Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 39-48
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2016.457

Weak and (-1)-weak amenability of second dual of Banach algebras

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(Communicated by A. Ebadian)

Abstract

For a Banach algebra A, A'' is (-1)-weakly amenable if A' is a Banach A''-bimodule and $H^1(A'', A') = \{0\}$. In this paper we prove some important properties of this notion, for instance if A'' is (-1)-weakly amenable then A is essential and there is no non-zero point derivation on A. We also give some examples, namely, the second dual of every C^*-algebras is (-1)-weakly amenable. Finally, we study the relationships between the (-1)-weakly amenability of A'' and the weak amenability of A'' or A.

Keywords: Banach algebra; Point derivation; (-1)-Weak amenability.
2010 MSC: 46H25.

1. Introduction and preliminaries

For a Banach algebra A, let X be a Banach A-bimodule. Then a bounded derivation from A into X is a bounded linear map $D : A \longrightarrow X$ such that

$$D(a \cdot b) = a \cdot Db + Da \cdot b \qquad (a, b \in A).$$

Easy examples of derivations are the inner derivations, which are given for each $x \in X$ by

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in A).$$

The set of all bounded derivations from A into X is denoted by $Z^1(A, X)$ and the set of all inner derivations from A into X is denoted by $N^1(A, X)$. The Banach algebra A is amenable if $H^1(A, X') =$

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Received: December 2015 *Revised:* March 2016

$Z^1(A,X')/N^1(A,X') = \{0\}$ for each Banach A-bimodule X, where $H^1(A,X')$ is the first cohomology group from A with coefficients in X'. This definition was introduced by B. E. Johnson in [8].

A Banach algebra A is weakly amenable if $H^1(A,A') = \{0\}$. W. G. Bade, P. C. Curtis and H. G. Dales introduced the concept of weak amenability for commutative Banach algebras in [2].

Consider now the second dual A'' of a Banach algebra A. Then A'' is also a Banach algebra with respect to the first and second Arens product which are denoted by $\cdot$ and $\times$ respectively. The algebra A is called Arens regular if for each $F, G \in A''$, $F \cdot G = F \times G$. For more details, see [4].

Let A be a Banach algebra, then the Banach algebra A'' is (-1)-weakly amenable if A' is a Banach A''-bimodule and $H^1(A'',A') = \{0\}$. This definition was introduced by A. Medghalchi and T. Yazdanpanah in [9]. In this paper A'' is considered with the first Arens product. It is proved that for the James algebra $\mathcal{J}$, $\mathcal{J}''$ is (-1)-weakly amenable , whereas $(l^1,*)$ is not (-1)-weakly amenable. We also know that $(lip_\alpha \mathbb{T})''$ is not (-1)-weakly amenable for $\alpha \in (\frac{1}{2},1)$, where $\mathbb{T}$ is the unit circle, see [6] and [7]. For the (-1)-weak amenability of $(Lip_\alpha K)''$ see 2.5. The following examples show that the notion of (-1)-weak amenability is disjoint from weak amenability or amenability in Banach algebras.

Example 1.1. l^p for $1 < p < \infty$, with pointwise multiplication is (-1)-weakly amenable which is not amenable since has no bounded approximate identity.

Example 1.2. In Corollary 3.7 we prove that the second dual of a C^*-algebra is a (-1)-weakly amenable Banach algebra. So, in the case A'' is a non-nuclear C^*-algebra, we can conclude that A'' is (-1)-weakly amenable but is not amenable.

Now we recall Theorem 2.1 of [7], which we use in this paper.

Theorem 1.3. For a Banach algebra A, in each of the following cases, A' is a Banach A''-bimodule

1) A is Arens regular,

2) A is a left ideal in A'',

3) A is a right ideal in A'' and $A'' \cdot A = A''$.

2. Some properties of (-1)-Weak amenability

Theorem 2.1. *Let A be a Banach algebra and A'' be (-1)-Weakly amenable, then A is essential.*

Proof . If $\overline{A^2} \neq A$ then there exists a_0 in $A \backslash \overline{A^2}$, and by Hahn-Banach theorem there exists $\lambda_0 \in A'$ such that $\lambda_0(a_0) = 1$ and $\lambda_0|_{A^2} = 0$. Now we define $D : A'' \longrightarrow A'$ with $DF = F(\lambda_0)\lambda_0$, for each $F \in A''$. D is a bounded derivation since for each F and G in A'' and the net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A with $\hat{a}_\alpha \xrightarrow{w^*} F$ and $\hat{b}_\beta \xrightarrow{w^*} G$ we have

$$\begin{aligned} D(F \cdot G) &= F \cdot G(\lambda_0)\lambda_0 = \lim_\alpha \lim_\beta \hat{a}_\alpha \cdot \hat{b}_\beta(\lambda_0)\lambda_0 \\ &= \lim_\alpha \lim_\beta \lambda_0(a_\alpha \cdot b_\beta)\lambda_0 = 0. \end{aligned}$$

Moreover for each $a \in A$ we have

$$\begin{aligned} DF \cdot G(a) &= (F(\lambda_0) \cdot \lambda_0) \cdot G(a) = F(\lambda_0)(\lambda_0 \cdot G(a)) \\ &= F(\lambda_0) \cdot G(a \cdot \lambda_0) = F(\lambda_0) \cdot \lim_\beta \lambda_0(b_\beta \cdot a) = 0 \end{aligned}$$

so $DF \cdot G = 0$ and similarly $F \cdot DG = 0$. Therefore $D \in Z^1(A'', A')$ and by the assumption D is inner so there exists $f_0 \in A'$ such that $DF = \delta_0 F$, for each F in A''. Since $\lambda_0(a_0) = 1$ we have

$$D(\hat{a}_0)(a_0) = \hat{a}_0(\lambda_0) \cdot \lambda_0(a_0) = 1.$$

On the other hand

$$D(\hat{a}_0)(a_0) = (\hat{a}_0 \cdot f_0 - f_0 \cdot \hat{a}_0)(a_0) = f_0(a_0^2 - a_0^2) = 0$$

which is a contradiction. □

Remark 2.2. The (−1)-weak amenability of A'' implies that A is essential, but we can not conclude that $A'' \cdot A = A''$ even if A is commutative and Arens regular. For example the James algebra $\mathcal{J}$ is a commutative Arens regular Banach algebra with the property $\mathcal{J}'' = \mathcal{J}^\#$. $\mathcal{J}''$ is (−1)-weakly amenable so $\mathcal{J}$ is essential, but $\mathcal{J}'' \cdot \mathcal{J} \neq \mathcal{J}''$ since $\mathcal{J}'' \cdot \mathcal{J} = \mathcal{J}^\# \cdot \mathcal{J} = \mathcal{J} \neq \mathcal{J}''$, see Example 2.2 in [6].

Example 2.3. Let $A = \mathbb{C}$ with zero multiplication. Then A' is an A''-bimodule, but it is not essential. Therefore A'' is not (−1)-Weakly amenable.

Example 2.4. Let S be a discrete semigroup for which $S^2 \neq S$. Then $l^1(S)$ is not essential and by using Theorem 2.1, $(l^1(S))''$ is not (−1)-weakly amenable.

Remark 2.5. Let A be a commutative Banach algebra and Φ_A be the set of all multiplicative linear functionals on A . Then for each $\varphi \in \Phi_A \cup \{0\}$, if we consider the following multiplications $\mathbb{C}$ is a Banach A-bimodule:

$$a \cdot z = \varphi(a) \cdot z \qquad , \qquad z \cdot a = \varphi(a) \cdot z \qquad (a \in A, z \in \mathbb{C}).$$

This module is denoted by $\mathbb{C}_\varphi$.

A derivation in $Z^1(A, \mathbb{C}_\varphi)$ is called a point derivation at φ. So for each point derivation at φ, say d, we have

$$d(a \cdot b) = \varphi(a)d(b) + \varphi(b)d(a) \qquad (a, b \in A).$$

As we know, there is no non-zero continuous point derivation on a weakly amenable Banach algebra, see 2.8.63 in [4] and one may ask about (−1)-Weakly amenable Banach algebras.

Theorem 2.6. *Let A be a Banach algebra and A'' be (−1)-Weakly amenable. Then there is no non-zero continuous point derivation on A.*

Proof . By Theorem 1.3, A is essential so for $\varphi = 0$ there are no non-zero point derivations. Since for each $a, b \in A$ we have $d(a \cdot b) = da \cdot \varphi b + db \cdot \varphi a = 0$, so $d|_{A^2} = 0$ and since A is essential, then we have $d = 0$.
Now let $d : A \longrightarrow \mathbb{C}_\varphi$ be a continuous point derivation at $\varphi \in \Phi_A$. We show first that $d'' : A'' \longrightarrow \mathbb{C}_\varphi$ is a derivation. Since for $F, G \in A''$, with $F = w^* - \lim_i \hat{a}_i$ and $G = w^* - \lim_j \hat{b}_j$ we have

$$\begin{aligned} d''(F \cdot G) &= d''(w^* - \lim_i \lim_j \hat{a}_i \hat{b}_j) = w^* - \lim_i \lim_j d''(\widehat{a_i b_j}) \\ &= \lim_i \lim_j d(a_i b_j) = \lim_i \lim_j ((da_i)(\varphi b_j) + (db_j)(\varphi a_i)) \\ &= \lim_i \lim_j d(a_i)\hat{b}_j(\varphi) + \lim_i \lim_j d(b_j)\varphi(a_i) \\ &= \lim_i d(a_i)G(\varphi) + \lim_i \varphi(a_i)d''(G) = d''(F)G(\varphi) + d''(G)F(\varphi). \end{aligned}$$

Since φ is a multiplicative functional on A, for each $a, b \in A$

$$(a \cdot \varphi)(b) = \varphi(b \cdot a) = \varphi b \cdot \varphi a = \varphi a \cdot \varphi(b)$$

so $a \cdot \varphi = \varphi(a) \cdot \varphi$ and similarly $\varphi \cdot a = \varphi a \cdot \varphi$. Then $D : A'' \longrightarrow A'$ with $DF(a) = d''F \cdot \varphi a$ is a bounded derivation. Since for each $a \in A$ and $F, G \in A''$ we have

$$\begin{aligned} D(F \cdot G)(a) &= d''(F \cdot G) \cdot \varphi a = d''F \cdot G\varphi \cdot \varphi a + d''G \cdot F\varphi \cdot \varphi a \\ &= d''F \cdot G(\varphi \cdot \varphi a) + d''G \cdot F(\varphi \cdot \varphi a) \\ &= d''F \cdot G(a \cdot \varphi) + d''G \cdot F(\varphi \cdot a) \\ &= G(a \cdot (d''F \cdot \varphi)) + F((d''G \cdot \varphi) \cdot a) \\ &= G(a \cdot DF) + F(DG \cdot a) = (DF \cdot G + F \cdot DG)(a). \end{aligned}$$

So D is a derivation and by the (-1)-Weak amenability of A'', there exists $f_0 \in A'$ such that for each F in A'', $DF = \delta_{f_0}(F)$.
Now for each $a \in A \backslash \ker \varphi$ we have

$$\begin{aligned} da \cdot \varphi a &= d''(\hat{a}) \cdot \varphi a = D\hat{a}(a) = \delta_{f_0}\hat{a}(a) \\ &= (\hat{a} \cdot f_0 - f_0 \cdot \hat{a})(a) = f_0(a^2 - a^2) = 0. \end{aligned}$$

Since $a \notin \ker \varphi$, then $da = 0$. On the other hand, for each $a \in \ker \varphi$ and $b \in A$, we have

$$\begin{aligned} da \cdot \varphi b &= d''(\hat{a}) \cdot \varphi b = D\hat{a}(b) = \delta_{f_0}\hat{a}(b) = (\hat{a} \cdot f_0 - f_0 \cdot \hat{a})(b) \\ &= (f_0 \cdot \hat{b} - \hat{b} \cdot f_0)(a) = -\delta_{f_0}\hat{b}(a) = -d''(\hat{b})\varphi a = 0. \end{aligned}$$

Since $\varphi a \neq 0$, then $da = 0$ and so there is no non-zero continuous point derivation on A. □

Example 2.7. Let (K, d) be an infinite compact metric space, then for $\alpha \in (0, 1)$ there is a non-zero continuous point derivation on $Lip_\alpha K$ at each non-isolated point of K, see 4.4.33 (i) in [4]. So by Theorem 2.6, $(Lip_\alpha K)''$ is not (-1)-Weakly amenable, for each $\alpha \in (0, 1)$.

Example 2.8. If G is an infinite compact group then $L^1(G)$ is an ideal in $(L^1(G))''$. Also, in the case G is a compact abelian group, we can conclude that $M(G) \simeq L^1(G)''/K$, where $K = \{F \in L^1(G)'' : L^1(G)'' \cdot F = 0\}$ is a closed ideal having zero product, see [5]

If G is not discrete, then $M(G)$ has non-zero continuous point derivation, see [3], which is lift to $L^1(G)''$. So by Theorem 2.6 $(L^1(G))^{(4)}$ is not (-1)-Weakly amenable, whereas $L^1(G)$ is weakly amenable for every locally compact group G.

We know that for a commutative weakly amenable Banach algebra A, there is no nonzero bounded derivation into any symmetric Banach A-module (symmetric means that the left and right module multiplications agree).

Now, it is natural to ask if $H^1(A'', E) = (0)$, for a commutative (-1)-Weakly amenable Banach algebra A'' and each Banach A-module E.

Remark 2.9. Let E be a Banach A''-bimodule, then E'' is a Banach A''-bimodule whenever for $F \in A''$ and $\varphi \in E''$ with $\varphi = w^* - \lim_\beta \hat{x}_\beta$ we define A''-module multiplications in E'' as follows:

$$F \cdot \varphi = w^* - \lim_\beta F \cdot x_\beta \qquad \text{and} \qquad \varphi \cdot F = w^* - \lim_\beta x_\beta \cdot F.$$

Moreover, E' is a Banach left and right A''-module, by:

$$F \cdot \lambda(x) = \lambda(x \cdot F) \qquad \text{and} \qquad \lambda \cdot F(x) = \lambda(F \cdot x)$$

for each $x \in E$, $F \in A''$ and $\lambda \in E'$, see [4].

Lemma 2.10. *Let A be a commutative Banach algebra, $A'' \cdot A = A''$ and A be essential. Then for each Banach left A''-module E and each non-zero $D \in Z^1(A'', E)$ we have $\hat{A} \cdot \mathrm{Im}\ D \neq 0$.*

Proof . Let E be a Banach left A''-module, then with $x \cdot F =: F \cdot x$, E is a Banach A''-bimodule. Now suppose that for some $D \in Z^1(A'', E)$, $\hat{A} \cdot \mathrm{Im}\ D = 0$. Then for each $a, b \in A$ and $F \in A''$ we have $D\widehat{a.b} = \hat{a} \cdot D\hat{b} + D\hat{a} \cdot \hat{b} = 0$. Since A is essential, $D|_{\hat{A}} = 0$. Moreover $D(\hat{a} \cdot F) = \hat{a} \cdot DF + D(\hat{a}) \cdot F = 0$, then $D|_{A \cdot A''} = 0$ so $D = 0$.
For a Banach right A''-module E, with similar argument we have $\mathrm{Im}\ D \cdot \hat{A} \neq \{0\}$. □

Theorem 2.11. *Let A be a Banach algebra and A'' be commutative and (-1)-Weakly amenable, for which $A'' \cdot A = A''$. Then $Z^1(A'', E) = \{0\}$, for each Banach A''-module E.*

Proof . Let E be a Banach left A''-module and define $x \cdot F =: F \cdot x$ for each $F \in A''$ and $x \in E$. Then E is a Banach right A''-module and the commutativity of A'' implies that E is a Banach A''-bimodule.
Let $D \in Z^1(A'', E)$ and $D \neq 0$ Since A'' is (-1)-weakly amenable then A is essential and by previous lemma $\hat{A} \cdot \mathrm{Im}\ D \neq 0$, so there are $a_0 \in A$ and $F_0 \in A''$ such that $\hat{a}_0 \cdot DF_0 \neq 0$. Then there exists $\lambda \in E'$ such that $\lambda(a_0 \cdot DF) = 1$.

Now, we define:

$$R : E \longrightarrow A'$$
$$R(x)(a) = \lambda(\ a \cdot x) \qquad (a \in A, x \in X).$$

So $R \circ D : A'' \longrightarrow A'$ is a bounded derivation since:

$$\begin{aligned} R \circ D(F \cdot G)(a) &= R(DF \cdot G + F \cdot DG)(a) = \lambda(\ a \cdot (DF \cdot G) +\ a \cdot (F \cdot DG)) \\ &= \lambda((\ a \cdot DF) \cdot G) + \lambda((\ a \cdot DG) \cdot F) \\ &= G \cdot \lambda(\hat{a} \cdot DF) + F \cdot \lambda(\hat{a} \cdot DG). \end{aligned}$$

On the other hand for $G = w^* - \lim_\alpha \hat{b}_\alpha$ and $x \in E$, the net $(\hat{b}_\alpha \cdot x)_\alpha$ is a bounded net in E'', so $\widehat{b_\alpha \cdot x} \xrightarrow{w^*} G \cdot x$, especially $\lambda(G \cdot x) = \lim \lambda(\hat{b}_\alpha \cdot x)$ and we have

$$\begin{aligned} (R(DF) \cdot G)\,(a) &= G(R(DF) \cdot a) \\ &= \lim_\alpha \hat{b}_\alpha(R(DF) \cdot a) = \lim_\alpha (R(DF) \cdot a)(b_\alpha) \\ &= \lim \lambda(\ b_\alpha \cdot\ a \cdot DF) = \lambda(G \cdot (\hat{a} \cdot DF)) \\ &= \lambda \cdot G(\hat{a} \cdot DF) = G \cdot \lambda(\hat{a} \cdot DF). \end{aligned}$$

Similarly $(F \cdot R(DG))\,(a) = F \cdot \lambda(a \cdot DG)$. Since A'' is (-1)-weakly amenable and commutative then $R \circ D = 0$. On the other hand, $R \circ D(F_0)(a_0) = R(DF_0)(a_0) = \lambda(a_0 \cdot DF_0) = 1$, which is a contradiction, so $D = 0$. □

3. The relationships between weak amenability and (-1)-Weak amenability

Theorem 3.1. *Let A be a Banach algebra and $D \in Z^1(A, A')$. Then in each of the following cases there exists $\tilde{D} \in Z^1(A'', A')$ such that $\tilde{D}|_A = D$:*

(1) *A is a left ideal in A'',*

(2) *A is a right ideal in A'' and $A'' \cdot A = A''$,*

(3) *A' is a Banach A''-bimodule and D is weakly compact.*

Proof . By Theorem 1.3 in each of the cases A' is a Banach A''-bimodule. Since D is a bounded derivation, then its second adjoint $D'' : (A'', w^*) \longrightarrow (A''', w^*)$ is a bounded linear map and continuous.

Now we show that $P \circ D'' : A'' \longrightarrow A'$ is a bounded derivation where $P : A''' \longrightarrow A'$ is the natural projection. Let $F, G \in A''$ with $F = w^* - \lim_\alpha \hat{a}_\alpha$ and $G = w^* - \lim_\beta \hat{b}_\beta$. By using the w^*-continuity of D'' we have

$$\begin{aligned} D''(F \cdot G) &= w^* - \lim_\alpha \lim_\beta D''(\hat{a}_\alpha \cdot \hat{b}_\beta) = w^* - \lim_\alpha \lim_\beta (Da_\alpha \cdot b_\beta + a_\alpha \cdot Db_\beta) \\ &= w^* - \lim_\alpha (Da_\alpha \cdot G) + w^* - \lim_\alpha \lim_\beta a_\alpha \cdot Db_\beta \\ &= D''F \cdot G + w^* - \lim_\alpha a_\alpha \cdot D''G. \end{aligned} \tag{3.1}$$

So $w^* - \lim_\alpha a_\alpha \cdot D''G$ exists, and for each $x \in A$ we have

$$\begin{aligned} P(w^* - \lim_\alpha a_\alpha \cdot D''G)(x) &= \lim_\alpha D''G(\hat{x} \cdot a_\alpha) = \lim_\alpha P(D''G)(x \cdot a_\alpha) \\ &= F(P(D''G) \cdot x) = (F \cdot P(D''G))(x). \end{aligned}$$

By using (3.1) we conclude that:

$$P \circ D''(F \cdot G) = P(D''F \cdot G) + F \cdot P \circ D''(G). \tag{3.2}$$

Now, in each of the given cases we show that $P(D''F \cdot G) = P(D''F) \cdot G$.

(1) For $x \in A$, since A is a left ideal in A'' then there exists $y \in A$ such that $G \cdot \hat{x} = \hat{y}$ and so we have

$$\begin{aligned} P(D''F \cdot G)(x) &= (D''F \cdot G)(\hat{x}) = D''F(G \cdot \hat{x}) = D''F(\hat{y}) \\ &= P(D''F)(y) = \hat{y}\ (P(D''F)) \\ &= G(x \cdot P(D''F)) = P(D''F) \cdot G(x). \end{aligned}$$

(2) Since $A'' = A'' \cdot A$, then there are $y \in A$, $F_1 \in A''$ such that $F = F_1 \cdot y$ and there exists a net $(c_i)_i$ in A such that $F_1 = w^* - \lim_i \hat{c}_i$, now we have

$$\begin{aligned} D''F = D''(F_1 \cdot y) &= D''(w^* - \lim_i c_i \cdot y) = w^* - \lim_i D''(\hat{c_i y}) \\ &= w^* - \lim_i (\hat{Dc_i} \cdot y + c_i \cdot \hat{Dy}) \\ &= D''F_1 \cdot y + w^* - \lim_i \hat{c}_i \cdot Dy. \end{aligned}$$

For $x \in A$, let $f = P(x \cdot D''F_1) \cdot y + (x \cdot F_1) \cdot Dy$. So we show that $\hat{f} = x \cdot D''F$.

Since A is a right ideal in A'', then for $H \in A''$ there are $z, d \in A$ such that $\hat{z} = y \cdot H$ and $x \cdot F_1 = \hat{d}$ then we have

$$\begin{aligned}
(x \cdot D''F)(H) &= \Big((x \cdot D''F_1) \cdot y + w^* - \lim_i x \cdot \hat{c}_i \cdot Dy\Big)(H) \\
&= x \cdot D''F_1(y \cdot H) + \lim_i H((x \cdot c_i) \cdot Dy) \\
&= P(x \cdot D''F_1)(z) + \lim_i \widehat{x \cdot c_i}(Dy \cdot H) \\
&= \hat{z}(P(x \cdot D''F_1)) + x \cdot F_1(Dy \cdot H) \\
&= y \cdot H(P(x \cdot D''F_1)) + Dy \cdot H(d) \\
&= H(P(x \cdot D''F_1) \cdot y) + H \cdot d(Dy) \\
&= H(P(x \cdot D''F_1) \cdot y) + H((x \cdot F_1) \cdot Dy) \\
&= H(f) = \hat{f}(H)
\end{aligned}$$

so $x \cdot D''F = \hat{f}$ and we have

$$\begin{aligned}
P(D''F \cdot G)(x) &= (D''F \cdot G)(\hat{x}) = x \cdot D''F(G) = \hat{f}(G) \\
&= G(f) = \lim_\beta \hat{b}_\beta(f) = \lim_\beta \hat{f}(\hat{b}_\beta) \\
&= \lim_\beta x \cdot D''F(\hat{b}_\beta) = \lim_\beta D''F(\hat{b}_\beta \cdot x) \\
&= \lim_\beta P(D''F)(b_\beta \cdot x) = \lim_\beta (x \cdot P(D''F))(b_\beta) \\
&= G(x \cdot P(D''F)) = P(D''F) \cdot G(x).
\end{aligned}$$

(3) Since D is weakly compact then $D''A'' \subseteq (\hat{A}')$, that is $D''F = \widehat{P(D''F)}$ so

$$\begin{aligned}
P(D''F \cdot G)(x) &= D''F \cdot G(\hat{x}) = D''F(G \cdot x) \\
&= \widehat{P(D''F)}(G \cdot x) = G \cdot x(P(D''F)) \\
&= (P(D''F) \cdot G)(x).
\end{aligned}$$

In each of the cases $P(D''F \cdot G) = P(D''F) \cdot G$, and by (3.2) $P \circ D'' \in Z^1(A'', A')$. So for $\tilde{D} = P \circ D''$ we have $\tilde{D} \in Z^1(A'', A')$ and $\tilde{D}|_A = D$.

□

Theorem 3.2. *Let A be a Banach algebra and A'' be (-1)-weakly amenable, then in each of the following cases A is weakly amenable:*

(1) *A is a left ideal in A'';*

(2) *A is a right ideal in A'' and $A'' \cdot A = A''$;*

(3) *Each derivation in $Z^1(A, A')$ is weakly compact.*

Proof . Let $D : A \longrightarrow A'$ be a bounded derivation. By Theorem 3.1 there exists $\tilde{D} \in Z^1(A'', A')$ such that $\tilde{D}|_A = D$. Since A'' is (-1)-weakly amenable, there exists $\lambda \in A'$ such that $\tilde{D}F = F \cdot \lambda - \lambda \cdot F$.
□

Remark 3.3. Let A be a Banach algebra and A' be a Banach A''-bimodule, for $D \in Z^1(A'', A')$ we say, D is w^*-continuous if for each $F \in A''$ with $F = w^* - \lim_\alpha \hat{a}_\alpha$, then $DF = w^* - \lim_\alpha D\hat{a}_\alpha$.

Theorem 3.4. *Let A be a weakly amenable Banach algebra and A' be a Banach A''-bimodule. If every $D \in Z^1(A'', A')$ is w^*-continuous, then A'' is (-1)-weakly amenable.*

Proof . Let $D \in Z^1(A'', A')$, then $D_1 : A \longrightarrow A'$ with $D_1 a = D\hat{a}$ is a bounded derivation and by the weak amenability of A there exists $f_0 \in A'$ such that for each $a \in A$, $D_1 a = \delta_{f_0} a$. Now, for $F \in A''$ with $F = w^* - \lim_\alpha \hat{a}_\alpha$ and $x \in A$ by the w^*-continuity of D we have

$$\begin{aligned} DF(x) &= (w^* - \lim_\alpha D\hat{a}_\alpha)(x) = \lim_\alpha D_1 a_\alpha(x) = \lim_\alpha \delta_{f_0}(a_\alpha)(x) \\ &= \lim_\alpha (a_\alpha \cdot f_0 - f_0 \cdot a_\alpha)(x) = (F \cdot f_0 - f_0 \cdot F)(x)) = \delta_{f_0}(F)(x). \end{aligned}$$

□

Let A be a Banach algebra and let $i : A' \longrightarrow A'''$ with $i(f)(G) = G(f)$ be the inclusion map. Then i is a bounded left A''-module homomorphism, that is for each $f \in A'$ and $F \in A''$, $i(F \cdot f) = F \cdot i(f)$.

Lemma 3.5. *Let A be a Banach algebra, then $i : A' \longrightarrow A'''$ is an A''-bimodule homomorphism if and only if A is Arens regular.*

Proof . The proof is straightforward. □

Theorem 3.6. *Let A be an Arens regular Banach algebra and A'' be weakly amenable. Then in each the following cases A'' is (-1)-weakly amenable:*

(1) *A is an ideal in A'';*

(2) *each $D \in Z^1(A'', A')$ is w^*-continuous.*

Proof . For $D \in Z^1(A'', A')$ we define $D_1 = i \circ D$ ($D_1 F(G) = G(DF)$ for each $F, G \in A''$). By Lemma 3.5, i is an A''-bimodule homomorphism so $D_1 = i \circ D$ is a bounded derivation in $Z^1(A'', A''')$ and for some $\eta \in A'''$, $D_1 = \delta_\eta$. Put $\lambda = P(\eta)$, that is $\lambda(a) = \eta(\hat{a})$ for $a \in A$. Then for $F \in A''$, $a \in A$,

$$DF(a) = i \circ D(F)(\hat{a}) = D_1 F(\hat{a}). \tag{3.3}$$

Since A is an ideal in A'', there exists $b \in A$ such that $\hat{a} \cdot F - F \cdot \hat{a} = \hat{b}$ and

$$\begin{aligned} DF(a) &= D_1 F(\hat{a}) = \delta_\eta F(\hat{a}) = \eta(\hat{b}) \\ &= \hat{b}(P(\eta)) = F \cdot P(\eta) - P(\eta) \cdot F(a) = \delta_\lambda F(a). \end{aligned}$$

For the case (2), let $F = w^* - \lim_i \hat{x}_i$ then by using (3.3) we have

$$\begin{aligned} DF(a) &= D_1 F(\hat{a}) = \lim_i D_1(\hat{x}_i)(\hat{a}) \\ &= \lim_i \eta(\widehat{a \cdot x_i - x_i \cdot a}) = \lim_i P(\eta)(a \cdot x_i - x_i \cdot a) \\ &= \lim_i \lambda \cdot a - a \cdot \lambda(x_i) = F(\lambda \cdot a - a \cdot \lambda) = \delta_\lambda F(a). \end{aligned}$$

□

Corollary 3.7. *The second dual of a C^*-algebra is (-1)-Weakly amenable.*

Proof . Let A be a C^*-algebra, then A is Arens regular and so A' is a Banach A''-bimodule. Since A'' is a C^*-algebra, then each bounded linear map from A'' into A' is weakly compact, see [1], so each derivation $D : A'' \longrightarrow A'$ is w^*-w^*-continuous. On the other hand each C^*-algebra is weakly amenable. Therefore by Theorem 3.6 A'' is (-1)-Weakly amenable. □

Example 3.8. Let A'' be a non-nuclear C^*-algebra. Then A'' is a (-1)-weakly amenable Banach algebra which is not amenable.

Theorem 3.9. *Let A be a Banach algebra, A'' be (-1)-weakly amenable and for each $D \in Z^1(A'', A''')$ and each $F \in A''$, DF is w^*-continuous. Then A'' is weakly amenable.*

Proof . Let $D \in Z^1(A'', A'')$. Then $P \circ D : A'' \longrightarrow A'$ is a bounded linear map and for each $a \in A$ and $F, G \in A''$ with $F = w^* - \lim_\alpha \hat{a}_\alpha$ and $G = w^* - \lim_\beta \hat{b}_\beta$,

$$\begin{aligned} P(DF) \cdot G(a) &= G(a \cdot P(DF)) = \lim_\beta a \cdot P(DF)(b_\beta) \\ &= \lim_\beta P(DF)(b_\beta \cdot a) = \lim_\beta DF(\widehat{b_\beta \cdot a}) = DF(G \cdot a) \\ &= DF \cdot G(\hat{a}) = P(DF \cdot G)(a). \end{aligned}$$

Similarly $P(F \cdot DG) = F \cdot P(DG)$ and so $P \circ D$ is a derivation in $Z^1(A'', A')$. By the (-1)-Weakly amenability of A'' there is $\lambda_0 \in A'$ such that $P \circ D(F) = F \cdot \lambda_0 - \lambda_0 \cdot F$, for $F \in A''$. So,

$$\begin{aligned} DF(G) &= \lim_\beta DF(\hat{b}_\beta) = \lim_\beta P(DF)(b_\beta) = \lim_\beta \delta_{\lambda_0} F(b_\beta) = \lim_\beta \hat{b}_\beta(\delta_{\lambda_0} F) \\ &= G(F \cdot \lambda_0 - \lambda_0 \cdot F) = \hat{\lambda}_0(G \cdot F - F \cdot G) = \delta_{\hat{\lambda}_0}(F)(G). \end{aligned}$$

□

Remark 3.10. In the proof of Theorem 3.9 we didn't make use of the essentiality of A''. But in the case A'' is not essential, the w^*-continuity of DF will be failed ($D \in Z^1(A'', A''')$ and $F \in A''$). Since, if there exists $F_0 \in A'' \backslash \overline{A''^2}$, we can choose $\Lambda_0 \in A'''$ with $\Lambda_0(F_0) = 1$ and $\Lambda_0|_{A''^2} = 0$.

We define $D : A'' \longrightarrow A'''$ by $DF = \Lambda_0 F \cdot \Lambda_0$, then D is a bounded derivation which is not w^*-continuous. Since for $F_0 = w^* - \lim_\alpha \hat{a}_\alpha$ we have $DF_0(F_0) = \Lambda_0 F_0 \cdot \Lambda_0 F_0 = 1$, while $\lim_\alpha DF_0(\hat{a}_\alpha) = \Lambda_0 f_0 - \Lambda_0(\hat{a}_\alpha) = 0$. So DF_0 is not w^*-continuous.

Theorem 3.11. *Let A be an Arens regular commutative Banach algebra and A'' be weakly amenable. Then A'' is (-1)-weakly amenable.*

Proof . Since A is a commutative and Arens regular Bnach algebra then A' is a symmetric Banach A''-bimodule ($F \cdot f = f \cdot F$, for each $f \in A'$, $F \in A''$). Moreover, A'' is a commutative weakly amenable Banach algebra, so by using 1.5 in [2] we can conclude that $H^1(A'', A') = \{0\}$. □

Corollary 3.12. *Let A be an Arens regular commutative Banach algebra and $A'' \cdot A = A''$. Then the following conditions are equivalent:*

(1) A'' is weakly amenable;

(2) A'' is (-1)-Weakly amenable.

Proof .

(1 → 2) Theorem 3.9.
(2 → 1) Theorem 2.11. □

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