A, \mathcal{R})\} \cup \{(0, \psi) : \psi \in \mathbb{HOM}_{\mathcal{R}}(B, \mathcal{R})\},$$

where the above equation, topologically holds.

Let $\mathcal{R}$ be a commutative Banach algebra and let A be a Banach algebra such that is a Banach $\mathcal{R}$-bimodule. By $B^n_{\mathcal{R}}(A, A)$, we mean that the space of bounded n-$\mathcal{R}$-linear maps form A into A. A 2-$\mathcal{R}$-linear map $\gamma \in B^2(A, X)$ is called *2-$\mathcal{R}$-cocycle* if it satisfies in the following equation

$$a\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b)c = 0,$$

for every $a, b, c \in A$. The space of 2-$\mathcal{R}$-cocycles is a subspace of $B^2_{\mathcal{R}}(A, A)$, which denoted by $Z^2_{\mathcal{R}}(A, A)$. Now, we can write the main result of the Section 3 for Banach algebra case as follows:

Theorem 4.1. Let $\mathcal{R}$ be a unital commutative Banach algebra, let A and B be Banach algebras such that are Banach $\mathcal{R}$-bimodules and let $\delta : A \times_\varphi B \longrightarrow A \times_\varphi B$ be a bounded generalized 2-$\mathcal{R}$-cocycle derivation associate with $\gamma : (A \times_\varphi B) \times (A \times_\varphi B) \longrightarrow A \times_\varphi B$. Then there are corresponding 2-$\mathcal{R}$-cocycles $\gamma_A : A \times A \longrightarrow A$, $\gamma_B : B \times B \longrightarrow B$, generalized 2-$\mathcal{R}$-cocycle derivations $\delta_A : A \longrightarrow A$, $\delta_B : B \longrightarrow B$ associate with γ_A and γ_B, respectively, and a φ-derivation $\tau : B \longrightarrow A$.

Now, this question arise that if there are generalized 2-$\mathcal{R}$-cocycle derivations $\delta_A : A \longrightarrow A$ and $\delta_B : B \longrightarrow B$ associate with 2-$\mathcal{R}$-cocycles γ_A and γ_B, respectively, are there 2-$\mathcal{R}$-cocycle γ : $(A \times_\varphi B) \times (A \times_\varphi B) \longrightarrow A \times_\varphi B$ and generalized 2-$\mathcal{R}$-cocycle derivation $\delta : A \times_\varphi B \longrightarrow A \times_\varphi B$ related to γ?

Lemma 4.2. *Let $\gamma_A : A \times A \longrightarrow A$ and $\gamma_B : B \times B \longrightarrow B$ be continuous 2-$\mathcal{R}$-cocycles and $\tau : B \longrightarrow A$ be a φ-derivation such that $\varphi(\gamma_B(b_1, b_2)) = \gamma_A(\varphi(b_1), \varphi(b_2))$, for every $b_1, b_2 \in B$. Then $\gamma : (A \times_\varphi B) \times (A \times_\varphi B) \longrightarrow A \times_\varphi B$ defined by*

$$\gamma((a_1, b_1), (a_2, b_2)) = (\gamma_1((a_1, b_1), (a_2, b_2)), \gamma_2((a_1, b_1), (a_2, b_2))),$$

for every $(a_1, b_1), (a_2, b_2) \in A \times_\varphi B$, where

1. $\gamma_1((a_1, b_1), (a_2, b_2)) = \gamma_A(a_1, a_2) + \gamma_A(\varphi(b_1), a_2) + \gamma_A(a_1, \varphi(b_2)) - \tau(b_1)a_2 - a_1\tau(b_2)$,
2. $\gamma_2((a_1, b_1), (a_2, b_2)) = \gamma_B(b_1, b_2)$,

is a continuous 2-$\mathcal{R}$-cocycle on $A \times_\varphi B$.

Proof . The continuity of γ is clear from its definition. Thus, we show that it is a 2-$\mathcal{R}$-cocycle on $A \times_\varphi B$. For every $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times_\varphi B$, we have

$$\begin{aligned}
&(a_1, b_1)\ \gamma((a_2, b_2), (a_3, b_3)) = (a_1, b_1)(\gamma_A(a_2, a_3) + \gamma_A(\varphi(b_2), a_3) \\
&\quad + \gamma_A(a_2, \varphi(b_3)) - \tau(b_2)a_3 - a_2\tau(b_3), \gamma_B(b_2, b_3)) \\
&= (a_1\gamma_A(a_2, a_3) + a_1\gamma_A(\varphi(b_2), a_3) + a_1\gamma_A(a_2, \varphi(b_3)) \\
&\quad - a_1\tau(b_2)a_3 - a_1a_2\tau(b_3) - \varphi(b_1)a_2\tau(b_3) - \varphi(b_1)\tau(b_2)a_3 \\
&\quad + \varphi(b_1)\gamma_A(a_2, a_3) + \varphi(b_1)\gamma_A(\varphi(b_2), a_3) \\
&\quad + \varphi(b_1)\gamma_A(a_2, \varphi(b_3)) + a_1\varphi(\gamma_B(b_2, b_3)), b_1\gamma_B(b_2, b_3)) \\
&= (a_1\gamma_A(a_2, a_3) + a_1\gamma_A(\varphi(b_2), a_3) + a_1\gamma_A(a_2, \varphi(b_3)) \\
&\quad - a_1\tau(b_2)a_3 - a_1a_2\tau(b_3) - \varphi(b_1)a_2\tau(b_3) - \varphi(b_1)\tau(b_2)a_3 \\
&\quad + \varphi(b_1)\gamma_A(a_2, a_3) + \varphi(b_1)\gamma_A(\varphi(b_2), a_3) + \varphi(b_1)\gamma_A(a_2, \varphi(b_3)) \\
&\quad + a_1\gamma_A(\varphi(b_2), \varphi(b_3)), b_1\gamma_B(b_2, b_3)),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\gamma((a_1, b_1)(a_2, b_2), (a_3, b_3)) = \gamma((a_1a_2 + \varphi(b_1)a_2 + a_1\varphi(b_2), b_1b_2), (a_3, b_3)) \\
&= (\gamma_A(a_1a_2, a_3) + \gamma_A(a_1\varphi(b_2), a_3) + \gamma_A(\varphi(b_1)a_2, a_3) \\
&\quad + \gamma_A(\varphi(b_1b_2), a_3) + \gamma_A(a_1a_2, \varphi(b_3)) + \gamma_A(a_1\varphi(b_2), \varphi(b_3)) \\
&\quad + \gamma_A(\varphi(b_1)a_2, \varphi(b_3)) - \tau(b_1b_2)a_3 - a_1a_2\tau(b_3) \\
&\quad - a_1\varphi(b_2)\tau(b_3) - \varphi(b_1)a_2\tau(b_3), \gamma_B(b_1b_2, b_3)) \\
&= (\gamma_A(a_1a_2, a_3) + \gamma_A(a_1\varphi(b_2), a_3) + \gamma_A(\varphi(b_1)a_2, a_3) + \gamma_A(\varphi(b_1)\varphi(b_2), a_3) \\
&\quad + \gamma_A(a_1a_2, \varphi(b_3)) + \gamma_A(a_1\varphi(b_2), \varphi(b_3)) + \gamma_A(\varphi(b_1)a_2, \varphi(b_3)) \\
&\quad - \varphi(b_1)\tau(b_2)a_3 - \tau(b_1)\varphi(b_2)a_3 - a_1a_2\tau(b_3) \\
&\quad - a_1\varphi(b_2)\tau(b_3) - \varphi(b_1)a_2\tau(b_3), \gamma_B(b_1b_2, b_3)),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\gamma((a_1, b_1), (a_2, b_2)(a_3, b_3)) = \gamma((a_1, b_1), (a_2a_3 + \varphi(b_2)a_3 + a_2\varphi(b_3), b_2b_3)) \\
&= (\gamma_A(a_1, a_2a_3) + \gamma_A(a_1, a_2\varphi(b_3)) + \gamma_A(a_1, \varphi(b_2)a_3) \\
&\quad + \gamma_A(\varphi(b_1), a_2a_3) + \gamma_A(\varphi(b_1), a_2\varphi(b_3)) + \gamma_A(\varphi(b_1), \varphi(b_2)a_3) \\
&\quad + \gamma_A(a_1, \varphi(b_2)\varphi(b_3)) - \tau(b_1)a_2a_3 - \tau(b_1)a_2\varphi(b_3) \\
&\quad - \tau(b_1)\varphi(b_2)a_3 - a_1\tau(b_2b_3), \gamma_B(b_1, b_2b_3)) \\
&= (\gamma_A(a_1, a_2a_3) + \gamma_A(a_1, a_2\varphi(b_3)) + \gamma_A(a_1, \varphi(b_2)a_3) + \gamma_A(\varphi(b_1), a_2a_3) \\
&\quad + \gamma_A(\varphi(b_1), a_2\varphi(b_3)) + \gamma_A(\varphi(b_1), \varphi(b_2)a_3) + \gamma_A(a_1, \varphi(b_2)\varphi(b_3)) \\
&\quad - \tau(b_1)a_2a_3 - \tau(b_1)a_2\varphi(b_3) - \tau(b_1)\varphi(b_2)a_3 \\
&\quad - a_1\varphi(b_2)\tau(b_3) - a_1\tau(b_2)\varphi(b_3), \gamma_B(b_1, b_2b_3)),
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\gamma((a_1,b_1),(a_2,b_2))(a_3,b_3) = (\gamma_A(a_1,a_2)+\gamma_A(\varphi(b_1),a_2)+\gamma_A(a_1,\varphi(b_2))\\
&\quad +\tau(b_1)a_2+a_1\tau(b_2),\gamma_B(b_1,b_2))(a_3,b_3)\\
&= (\gamma_A(a_1,a_2)a_3+\gamma_A(\varphi(b_1),a_2)a_3+\gamma_A(a_1,\varphi(b_2))a_3\\
&\quad -\tau(b_1)a_2a_3-a_1\tau(b_2)a_3+\varphi(\gamma_B(b_1,b_2))a_3+\gamma_A(a_1,a_2)\varphi(b_3)\\
&\quad +\gamma_A(\varphi(b_1),a_2)\varphi(b_3)+\gamma_A(a_1,\varphi(b_2))\varphi(b_3)-\tau(b_1)a_2\varphi(b_3)-a_1\tau(b_2)\varphi(b_3),\gamma_B(b_1,b_2)b_3)\\
&= (\gamma_A(a_1,a_2)a_3+\gamma_A(\varphi(b_1),a_2)a_3+\gamma_A(a_1,\varphi(b_2))a_3\\
&\quad -\tau(b_1)a_2a_3-a_1\tau(b_2)a_3+\gamma_A(\varphi(b_1),\varphi(b_2))a_3+\gamma_A(a_1,a_2)\varphi(b_3)\\
&\quad +\gamma_A(\varphi(b_1),a_2)\varphi(b_3)+\gamma_A(a_1,\varphi(b_2))\varphi(b_3)-\tau(b_1)a_2\varphi(b_3)-a_1\tau(b_2)\varphi(b_3),\gamma_B(b_1,b_2)b_3).
\end{aligned} \tag{4.4}$$

Then by relations (4.1), (4.2), (4.3) and (4.4), γ is a 2-cocycle. □

Theorem 4.3. *Let $\delta_A : A \longrightarrow A$ and $\delta_B : B \longrightarrow B$ be two generalized continuous 2-$\mathcal{R}$-cocycle derivations associate with γ_A and γ_B and $\tau : B \longrightarrow A$ be a continuous φ-derivation such that $\varphi\circ\delta_B = \delta_A\circ\varphi$ and $\varphi(\gamma_B(b_1,b_2)) = \gamma_A(\varphi(b_1),\varphi(b_2))$, for every $b_1,b_2 \in B$. Then there is a continuous 2-$\mathcal{R}$-cocycle $\gamma : (A\times_\varphi B)\times(A\times_\varphi B) \longrightarrow A\times_\varphi B$ and there is a generalized continuous 2-$\mathcal{R}$-cocycle derivation $\delta : A\times_\varphi B \longrightarrow A\times_\varphi B$ associate with γ defined by*

$$\delta((a,b)) = (\delta_A(a)+\tau(b),\delta_B(b)) \qquad ((a,b)\in A\times_\varphi B). \tag{4.5}$$

Proof . Define $\gamma : (A\times_\varphi B)\times(A\times_\varphi B) \longrightarrow A\times_\varphi B$ by

$$\gamma((a_1,b_1),(a_2,b_2)) = (\gamma_1((a_1,b_1),(a_2,b_2)),\gamma_2((a_1,b_1),(a_2,b_2))), \tag{4.6}$$

for every $(a_1,b_1),(a_2,b_2)\in A\times_\varphi B$, where

1. $\gamma_1((a_1,b_1),(a_2,b_2)) = \gamma_A(a_1,a_2)+\gamma_A(\varphi(b_1),a_2)+\gamma_A(a_1,\varphi(b_2))-\tau(b_1)a_2-a_1\tau(b_2)$,
2. $\gamma_2((a_1,b_1),(a_2,b_2)) = \gamma_B(b_1,b_2)$.

Thus Lemma 4.2 implies that γ is a 2-cocycle. Now, we shall show that δ is a generalized 2-cocycle derivation associate with γ. For every $(a_1,b_1),(a_2,b_2)\in A\times_\varphi B$, we have

$$\begin{aligned}
&\delta((a_1,b_1)(a_2,b_2)) = \delta((a_1a_2+a_1\varphi(b_2)+\varphi(b_1)a_2,b_1b_2))\\
&= (\delta_A(a_1a_2+a_1\varphi(b_2)+\varphi(b_1)a_2)+\tau(b_1b_2),\delta_B(b_1b_2))\\
&= (\delta_A(a_1a_2)+\delta_A(a_1\varphi(b_2))+\delta_A(\varphi(b_1)a_2)+\tau(b_1b_2),\delta_B(b_1b_2))\\
&= (\delta_A(a_1)a_2+a_1\delta_A(a_2)+\gamma_A(a_1,a_2)+\delta_A(a_1)\varphi(b_2)+a_1\delta_A(\varphi(b_2))\\
&\quad +\gamma_A(a_1,\varphi(b_2))+\delta_A(\varphi(b_1))a_2+\varphi(b_1)\delta_A(a_2)\\
&\quad +\gamma_A(\varphi(b_1),a_2)+\tau(b_1)\varphi(b_2)+\varphi(b_1)\tau(b_2),\delta_B(b_1)b_2+b_1\delta_B(b_2)+\gamma_B(b_1,b_2))\\
&= (\delta_A(a_1)a_2+a_1\delta_A(a_2)+\delta_A(a_1)\varphi(b_2)+a_1\delta_A(\varphi(b_2))+\delta_A(\varphi(b_1))a_2+\varphi(b_1)\delta_A(a_2)\\
&\quad +\tau(b_1)\varphi(b_2)+\varphi(b_1)\tau(b_2),\delta_B(b_1)b_2+b_1\delta_B(b_2))+(\gamma_A(a_1,a_2)\\
&\quad +\gamma_A(\varphi(b_1),a_2)+\gamma_A(a_1,\varphi(b_2)),\gamma_B(b_1,b_2)).
\end{aligned} \tag{4.7}$$

On the other hand, for every $(a_1, b_1), (a_2, b_2) \in A \times_\varphi B$, we have

$$\begin{aligned}
&\delta((a_1,b_1))(a_2,b_2) + (a_1,b_1)\delta((a_2,b_2)) + \gamma((a_1,b_1),(a_2,b_2))\\
&= (\delta_A(a_1) + \tau(b_1), \delta_B(b_1))(a_2,b_2) + (a_1,b_1)\ (\delta_A(a_2) + \tau(b_2), \delta_B(b_2)) + \gamma((a_1,b_1),(a_2,b_2))\\
&= (\delta_A(a_1)a_2 + \tau(b_1)a_2 + \varphi(\delta_B(b_1))a_2 + \delta_A(a_1)\varphi(b_2)\\
+ \quad & \tau(b_1)\varphi(b_2), \delta_B(b_1)b_2) + (a_1\delta_A(a_2) + a_1\tau(b_2) + \varphi(b_1)\delta_A(a_2)\\
&\quad + \varphi(b_1)\tau(b_2) + a_1\varphi(\delta_B(b_2)), b_1\delta_B(b_2)) + (\gamma_1((a_1,b_1),(a_2,b_2)), \gamma_2((a_1,b_1),(a_2,b_2)))\\
&= (\delta_A(a_1)a_2 + a_1\delta_A(a_2) + \delta_A(a_1)\varphi(b_2) + a_1\varphi(\delta_B(b_2)) + \varphi(\delta_B(b_1))a_2 + \tau(b_1)a_2 + a_1\tau(b_2) \qquad (4.8)\\
&\quad + \varphi(b_1)\delta_A(a_2) + \tau(b_1)\varphi(b_2) + \varphi(b_1)\tau(b_2), \delta_B(b_1)b_2 + b_1\delta_B(b_2)) + (\gamma_A(a_1,a_2)\\
&\quad + \gamma_A(\varphi(b_1),a_2) + \gamma_A(a_1,\varphi(b_2)) - \tau(b_1)a_2 - a_1\tau(b_2), \gamma_B(b_1,b_2))\\
&= (\delta_A(a_1)a_2 + a_1\delta_A(a_2) + \delta_A(a_1)\varphi(b_2) + a_1\delta_A(\varphi(b_2)) + \delta_A(\varphi(b_1))a_2\\
&\quad + \varphi(b_1)\delta_A(a_2) + \tau(b_1)\varphi(b_2) + \varphi(b_1)\tau(b_2), \delta_B(b_1)b_2 + b_1\delta_B(b_2))\\
&\quad + (\gamma_A(a_1,a_2) + \gamma_A(\varphi(b_1),a_2) + \gamma_A(a_1,\varphi(b_2)), \gamma_B(b_1,b_2)).
\end{aligned}$$

By comparing the relations (4.7) and (4.8) we conclude that δ is a generalized 2-$\mathcal{R}$-cocycle derivation. Continuity is clearly hold. □

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