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# Some Inequalities Involving Lower Bounds of Operators on Weighted Sequence Spaces by a Matrix Norm

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**Abstract**

Let $A = (a_{n,k})_{n,k\geq 1}$ and $B = (b_{n,k})_{n,k\geq 1}$ be two non-negative matrices. Denote by $L_{v,p,q,B}(A)$, the supremum of those $L$, satisfying the following inequality:

$$\| Ax \|_{v,B(q)} \geq L \| x \|_{v,B(p)},$$

where $x \geq 0$ and $x \in l_p(v, B)$ and also $v = (v_n)_{n=1}^{\infty}$ is an increasing, non-negative sequence of real numbers. In this paper, we obtain a Hardy-type formula for $L_{v,p,q,B}(H_\mu)$, where $H_\mu$ is the Hausdorff matrix and $0 < q \leq p \leq 1$. Also for the case $p = 1$, we obtain $\|A\|_{w,B(1)}$, and for the case $p \geq 1$, we obtain $L_{w,B(p)}(A)$.

*Keywords:* Lower Bound. Weighted Block Sequence Space, Hausdorff Matrices, Euler Matrices, Cesàro Matrices, Matrix Norm.
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## 1. Introduction

Suppose that $v = (v_n)_{n=1}^{\infty}$ is an increasing, non-negative sequence of real numbers with $v_1 = v_2 = 1$ and $\sum_1^{\infty} \frac{v_n}{n} = \infty$. For $p \in R\backslash\{0\}$, let $l_p(v)$ denotes the space of all real sequences $x = \{x_k\}_{k=1}^{\infty}$, such that

$$\|x\|_{v,p} := \Big( \sum_{k=1}^{\infty} v_k |x_k|^p \Big)^{\frac{1}{p}} < \infty.$$

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Lashkaripour and Foroutannia in [10], defined the weighted block sequence space as follows. Assume that $F = (F_n)$ is a partition of positive integers where each $F_n$ is a finite interval of $N$ and

$$\max F_n < \min F_{n+1} \qquad (n = 1, 2, ...).$$

The weighted block sequence space $l_p(v, F)$ is defined as

$$l_p(v, F) := \Big\{ x = (x_n) : \sum_{n=1}^{\infty} v_n | < x, F_n > |^p < \infty \Big\},$$

where, $< x, F_n >= \sum_{j \in F_n} x_j$. The norm on $l_p(v, F)$ is denoted by $\|.\|_{p,v,F}$ and is defined by

$$\|x\|_{p,v,F} := \Big( \sum_{n=1}^{\infty} v_n | < x, F_n > |^p \Big)^{\frac{1}{p}}. \tag{1.1}$$

Note that with the above-mentioned definition $l_p(v, F)$ is not a norm sequence space. Indeed, one may consider $x = (1, -1, 0, 0, ...)$, $F_1 = \{1, 2\}$, $F_2 = \{3, 4\}$, ... and $v_n = 1$ then, $\|x\|_{p,v,F} = 0$ whereas $x \neq 0$.
We reform definition 1.1 as

$$l_p(v, F) := \Big\{ x = (x_n) : \sum_{n=1}^{\infty} v_n \Big( \sum_{j \in F_n} |x_j| \Big)^p < \infty \Big\},$$

and

$$\|x\|_{p,v,F} := \Big( \sum_{n=1}^{\infty} v_n \Big( \sum_{j \in F_n} |x_j| \Big)^p \Big)^{\frac{1}{p}}. \tag{1.2}$$

Of course, for non-negative sequences two definitions are coincide.

G. Bennett in [3] by a matrix $A$ with non-negative entries and $p > 0$, defined the sequence space

$$l_{A(p)} = \Big\{ x = (x_n) : \sum_n \Big( \sum_k a_{n,k} |x_k| \Big)^p < \infty \Big\}.$$

For $p \geq 1$ with the norm

$$\|x\|_{A(p)} = \Big( \sum_n \Big( \sum_k a_{n,k} |x_k| \Big)^p \Big)^{\frac{1}{p}}, \tag{1.3}$$

$l_{A(p)}$ is a norm sequence space.

By a partition $F = (F_n)$, we correspond a matrix $A = (a_{n,k})$ such that $a_{n,k} = 1$, for $k \in F_n$ and $a_{n,k} = 0$, otherwise. One may easily verifies that

$$\|x\|_{v,A(p)} = \|x\|_{p,v,F},$$

where,

$$\|x\|_{v,A(p)} = \Big( \sum_n v_n (\sum_k a_{n,k} |x_k|)^p \Big)^{\frac{1}{p}}. \tag{1.4}$$

For any partition, the corresponding matrix is a quasi-summability matrix, which is an upper triangular matrix which has column-sums 1.
For a certain $I_n$ such as $I_n = \{n\}$, $I = (I_n)$, is a partition of positive integers, $l_p(v, I) = l_{I(p)}(v) = l_p(v)$, and $\|x\|_{v,p,I} = \|x\|_{v,I(p)} = \|x\|_{v,p}$.

We write $x \geq 0$ if $x_k \geq 0$ for all $k$. For $p, q \in R\backslash\{0\}$, the lower bound involved hear is the number $L_{v,p,q,B}(A)$, which is defined as the supremum of those $L$ obeying the following inequality:

$$\|Ax\|_{v,B(q)} \geq L\|x\|_{v,B(p)},$$

where $x \geq 0$, $x \in l_{B(p)}(v)$ and $A = (a_{n,k})_{n,k\geq 1}$ is a non-negative matrix operator from $l_{B(p)}(v)$ into $l_{B(q)}(v)$. Also $B = (b_{n,k})_{n,k\geq 1}$ is a non-negative matrix.

In this study, $d\mu$ is a Borel probability measure on $[0, 1]$ and $H_\mu = (h_{n,k})_{n,k\geq 0}$ is the Hausdorff matrix associated with $d\mu$, defined by

$$h_{n,k} = \begin{cases} \binom{n}{k} \int_0^1 \theta^k (1-\theta)^{n-k} d\mu(\theta) & (n \geq k), \\ 0 & (n < k). \end{cases}$$

Clearly, $h_{n,k} = \binom{n}{k} \Delta^{n-k} \mu_k$ for $n \geq k \geq 0$, where

$$\mu_k = \int_0^1 \theta^k d\mu(\theta) \qquad (k = 0, 1, \cdots),$$

and $\Delta\mu_k = \mu_k - \mu_{k+1}$.

The Hausdorff matrix contains some famous classes of matrices. These classes are as follows:
i) Choosing $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1} d\theta$ gives the Cesàro matrix of order $\alpha$;
ii) Choosing $d\mu(\theta) =$point evaluation at $\theta = \alpha$ gives the Euler matrix of order $\alpha$;
iii) Choosing $d\mu(\theta) = |\log\theta|^{\alpha-1}/\Gamma(\alpha) d\theta$ gives the Hölder matrix of order $\alpha$;
iv) Choosing $d\mu(\theta) = \alpha\theta^{\alpha-1} d\theta$ gives the Gamma matrix of order $\alpha$.

The Cesàro, Hölder and Gamma matrices have non-negative entries whenever $\alpha > 0$, and also the Euler matrix has non-negative entries when $0 \leq \alpha \leq 1$.

The study of $L_{p,q}(A)$ goes back to the work of Copson. In [7](see also[[8] Theorem 344]) he proved that $L_{p,q}(C^t(1)) = p$ for $0 < p \leq 1$, where $C(1) = (a_{n,k})_{n,k\geq 0}$ is the Cesàro matrix defined by

$$a_{n,k} = \begin{cases} \frac{1}{n+1} & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases}$$

These results extended by Bennett in many ways (cf, [1],[2],[3],[4]). In particular, in ([3], Theorem 7.18), he proved that

$$L_{p,p}({H_\mu}^t) = \int_0^1 \theta^{-\frac{1}{p^*}} d\mu(\theta) \qquad (0 < p \leq 1), \tag{1.5}$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$. According to [[3], Proposition 7.9], 1.5 also gives

$$L_{p,p}(H_\mu) = \int_0^1 \theta^{-\frac{1}{p}} d\mu(\theta) \qquad (-\infty < p < 0). \tag{1.6}$$

This is a Hardy-type formula (cf. [[4], Eq. (1-8)]). The difference between them is that (1.6) is about $L_{p,p}(H_\mu)$, while Eq. (1-8) in [4] is about $\|H_\mu\|_{p,p}$.

Chen and Wang in [5] proved that $L_{p,p}(H_\mu) = \mu(\{1\})$ and $L_{p,p}(H_\mu{}^t) = \Big((\mu(\{0\})^q + ((\mu(\{1\})^q\Big)^{\frac{1}{q}}$, where $1 < q \leq p \leq \infty$. The case $0 < q \leq 1 \leq p \leq \infty$ is also examined there. Also in [6], they computed the exact values of $L_{p,p}(H_\mu)$ $(0 < p < 1)$ and $L_{p,p}(H_\mu)^t$ $(-\infty < p < 0)$ as follows:

$$L_{p,q}(H_\mu) \geq \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \qquad (0 < q \leq p \leq 1) \tag{1.7}$$

and

$$L_{p,q}(H_\mu{}^t) \geq \int_{(0,1]} \theta^{-\frac{1}{p^*}} d\mu(\theta) \qquad (-\infty < q \leq p < 1).$$

Lashkaripour and G. talebi in [11] proved the following theorem.

**Theorem 1.1.** *( [11], Theorem 2.4.) For the Hausdorff matrix $H_\mu$ and partition $F = (F_n)$ we have*

$$L_{v,p,q,F}(H_\mu) \geq \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \qquad (0 < q \leq p \leq 1). \tag{1.8}$$

*Moreover, the following statements are true:*
*i) For $p = q = 1$, (1.8) is an equality.*
*ii) For $0 < q < p \leq 1$ and $F_n = I_n$, (1.8) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right-hand side of (1.8) is infinity.*

In this paper, we improve and generalize the above-mentioned theorem. Also,we generalize some theorems on $l_p(w, F)$,which have proved by Lashkaripour and Foroutannia to the space $l_{w,B(p)}$.

## 2. New results

**Proposition 2.1.** *Suppose that $0 < p < 1$, and let $A = (a_{n,k})$ and $B = (b_{n,k})$ be two matrices with non-negative entries. If we take*

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} a_{n,k} = R_A, \qquad \inf_{k\geq 1} \sum_{n=1}^{\infty} a_{n,k} = C_A$$

*and*

$$\sup_{i\geq 1} \sum_{j=1}^{\infty} b_{i,j} = R_B, \qquad \inf_{j\geq 1} \sum_{i=1}^{\infty} b_{i,j} = C_B$$

*then for $x \geq 0$, we have*

$$\| Ax \|_{v,B(p)} \geq L \| x \|_{v,p}$$

*with*

$$L \geq (C_B C_A)^{\frac{1}{p}} (R_A R_B)^{\frac{1}{p^*}}.$$

**Proof .** By taking $y_j = (Ax)_j = \sum_{k=1}^{\infty} a_{j,k}x_k$ and applying Hölder's inequality, we have

$$\sum_{k=1}^{\infty} a_{n,k}v_k y_k^p = \sum_{k=1}^{\infty} a_{n,k}^{1-p}(a_{n,k}v_k^{\frac{1}{p}}y_k)^p$$

$$\leq \Big(\sum_{k=1}^{\infty} a_{n,k}\Big)^{1-p}\Big(\sum_{k=1}^{\infty} a_{n,k}v_k^{\frac{1}{p}}y_k\Big)^p$$

$$\leq R_A^{1-p}\Big(\sum_{k=1}^{\infty} a_{n,k}v_k^{\frac{1}{p}}y_k\Big)^p.$$

By similar way

$$\sum_{j=1}^{\infty} b_{i,j}v_j y_j^p \leq R_B^{1-p}\Big(\sum_{j=1}^{\infty} b_{i,j}v_j^{\frac{1}{p}}y_j\Big)^p.$$

Since $v$ is increasing, we have

$$R_A^{1-p}R_B^{1-p}\|Ax\|_{v,B(p)}^p = R_A^{1-p}R_B^{1-p}\Big(\sum_{i=1}^{\infty} v_i(\sum_{j=1}^{\infty} b_{i,j}y_j)^p\Big)$$

$$\geq R_A^{1-p}R_B^{1-p}\Big(\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} b_{i,j}v_j^{\frac{1}{p}}y_j)^p\Big)$$

$$\geq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_{i,j}(\sum_{k=1}^{\infty} a_{j,k}v_k x_k^p)$$

$$= \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_{j,k}(\sum_{i=1}^{\infty} b_{i,j})\Big)v_k x_k^p$$

$$\geq C_B C_A \sum_{k=1}^{\infty} v_k x_k^p,$$

and this leads us to the desired inequality. □

**Remark 2.2.** *By taking $B = I$ and $v_n = 1$ in above statement we obtain the following conclusion due Bennett ([3] Proposition 7.4. ):*
*Fix $p$, $0 < p < 1$, and suppose that $A$ is a matrix with non-negative entries. If $\sup_n \sum_{k=1}^{\infty} a_{n,k} = R$ and $\inf_k \sum_{n=1}^{\infty} a_{n,k} = C$, then $L_{p,q}(A) \geq R^{\frac{1}{p^*}}C^{\frac{1}{p}}$.*

*For $\alpha \geq 0$, let $E(\alpha) = (e_{n,k}(\alpha))_{n,k\geq 1}$ denotes the Euler matrix, defined by*

$$e_{n,k}(\alpha) = \begin{cases} \binom{n-1}{k-1}\alpha^k(1-\alpha)^{n-k} & (n \geq k), \\ 0 & (n < k). \end{cases}$$

*(cf. [6]). For $\Omega \subset (0,1]$, we have*

$$\int_{\Omega} e_{n,k}(\theta) d\mu(\theta) = \mu(\Omega) \times \int_0^1 e_{n,k}(\theta) d\lambda(\theta),$$

*where, $d\lambda = \frac{\chi_\Omega}{\mu(\Omega)} d\mu$ is a Borel probability measure on $[0,1]$ with $\lambda(\{0\}) = 0$. Hence the second part of ([3], Proposition 19.2) can be generalized in the following way.*

**Proposition 2.3.** *Suppose that $0 < p \leq 1, \Omega \subseteq [0,1]$ and $d\mu$ is any Borel probability measure on $[0,1]$. If $\mu(\{0\}) = 0$ or $\Omega \subset (0,1]$, then the sequence $\left\| \left\{ \int_\Omega e_{n,k}(\theta) d\mu(\theta) \right\}_{n=k}^{\infty} \right\|_{v,p}$ increase with respect to $k$.*

**Proposition 2.4.** *Suppose that $0 < p \leq 1$ and $B$ is a matrix with non-negative entries, then for $0 < \alpha \leq 1$, we have*

$$L_{v,B(p)}(E(\alpha)) \geq C_B^{\frac{1}{p}} R_B^{\frac{1}{p^*}} \alpha^{\frac{-1}{p}}.$$

**Proof .** One may easily verifies that $\sum_{k=1}^{\infty} e_{n,k}(\alpha) = 1 (n \geq 1)$ and $\sum_{n=1}^{\infty} e_{n,k}(\alpha) = \alpha^{-1} (k \geq 1)$. Applying Proposition 2.1 to case that $R_A = 1$ and $C_A = \alpha^{-1}$,for $0 < p < 1$, we deduce that

$$L_{v,B(p)}(E(\alpha)) \geq C_B^{\frac{1}{p}} R_B^{\frac{1}{p^*}} \alpha^{\frac{-1}{p}}.$$

For $p = 1$,by the Fubini's theorem and monotonicity of $(v_n)$, we deduce that

$$\begin{aligned}
\|E(\alpha)x\|_{v,B(1)} &= \sum_{i=1}^{\infty} v_i \Big( \sum_{j=1}^{\infty} b_{i,j} y_j \Big) \\
&\geq \sum_{j=1}^{\infty} v_j y_j \Big( \sum_{i=1}^{\infty} b_{i,j} \Big) \\
&\geq C_B \sum_{j=1}^{\infty} v_j \Big( \sum_{k=1}^{\infty} e_{j,k}(\alpha) x_k \Big) \\
&\geq C_B \sum_{k=1}^{\infty} v_k x_k \Big( \sum_{j=1}^{\infty} e_{j,k}(\alpha) \Big) \\
&= C_B \alpha^{-1} \|x\|_{v,1},
\end{aligned}$$

which gives the desired inequality. This completes the proof.
□

**Theorem 2.5.** *By the previous assumptions on $B$ and $v$, we have*

$$L_{v,p,q,B}(H_\mu) \geq C_B^{\frac{1}{q}} R_B^{\frac{1}{q^*}} \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \qquad (0 < q \leq p \leq 1). \tag{2.1}$$

*Moreover, the following statements are true:*
*(i) For $p = q = 1$, (2.1) is an equality, if $B$ is a quasi-summability matrix.*
*(ii) For $0 < q < p \leq 1$ or $B = I$(the identity matrix), (2.1) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right-hand side of 2.1 is infinity.*

**Proof .** Suppose that $x \geq 0$ with $\|x\|_{v,B(p)} = 1$, then $\|x\|_{v,B(q)} \geq \|x\|_{v,B(p)} = 1$. Applying Minkowski's inequality and Proposition 2.3, we have

$$\begin{aligned}
\|H_\mu(x)\|_{v,B(q)} &= \Big(\sum_{n=1}^{\infty} v_n(\sum_{k=1}^{\infty} b_{n,k}(H_\mu(x))_k)^q\Big)^{\frac{1}{q}} \\
&= \Bigg(\sum_{n=1}^{\infty} v_n\Bigg(\sum_{k=1}^{\infty} b_{n,k}\Big(\sum_{j=1}^{\infty}\binom{k-1}{j-1}\int_0^1 \theta^{j-1}(1-\theta)^{k-j}d\mu(\theta)x_k\Big)\Bigg)^q\Bigg)^{\frac{1}{q}} \\
&= \Bigg(\sum_{n=1}^{\infty} v_n\Big(\int_0^1\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} b_{n,k}e_{j,k}(\theta)x_k d\mu(\theta)\Big)^q\Bigg)^{\frac{1}{q}} \\
&\geq \int_0^1 \Bigg(\sum_{n=1}^{\infty} v_n\Big(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} b_{n,k}e_{j,k}(\theta)x_k\Big)^q\Bigg)^{\frac{1}{q}} d\mu(\theta) \\
&= \int_0^1 \|E_(\theta)x\|_{v,B(q)}d\mu(\theta) \\
&\geq C_B^{\frac{1}{q}} R_B^{\frac{1}{q^*}} \|x\|_{v,B(q)} \int_0^1 \theta^{-\frac{1}{q}}d\mu(\theta) \\
&\geq C_B^{\frac{1}{q}} R_B^{\frac{1}{q^*}} \int_0^1 \theta^{-\frac{1}{q}}d\mu(\theta).
\end{aligned}$$

Now, consider (i). Let $e_2 = (0, 1, 0, ...)$, then $e_2 \geq 0$ and $\|e_2\|_{v,B(1)} = v_1b_{2,1} + v_2b_{2,2} = 1$.

$$\begin{aligned}
\|H_\mu e_2\|_{v,B(1)} &= \sum_{n=1}^{\infty} v_n(\sum_{k=1}^{\infty} b_{n,k}h_{k,2}(\theta)) \\
&\geq \int_0^1 \sum_{n=1}^{\infty} e_{n,2}(\theta)d\mu(\theta) \\
&= \int_{(0,1]} \theta^{-1}d\mu(\theta). \\
&\geq C_B \int_{(0,1]} \theta^{-1}d\mu(\theta).
\end{aligned}$$

Hence

$$L_{v,B(1)}(H_\mu) \leq C_B \int_{(0,1]} \theta^{-1}d\mu(\theta).$$

Combining this with (2.1), we obtain (i). Now, consider (ii). Obviously, (2.1) is an equality if its right-hand side is infinity. For the case that $\mu(\{0\}) + \mu(\{1\}) = 1$, we have

$$\|H_\mu e_2\|_{v,B(q)} = \Big(\sum_{n=1}^{\infty} v_n(\sum_{k=1}^{\infty} b_{n,k}h_{k,2}(\theta))^q\Big)^{\frac{1}{q}}$$

$$= \Big(\sum_{n=2}^{\infty} v_n(\sum_{k=1}^{\infty} h_{k,2}(\theta))^q\Big)^{\frac{1}{q}}$$

$$\geq \Big(\sum_{n=2}^{\infty} v_n(\sum_{k=1}^{\infty} h_{k,2}^q(\theta))\Big)^{\frac{1}{q}}$$

$$\geq \Big(\sum_{n=2}^{\infty} v_n h_{n,2}^q(\theta)\Big)^{\frac{1}{q}}$$

$$= \Big(\sum_{n=2}^{\infty} v_n\Big(\binom{n-1}{1}\int_0^1 \theta(1-\theta)^{n-2}d\mu(\theta)\Big)^q\Big)^{\frac{1}{q}}$$

$$= \mu(\{1\}) = \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta).$$

this follows that

$$L_{v,p,q,B}(H_\mu) \leq \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta),$$

so (2.1) is an equality.
Conversely, let $0 < q < p \leq 1$, $B = I$ and assume that $\mu(\{0\}) + \mu(\{1\}) \neq 1$ and also

$$\int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) < \infty,$$

then $\mu((0,1)) \neq 0$. Since $0 < q < 1$, we have

$$\sum_{n=0}^{\infty}(1-\theta)^n < \sum_{n=0}^{\infty}(1-\theta)^{nq}. \quad (\theta \in (0,1)) \tag{2.2}$$

Applying (2.2), Minkowski's inequality and monotonicity of $v$ we have

$$\int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) = \int_{(0,1]} \Big(\sum_{n=1}^{\infty}(1-\theta)^n\Big)^{\frac{1}{q}} d\mu(\theta)$$

$$< \int_{(0,1]} \Big(\sum_{n=1}^{\infty}(1-\theta)^{nq}\Big)^{\frac{1}{q}} d\mu(\theta)$$

$$\leq \Big\|\Big\{\int_{(0,1]}(1-\theta)^n d\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_q$$

$$\leq \Big\|\Big\{\int_{(0,1]}(1-\theta)^n d\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}. \tag{2.3}$$

From 2.3 we can find $0 < \beta < 1$ such that

$$\int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) < \beta\Big\|\Big\{\int_{(0,1]}(1-\theta)^n d\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}. \tag{2.4}$$

We claim that

$$L_{v,p,q,B}(H_\mu) \geq \min\Big\{\beta^{\frac{q-p}{q}}\int_{(0,1]}\theta^{-\frac{1}{q}}d\mu(\theta), \beta\Big\|\Big\{\int_{(0,1]}(1-\theta)^n d\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}\Big\}. \tag{2.5}$$

Let $x \geq 0$, with $\|x\|_{v,B(p)} = 1$. We divide the proof into two cases: $x_{k_0} \geq \beta$ for some $k_0$ or $x_k < \beta$ for all $k$. For the first case, applying Proposition 2.3, it follows that

$$\begin{aligned}
\|H_\mu x\|_{v,B(q)} &= \Big(\sum_{n=1}^{\infty} v_n(\sum_{k=1}^{\infty} b_{n,k}(H_\mu(x))_k)^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{n=1}^{\infty} v_n\Big(H_\mu x\Big)_n^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{n=1}^{\infty} v_n\Big(\sum_{k=1}^{\infty} h_{n,k}x_k\Big)^q\Big)^{\frac{1}{q}} \\
&\geq x_{k_0}\Big(\sum_{n=1}^{\infty} v_n h_{n,k_0}^q\Big)^{\frac{1}{q}} \\
&\geq \beta\Big\|\Big\{\int_{(0,1]} e_{n,k_0}(\theta)d\mu(\theta)\Big\}_{n=k_0}^{\infty}\Big\|_{v,q} \\
&\geq \beta\Big\|\Big\{\int_{(0,1]} e_{n,1}(\theta)d\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q} \\
&= \beta\Big\|\Big\{\int_{(0,1]} (1-\theta)^n d\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}.
\end{aligned}$$

As for the second case,we have

$$x_k^q \geq \beta^{q-p}x_k^p,$$

so

$$\|x\|_{v,q} = \Big(\sum_{k=1}^{\infty} v_k x_k^q\Big)^{\frac{1}{q}} \geq \beta^{\frac{q-p}{q}}\Big(\sum_{k=1}^{\infty} v_k x_k^p\Big)^{\frac{1}{q}} = \beta^{\frac{q-p}{q}}.$$

Applying (2.1), for the case $B = I$, we deduce that

$$\begin{aligned}
\|H_\mu x\|_{v,B(q)} &\geq \Big(\int_{(0,1]}\theta^{-\frac{1}{q}}d\mu(\theta)\Big)\|x\|_{v,B(q)} \\
&= \Big(\int_{(0,1]}\theta^{-\frac{1}{q}}d\mu(\theta)\Big)\|x\|_{v,q} \\
&= \beta^{\frac{q-p}{q}}\Big(\int_{(0,1]}\theta^{-\frac{1}{q}}d\mu(\theta)\Big).
\end{aligned}$$

Hence, $\|H_\mu x\|_{v,B(q)}$ is always grater than or equal to the minimum stated at the right-hand side of (2.5). It is clear that $\beta^{\frac{q-p}{q}} > 1$. Considering (2.4) and (2.5) together, (ii) is obtained.
□

**Corollary 2.6.** *If $F=(F_n)$ is a partition of natural numbers which $N$ is the largest cardinal numbers of $F_n$'s. Then*

$$L_{v,p,q,F}(H_\mu) \geq N^{\frac{1}{q^*}} \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \qquad (0 < q \leq p \leq 1).$$

*So, Theorem 1.1 is improved.*

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