Strong convergence of modified iterative algorithm for family of asymptotically nonexpansive mappings

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Abstract

In this paper we introduce new modified implicit and explicit algorithms and prove strong convergence of the two algorithms to a common fixed point of a family of uniformly asymptotically regular asymptotically nonexpansive mappings in a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Our result is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ and consequently in Sobolev spaces.

Keywords: Fixed point; Banach space; Asymptotically nonexpansive mapping.

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1. Introduction

Let $E$ be a real Banach space and $E^*$ be the dual space of $E$. The normalised duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \|x\| \ \forall \ x \in E \},$$

where $\langle ., . \rangle$ denotes the pairing between the elements of $E$ and those of $E^*$.

Let $S(E) := \{ x \in E : \|x\| = 1 \}$ be the unit sphere of $E$. Then space $E$ is said to have Gâteaux differentiable norm if for any $x \in S(E)$ the limit

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists $\forall y \in S(E)$. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit (1.2) is attained uniformly for $x \in S(E)$.
A mapping $T : E \to E$ is said to be $L$-Lipschitz if there exists a constant $L > 0$ such that
\[ ||Tx - Ty|| \leq L||x - y|| \quad \text{for all } x, y \in E. \] (1.3)
If (1.3) is satisfied with $L \in [0, 1)$, respectively $L = 1$, then the mapping $T$ is called a contraction, respectively nonexpansive. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\rho_n \in [1, \infty)$, $\lim_{n \to \infty} \rho_n = 1$ such that for all $x, y \in K$
\[ ||T^n x - T^n y|| \leq \rho_n||x - y|| \quad \text{for all } n \in N. \] (1.4)
The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] as an important generalization of the class of nonexpansive mappings. Goebel and Kirk [14] proved that if $K$ is a nonempty, bounded, closed and convex subset of a real uniformly convex Banach space and $T$ is a self asymptotically nonexpansive mapping of $K$, then $T$ has a fixed point. $T$ is said to be uniformly $L$-Lipschitzian if there exists $L \geq 0$ such that
\[ ||T^n x - T^n y|| \leq L||x - y||, \forall x, y \in E. \] (1.5)
A point $x \in K$ is called a fixed point of $T$ provided $Tx = x$. We denote by $F(T)$ the set of all fixed point of $T$ (i.e., $F(T) = \{ x \in E : Tx = x \}$). $T$ is said to be demiclosed at $p$ if whenever $\{x_n\}$ is a sequence in $K$ which converges weakly to $x^* \in K$ and $\{Tx_n\}$ converges strongly to $p$, then $Tx^* = p$. It is well known that if $T : K \to K$ is asymptotically nonexpansive, then $T$ is uniformly $L$-Lipschitzian; $(I - T)$ is demiclosed at 0, and $F(T)$ is closed and convex (see for example [15, 22]).

The mapping $T$ is said to be asymptotically regular if
\[ \lim_{n \to \infty} ||T^{n+1}x - T^n x|| = 0 \]
for all $x \in K$. It is said to be uniformly asymptotically regular if for any bounded subset $C$ of $K$,
\[ \lim_{n \to \infty} \sup_{x \in C} ||T^{n+1}x - T^n x|| = 0. \]

Let $C$ be a closed subset of a Hilbert space $H$ and $T$ be a self-nonexpansive mapping. The classical Mann iteration method [20] is given by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1, \] (1.6)
where $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$, has extensively been investigated in literature (see, e.g., [6, 28, 37] and references therein). If the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (1.6) converges weakly to a fixed point of $T$ (this is indeed true in a uniformly convex Banach space with Fréchet differentiable norm [28]). Related works can also be found in [11, 2, 4, 9, 11, 17, 23, 19, 24, 30, 33]. However, this convergence is in general not strong (see the counter example in [12], see also [13]). Attempts to modify the Mann iteration method (1.6) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [29] proposed the following modification of the Mann iteration method
\[
\begin{align*}
  x_0 \in C & \quad \text{chosen arbitrarily} \\
  y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
  C_n &= \{ z \in C : ||y_n - z|| \leq ||x_n - z|| \}, \\
  Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \} \\
  x_{n+1} &= P_{C_n \cap Q_n}(x_0), \quad n \geq 0.
\end{align*}
\] (1.7)
They proved that the sequence \( \{x_n\} \) defined by (1.7) converges strongly to the fixed point of nonexpansive \( T \).

It is worth mentioning that Scheme (1.7) involves computation of closed convex subsets \( C_n \) and \( Q_n \) of \( C \) for each \( n \geq 1 \) and hence is not easy to compute.

In [31], Schu introduced a Mann type process given by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^n x_n, \quad n \geq 1,
\]
(1.8)
to approximate fixed point of asymptotically nonexpansive self-mapping. He proved that, if \( C \) is a nonempty, closed and bounded and \( T \) is completely continuous asymptotically nonexpansive self-mapping with sequence \( \{k_n\} \subset [1, \infty) \), for all \( n \geq 1 \), and \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \) then the sequence \( \{x_n\} \) given by (1.9) converges strongly to some fixed point of \( T \).

Rhoades [25] and Chidume et al. [8] extended the results of Schu [31] to uniformly convex Banach spaces which are more general than Hilbert spaces using a modified Ishikawa iteration method [18] under different settings. In [21], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on \( C \), provided that \( F(T) \neq \emptyset \).

Recently, Chidume et al. [10] proved that, if \( T \) is completely continuous and asymptotically nonexpansive mapping in the intermediate sense with a sequence \( \{\nu_n\} \) such that \( \sum \nu_n < \infty \) with \( F(T) \neq \emptyset \), then, for arbitrary \( x_0 \in C \), the sequence defined by:
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^n x_n, \quad n \geq 1,
\]
(1.9)
where \( \{\alpha_n\} \) is a sequence in \( [\epsilon, 1 - \epsilon] \), for some \( \epsilon > 0 \), converges strongly to some fixed point of \( T \). They also proved weak convergence of the scheme without the assumption that \( T \) is completely continuous.

But it is worth mentioning that in all the above results, either compactness assumption or complete continuity, is imposed on the map \( T \) or the convergence is weak. A natural question arises:

**Question.** Besides the concepts mentioned before, could one construct a new Mann iterative algorithm in order to get strong convergence?

In 2009, Yao et al. [35] introduced a new modified Mann iterative algorithm which is different from those in the literature for a nonexpansive mapping in a real Hilbert space. To be more precise, they proved the following theorem.

**Theorem 1.1.** Let \( C \) be a nonempty, closed and convex subset of a real Hilbert space. Let \( T : C \to C \) be a nonexpansive mapping such that \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be two real sequences in \( (0, 1) \). For \( x_0 \in C \) given arbitrarily, let the sequence \( \{x_n\}, n \geq 0 \) be generated iteratively by
\[
\begin{align*}
  v_n &= P_C[(1 - \alpha_n)x_n], \\
h_n &= (1 - \beta_n)v_n + \beta_nTv_n.
\end{align*}
\]
(1.10)
Suppose that the following conditions are satisfied:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} = \infty \);

(ii) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \);
then the sequence \( \{x_n\} \) generated by (1.10) converges strongly to a fixed point of \( T \).

Recently, Shehu and Ugwunnadi \[36\], extended the result of Yao et al. \[35\] to uniformly convex Banach space which is also uniformly smooth. Under some assumption on \( \{\alpha_n\}, \{\beta_n\} \), they proved that the sequence \( \{x_n\} \) generated by (1.10), under their assumption converges strongly to the unique some fixed point \( T \).

It is our purpose in this paper to modified the algorithm (1.10) and prove strong convergence of both implicit and explicit of the modified algorithm to a common fixed point of a family of uniformly asymptotically regular asymptotically nonexpansive mappings in a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Our result is applicable in \( L_p(\ell_p) \) spaces, \( 1 < p < \infty \) and consequently in sobolev spaces.

2. Preliminaries

Let \( K \) be a nonempty, closed, convex and bounded subset of a Banach space \( E \) and let the diameter of \( K \) be defined by \( d(K) := \sup\{\|x - y\| : x, y \in K\} \). For each \( x \in K \), let \( r(x, K) := \sup\{\|x - y\| : y \in K\} \) and let \( r(K) := \inf\{r(x, K) : x \in K\} \) denote the Chebyshev radius of \( K \) relative to itself. The normal structure coefficient \( N(E) \) of \( E \) (introduced in 1980 by Bynum \[5\], see also Lim \[26\] and the references contained therein) is defined by \( N(E) := \inf\{\frac{d(K)}{r(K)} : K \) is a closed convex and bounded subset of \( E \) with \( d(K) > 0 \}. A space \( E \) such that \( N(E) > 1 \) is said to have uniform normal structure . It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g., \[7, 27\]).

The following lemmas are used for our main result.

**Lemma 2.1.** Let \( E \) be a real normed space. Then

\[
\|x + y\|^2 \leq \|x\|^2 + 2(y, J(x + y)),
\]

for all \( x, y \in E \) and for all \( y(x + y) \in J(x + y) \).

**Lemma 2.2.** (Suzuki \[32\]) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( E \) and let \( \{\beta_n\} \) be a sequence in \([0, 1]\) with \( 0 < \lim \inf \beta_n \leq \lim \sup \beta_n < 1 \). Suppose that \( x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n \) for all integer \( n \geq 1 \) and \( \lim \sup (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).

**Lemma 2.3.** (Xu \[34\]) Let \( \{a_n\} \) be a sequence of nonegative real numbers satisfying the following relation:

\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0
\]

where, (i) \( \{\alpha_n\} \subset [0, 1], \sum \alpha_n = \infty \); (ii) \( \lim \sup \sigma_n \leq 0 \); (iii) \( \gamma_n \geq 0 \); \( n \geq 0 \), \( \sum \gamma_n < \infty \). Then, \( a_n \to 0 \) as \( n \to \infty \).

3. The main results

In the sequel we assume for the sequences \( \{\beta_n\}, \{\sigma_i\} \subset (0, 1) \), that \( \sum_{i \geq 1} \sigma_i := 1 - \beta_n \) for each \( n \in \mathbb{N} \).
Theorem 3.1. Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $E$ with sequences $\{v_n\}$ such that $v_n \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:

1. $\lim \alpha_n = 0$ and $\lim \frac{\alpha_n}{\alpha_{n+1}} = 0$, where $v_n := \sup_{i \geq 1} \{v_n\}$
2. $\sum_{i=0}^{\infty} \alpha_n = \infty$
3. $\beta_n \in [a, b]$ for some $a, b \in (0, 1)$.

For some fixed $\delta \in (0, 1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in C$ chosen arbitrarily,

$$
\begin{align*}
\begin{cases}
  y_n = (1 - \alpha_n)x_n \\
  x_n = [1 - \delta(1 - \beta_n)]y_n + \delta \sum_{i \geq 1} \sigma_{i,n} T_i^n y_n, & n \geq 0.
\end{cases}
\end{align*}
$$

(3.1)

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$.

**Proof.** First, we show that $\{x_n\}$ defined by (3.1) is well defined. For all $n \in \mathbb{N}$, let define the mapping

$$T_n^\delta x := [1 - \delta(1 - \beta_n)](1 - \alpha_n)x + \delta \sum_{i \geq 1} \sigma_{i,n} T_i^n (1 - \alpha_n)x.$$

Indeed, for all $x, y \in E$, we have

$$
\begin{align*}
||T_n^\delta x - T_n^\delta y|| &\leq [1 - \delta(1 - \beta_n)](1 - \alpha_n)||x - y|| \\
&\quad + \delta \sum_{i \geq 1} \sigma_{i,n}||T_i^n (1 - \alpha_n)x - T_i^n (1 - \alpha_n)y|| \\
&\leq [1 - \delta(1 - \beta_n)](1 - \alpha_n)||x - y|| \\
&\quad + \delta(1 - \beta_n)(1 + v_n)(1 - \alpha_n)||x - y|| \\
&\leq [1 - \alpha_n + \delta(1 - \beta_n)v_n]||x - y|| \\
&\leq (1 - \alpha_n[1 - \delta(1 - \beta_n)v_n/\alpha_n])||x - y||.
\end{align*}
$$

Since $\lim_{n \to \infty} \delta(1 - \beta_n)v_n/\alpha_n = 0$, then there exist $n_0 \in \mathbb{N}$ such that $\delta(1 - \beta_n)v_n/\alpha_n < 1/2$ for all $n \geq n_0$. Therefore, for $n \geq n_0$, we have

$$1 - \alpha_n[1 - \delta(1 - \beta_n)v_n/\alpha_n] < 1.$$  

Hence,

$$||T_n^\delta x - T_n^\delta y|| \leq ||x - y||, \quad n \geq n_0.$$  

Thus, $\{x_n\}$ defined by (3.1) is well defined. Therefore, by contraction mapping principle, there exists a unique fixed point $x_n \in E$ of $T_n^\delta$ for each $n \geq 0$ such that (3.1) holds.
Let \( p \in F \), then from (3.1), we obtain
\[
||x_n - p|| \leq [1 - \delta(1 - \beta_n)]||y_n - p|| + \delta \sum_{i \geq 1} \sigma_{in}||T_i^n y_n - p||
\]
\[
\leq [1 - \delta(1 - \beta_n)]||y_n - p|| + \delta(1 - \beta_n)(1 + v_n)||y_n - p||
\]
\[
= [1 + \delta(1 - \beta_n)v_n]||y_n - p||
\]
\[
\leq [1 + \delta(1 - \beta_n)v_n](1 - \alpha_n)||x_n - p|| + \alpha_n||x_n - p||
\]
\[
+ \alpha_n[1 + \delta(1 - \beta_n)v_n]||p||
\]
\[
= [1 - \alpha_n + \delta(1 - \alpha_n)(1 - \beta_n)v_n]||x_n - p||
\]
\[
+ \alpha_n[1 + \delta(1 - \beta_n)v_n]||p||
\]
\[
\leq [1 - \alpha_n + \delta(1 - \beta_n)v_n]||x_n - p||
\]
\[
+ \alpha_n[1 + \delta(1 - \beta_n)v_n]||p||
\]
\[
= \left(1 - \alpha_n[1 - \delta(1 - \beta_n)v_n/\alpha_n]\right)||x_n - p||
\]
\[
+ \alpha_n[1 + \delta(1 - \beta_n)v_n]||p||.
\]
Therefore
\[
||x_n - p|| \leq \frac{1 + \delta(1 - \beta_n)v_n||p||}{1 - \delta(1 - \beta_n)v_n/\alpha_n}.
\]

Since \( \delta(1 - \beta_n)v_n \to 0 \) and \( \delta(1 - \beta_n)v_n/\alpha_n \to 0 \) as \( n \to \infty \), then there exists \( n_0 \in \mathbb{N} \) such that \( \delta(1 - \beta_n)v_n < 1/2 \) and \( \delta(1 - \beta_n)(v_n/\alpha_n) < 1/2 \) respectively for all \( n \geq n_0 \).

Hence \( ||x_n - p|| \leq 3||p|| \), for all \( n \geq n_0 \). Thus \( \{x_n\} \) is bounded, which imply that \( \{y_n\} \) is also bounded. From (3.1), we also obtain that
\[
||y_n - x_n|| = \alpha_n||x_n|| \to 0 \quad \text{as} \quad n \to \infty
\]
which implies
\[
\sum_{i \geq 1} \sigma_{in}||T_i^n y_n - y_n|| = ||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty
\]
hence
\[
||T_i^n y_n - y_n|| = ||x_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty
\]
for each \( i \geq 1 \). Therefore
\[
||T_i^n x_n - x_n|| = ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - y_n|| + ||y_n - x_n||
\]
\[
\leq (2 + v_n)||x_n - y_n|| + ||T_i^n y_n - y_n||
\]
From (3.2) and (3.3), we obtain
\[
||T_i^n x_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \geq 1.
\]
For each \( i \geq 1 \), using the asymptotic regularity of \( T_i \), we obtain
\[
\lim_{n \to \infty} ||T_i^n x_n - x_n|| = \lim_{n \to \infty} ||x_n - T_i^n x_n|| + \lim_{n \to \infty} ||T_i^n x_n - T_i^{n+1} x_n||
\]
\[
+ \lim_{n \to \infty} ||T_i^{n+1} x_n - T_i x_n||
\]
\[
\leq (1 + L) \lim_{n \to \infty} ||x_n - T_i^n x_n||
\]
\[
+ \lim_{n \to \infty} ||T_i^{n+1} x_n - T_i^n x_n|| = 0
\]
(3.5)
where \( L = \sup_{i \geq 1} L_i \), hence

\[
\lim_{n \to \infty} \|y_n - T_i y_n\| \leq \lim_{n \to \infty} \|y_n - x_n\| + \lim_{n \to \infty} \|x_n - T_i x_n\|
\]
\[
+ \lim_{n \to \infty} \|T_i x_n - T_i y_n\|
\]
\[
\leq (1 + L) \lim_{n \to \infty} \|x_n - y_n\|
\]
\[
+ \lim_{n \to \infty} \|x_n - T_i x_n\| = 0
\]

(3.6)

We next show that \( x_n \to p \) (as \( n \to \infty \)). Indeed, define a map \( \phi : E \to \mathbb{R} \) by

\[
\phi(y) := \mu_n \|y_n - y\|^2, \quad \forall y \in E.
\]

Then, \( \phi(y) \to \infty \) as \( \|y\| \to \infty \), \( \phi \) is continuous and convex, so as \( E \) is reflexive, there exists \( q \in E \) such that \( \phi(q) = \min_{u \in E} \phi(u) \). Hence, the set

\[
K^* := \{ y \in E : \phi(y) = \min_{u \in E} \phi(u) \} \neq \emptyset.
\]

Since \( \lim_{n \to \infty} \|y_n - T_i y_n\| = 0 \), \( \lim_{n \to \infty} \|y_n - T_i^m y_n\| = 0 \), for any \( m \geq 1 \) and each \( i \geq 1 \), by induction. Now let \( v \in K^* \), we have

\[
\lim_{n \to \infty} \phi(T_i v) = \lim_{n \to \infty} \mu_n \|y_n - T_i v\|^2
\]
\[
= \lim_{n \to \infty} \mu_n \|y_n - T_i y_n + T_i y_n - T_i v\|^2
\]
\[
\leq \lim_{n \to \infty} \mu_n [(1 + v_n)\|y_n - v\|^2 = \lim_{n \to \infty} \phi(v),
\]

and hence \( T_i v \in K^* \).

Now let \( z \in F \), then \( z = T_i z \). Since \( K^* \) is a closed convex set, there exists a unique \( v^* \in K^* \) such that

\[
\|z - v^*\| = \min_{u \in K^*} \|z - u\|.
\]

But

\[
\lim_{n \to \infty} \|z - T_i v^*\| = \lim_{n \to \infty} \|T_i z - T_i v^*\| \leq \lim_{n \to \infty} (1 + v_n)\|z - v^*\|,
\]

which implies \( v^* = T_i v^* \) and so \( K^* \cap F \neq \emptyset \).

Let \( p \in K^* \cap F \) and \( t \in (0, 1) \), then it follows that \( \phi(p) \leq \phi(p - tp) \) and using Lemma 2.1, we obtain that

\[
\|y_n - p + tp\|^2 \leq \|y_n - p\|^2 + 2t\langle p, j(y_n - p + tp) \rangle
\]

which implies that

\[
\mu_n \langle -p, j(y_n - p + tp) \rangle \leq 0.
\]

Moreover

\[
\mu_n \langle -p, j(y_n - p) \rangle = \mu_n \langle -p, j(y_n - p) - j(y_n - p + tp) \rangle
\]
\[
+ \mu_n \langle -p, j(y_n - p + tp) \rangle \leq \mu_n \langle -p, j(y_n - p) - j(y_n - p + tp) \rangle.
\]
Since $j$ is norm-to-weak* uniformly continuous on bounded subsets of $E$, we have that

$$\mu_n\langle-p, j(y_n-p)\rangle \leq 0.$$  \hfill(3.7)

Since $\delta(1-\beta_n)v_n \rightarrow 0$ and $\delta(1-\beta_n)v_n/\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, if we denote by $w_n$ the value of $2v_n + v_n^2$, it implies that $\delta(1-\beta_n)w_n \rightarrow 0$ and $\delta(1-\beta_n)w_n/\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\delta(1-\beta_n)w_n < 1/2$ and $\delta(1-\beta_n)(w_n/\alpha_n) < 1/2$, for all $n \geq n_0$. From recursion formula (3.1), we obtain

$$||x_n - p||^2 = ||[1 - \delta(1 - \beta_n)](y_n - p) + \delta \sum_{i \geq 1} \sigma_{in}(T_i^n y_n - p)||^2$$

$$\leq [1 - \delta(1 - \beta_n)]||y_n - p||^2 + \delta \sum_{i \geq 1} \sigma_{in}||T_i^n y_n - p||^2$$

$$\leq [1 - \delta(1 - \beta_n)]||y_n - p||^2 + \delta(1 - \beta_n)(1 + v_n)^2||y_n - p||^2$$

$$= [1 - \delta(1 - \beta_n) + \delta(1 - \beta_n)(1 + w_n)]||y_n - p||^2$$

$$= [1 + \delta(1 - \beta_n)w_n][1 - \alpha_n]||y_n - p - \alpha_n p||^2$$

$$\leq [1 + \delta(1 - \beta_n)w_n]\left((1 - \alpha_n)||y_n - p||^2 + 2\alpha_n\langle-p, j(y_n-p)\rangle\right)$$

$$= [1 + \delta(1 - \beta_n)w_n](1 - \alpha_n)||y_n - p||^2$$

$$+ 2\alpha_n[1 + \delta(1 - \beta_n)w_n]\langle-p, j(y_n-p)\rangle$$

$$= [1 - \alpha_n + \delta(1 - \beta_n)w_n](1 - \beta_n)w_n$$

$$+ 2\alpha_n[1 + \delta(1 - \beta_n)w_n]\langle-p, j(y_n-p)\rangle$$

$$\leq \left[1 - \alpha_n \left(1 - \delta(1 - \beta_n)w_n/\alpha_n\right)\right]$$

$$+ 2\alpha_n[1 + \delta(1 - \beta_n)w_n]\langle-p, j(y_n-p)\rangle.$$  

Therefore

$$||x_n - p||^2 \leq \frac{2[1 + \delta(1 - \beta_n)w_n]\langle-p, j(y_n-p)\rangle}{(1 - \delta(1 - \beta_n)w_n/\alpha_n)}$$

hence

$$\mu_n||x_n - p||^2 \leq 3\mu_n\langle-p, j(y_n-p)\rangle.$$  \hfill(3.8)

Therefore, from (3.7) we obtain $\mu_n||x_n - p|| \leq 0$. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$. To complete the proof, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightarrow z$ as $j \rightarrow \infty$, from (3.8) we obtain

$$\mu_n||z - p||^2 \leq 0.$$

which implies that $z = p$ and hence $\{x_n\}$ converges strongly to $p \in F$ as $n \rightarrow \infty$. This complete the proof. \hfill $\Box$

**Theorem 3.2.** Let $E$ be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, $K$ a nonempty closed convex subset of $E$. Let $\{T_i\}_{i=1}^\infty$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of $E$ with sequences $\{v_{in}\}$ such that $v_{in} \rightarrow 0$ as $n \rightarrow \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \frac{v_n}{\alpha_n} = 0$, where $v_n := \sup_{i \geq 1} \{v_{in}\}$ and $\sum_{n=1}^\infty v_n < \infty$
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \sum_{n \geq 1} |\sigma_{i,n+1} - \sigma_{in}| = 0$

(C3) $\sum_{n=1}^{\infty} \beta_n < \infty$

For some fixed $\delta \in (0, 1)$, let \( \{x_n\}_{n=1}^{\infty} \) be a sequence defined iteratively by $x_0 \in C$ chosen arbitrarily,

\[
\begin{align*}
\begin{cases}
y_n = (1 - \alpha_n) x_n \\
x_{n+1} = [1 - \delta(1 - \beta_n)] y_n + \delta \sum_{i \geq 1} \sigma_{in} T_i^n y_n, & n \geq 0.
\end{cases}
\end{align*}
\] (3.9)

Then, \( \{x_n\}_{n=1}^{\infty} \) converges strongly to $p \in F$.

**Proof**. Let $p \in F$ be arbitrary, we obtain from (3.9)

\[
||x_{n+1} - p|| = ||[1 - \delta(1 - \beta_n)](y_n - p) + \delta \sum_{i \geq 1} \sigma_{i,n} (T_i^n y_n - p)||
\]

\[
\leq [1 - \delta(1 - \beta_n)]||y_n - p|| + \delta(1 - \beta_n)(1 + v_n)||y_n - p||
\]

\[
= [1 - \delta(1 - \beta_n) + \delta(1 - \beta_n)(1 + v_n)||y_n - p||
\]

\[
= [1 + \delta(1 - \beta_n) v_n]||(1 - \alpha_n)(x_n - p) - \alpha_n p||
\]

\[
\leq [1 + \delta(1 - \beta_n) v_n] (1 - \alpha_n) ||x_n - p|| + \alpha_n ||p||
\]

\[
\leq [1 + \delta(1 - \beta_n) v_n] \max\{||x_n - p||, ||p||\}
\]

\[
\vdots
\]

\[
\leq \prod_{j=1}^{n} [1 + \delta(1 - \beta_j) v_j] \max\{||x_1 - p||, ||p||\}.
\] (3.10)

Since $\sum_{n=1}^{\infty} v_n < \infty$. it follows from (3.10) that \( \{x_n\} \) is bounded. Hence \( \{y_n\} \) is also bounded. Furthermore, it follows from (3.9) that

\[
||y_n - x_n|| = \alpha_n ||x_n|| \to 0 \quad \text{as} \quad n \to \infty
\] (3.11)

Define two sequences by $\gamma_n := (1 - \delta) \beta_n + \delta$ and $z_n := x_n + (1 - \gamma_n)x_n + v_n x_n$. From the recursion formula (3.9), we observe that

\[
z_n = \frac{[1 - \delta(1 - \beta_n)](y_n - x_n) + \delta \sum_{i \geq 1} \sigma_{in} T_i^n y_n + \beta_n x_n}{\gamma_n}
\]

which implies

\[
z_{n+1} - z_n = \frac{[1 - \delta(1 - \beta_{n+1})](y_{n+1} - x_{n+1}) + \delta \sum_{i \geq 1} \sigma_{i,n+1} T_i^{n+1} y_{n+1} + \beta_{n+1} x_{n+1}}{\gamma_{n+1}}
\]

\[
- \frac{[1 - \delta(1 - \beta_n)](y_n - x_n) + \delta \sum_{i \geq 1} \sigma_{in} T_i^n y_n + \beta_n x_n}{\gamma_n}
\]

\[
= \frac{[1 - \delta(1 - \beta_{n+1})](y_{n+1} - x_{n+1}) - [1 - \delta(1 - \beta_n)](y_n - x_n)}{\gamma_{n+1}}
\]

\[
+ \frac{\delta \sum_{i \geq 1} \sigma_{i,n+1} (T_i^{n+1} y_{n+1} - T_i^n y_n)}{\gamma_{n+1}}
\]

\[
+ \frac{\delta \sum_{i \geq 1} \sigma_{i,n+1} y_{n+1}}{\gamma_{n+1}} - \frac{\delta \sum_{i \geq 1} \sigma_{in} T_i^n y_n}{\gamma_n}
\]

\[
+ \frac{\beta_{n+1} x_{n+1}}{\gamma_{n+1}} - \frac{\beta_n x_n}{\gamma_n}
\]
therefore

\[
||z_{n+1} - z_n|| = \frac{[1 - \delta(1 - \beta_{n+1})]||y_{n+1} - x_{n+1}||}{\gamma_{n+1}} + \frac{[1 - \delta(1 - \beta_n)]||y_n - x_n||}{\gamma_n} + \frac{\delta \sum_{i \geq 1} \sigma_{i,n+1} ||T_i^{n+1} y_{n+1} - T_i^n y_n||}{\gamma_{n+1}} + \frac{\delta \sum_{i \geq 1} \sigma_{i,n+1} ||T_i^{n+1} y_n - T_i^n y_n||}{\gamma_n} + \frac{\delta \sum_{i \geq 1} \sigma_{i,n+1} ||T_i^{n+1} y_n - T_i^n y_n||}{\gamma_n} + \frac{\beta_{n+1} ||x_{n+1}|| + \beta_n ||x_n||}{\gamma_{n+1}} + \frac{\beta_n ||x_n||}{\gamma_n} + \frac{\beta_{n+1} ||x_{n+1}|| + \beta_n ||x_n||}{\gamma_{n+1}} + \frac{\beta_n ||x_n||}{\gamma_n} + \frac{\beta_{n+1} ||x_{n+1}|| + \beta_n ||x_n||}{\gamma_{n+1}} + \frac{\beta_n ||x_n||}{\gamma_n} \]

From (3.13) and (3.12), we obtain

\[
||y_{n+1} - y_n|| = (1 - \alpha_{n+1}) ||x_{n+1} - x_n|| + (\alpha_{n+1} - \alpha_n) ||x_n||.
\]

so that

\[
||y_{n+1} - y_n|| = (1 - \alpha_{n+1}) ||x_{n+1} - x_n|| + |\alpha_{n+1} - \alpha_n| ||x_n||.
\]
for some $M^* > 0$, thus
\[\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0,\]
and by Lemma 2.2 we have
\[\lim_{n \to \infty} \|z_n - x_n\| = 0.\]
Hence
\[\|x_{n+1} - x_n\| = (1 - \gamma_n)\|z_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty.\]
\[\text{(3.14)}\]
From the recursion formula (3.9), we obtain
\[
\delta \sum_{i \geq 1} \sigma_i \|T_i^n y_n - y_n\| = \|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]
\[\text{(3.15)}\]
For each $i \geq 1$, we get
\[
\lim_{n \to \infty} \|T_i^n y_n - y_n\| = 0.
\]
\[\text{(3.16)}\]
Therefore
\[
\|T_i^n x_n - x_n\| \leq \|T_i^n x_n - T_i^n y_n\| + \|T_i^n y_n - y_n\| + \|y_n - x_n\| \\
\leq (2 + v_n)\|x_n - y_n\| + \|T_i^n y_n - y_n\|
\]
From (3.11) and (3.15), we obtain
\[
\|T_i^n x_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \geq 1.
\]
\[\text{(3.17)}\]
For each $i \geq 1$, using the asymptotic regularity of $T_i$, we obtain
\[
\lim_{n \to \infty} \|T_i x_n - x_n\| \leq \lim_{n \to \infty} \|x_n - T_i^n x_n\| + \lim_{n \to \infty} \|T_i^n x_n - T_i^{n+1} x_n\| \\
+ \lim_{n \to \infty} \|T_i^{n+1} x_n - T_i x_n\| \\
\leq (1 + L) \lim_{n \to \infty} \|x_n - T_i^n x_n\| + \lim_{n \to \infty} \|T_i^{n+1} x_n - T_i^n x_n\| = 0
\]
\[\text{(3.18)}\]
For each $m \geq 0$, let $z_m \in E$ be the unique fixed point of the contraction mapping
\[z_m := [1 - \delta(1 - \alpha_m)](1 - \beta_m)z_m + \delta \sum_{i \geq 1} \sigma_{im} T_i^m (1 - \beta_n)z_m\]
on $E$, for $i \geq 1$ (see Theorem 3.1). Then we obtain by letting $y_m = (1 - \alpha_m)z_m$ and $w_m$ denote by $2v_m + v_m^2$

$$||z_m - y_n||^2 = ||[1 - \delta(1 - \beta_m)](y_m - y_n) + \delta \sum_{i \geq 1} \sigma_{im}(T_i^m y_m - y_n)||^2$$

$$\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2$$

$$\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2 + \delta(1 - \beta_m)(1 + v_m)^2||y_m - y_n||^2$$

$$+ 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$\leq [1 - \delta(1 - \beta_m)]||y_m - y_n||^2 + \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$\leq [1 + \delta(1 - \beta_m)w_m]||y_m - y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$\leq [1 + \delta(1 - \beta_m)w_m][1 - \alpha_m](z_m - y_n) - \alpha_m y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$\leq [1 + \delta(1 - \beta_m)w_m][1 - \alpha_m]||z_m - y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$\leq [1 + \delta(1 - \beta_m)w_m](1 - \alpha_m)^2||z_m - y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$+ \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2$$

$$\leq [1 + \delta(1 - \beta_m)w_m][1 - \alpha_m]||z_m - y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$+ \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2$$

$$\leq [1 + \delta(1 - \beta_m)w_m]||y_m - y_n||^2 + 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n||||T_i^m y_n - y_n||$$

$$+ \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_m - y_n||^2$$
\[ \leq [1 + \delta(1 - \beta_m)w_m](1 + \alpha_m^2)||z_m - y_n||^2 \\
+ 2\alpha_m[1 + \delta(1 - \beta_m)w_m]\langle -z_m, j(z_m - y_n) \rangle \\
+ 2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n|||T_i^m y_n - y_n|| \\
+ \delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_n - y_n||^2. \]

Therefore

\[ \langle -z_m, j(y_n - z_m) \rangle \leq \left\{ \delta(1 - \beta_m)(w_m/\alpha_m) + \alpha_m[\delta(1 - \beta_m)w_m] \right\} ||z_m - y_n||^2 \\
+ \frac{2\delta \sum_{i \geq 1} \sigma_{im}(1 + v_m)||y_m - y_n|||T_i^m y_n - y_n||}{2\alpha_m[1 + \delta(1 - \beta_m)w_m]} \\
+ \frac{\delta \sum_{i \geq 1} \sigma_{im}||T_i^m y_n - y_n||^2}{2\alpha_m[1 + \delta(1 - \beta_m)w_m]} \]

Now, taking limit superior as \( n \to \infty \) firstly, and then as \( m \to \infty \), we have

\[ \lim \sup_{m \to \infty} \lim \sup_{n \to \infty} \langle -z_m, j(y_n - z_m) \rangle \leq 0 \quad (3.19) \]

But by Theorem 3.1, \( z_m \to p \) as \( m \to \infty \) and the fact that \( E \) has a uniformly Gâteaux differentiable norm implies that \( j \) is norm-to-weak* uniformly continuous on bounded sets. Thus, since

\[ \langle -p, j(y_n - z_m) \rangle = \langle -p, j(y_n - p) - j(y_n - z_m) \rangle + \langle z_m - p, j(y_n - z_m) \rangle \]

\[ \leq \langle -p, j(y_n - p) - j(y_n - z_m) \rangle + ||z_m - p||||y_n - z_m|| \\
+ \langle z_m, j(y_n - z_m) \rangle \]

we get that

\[ \lim \sup_{n \to \infty} \langle -p, j(y_n - p) \rangle \leq \lim \sup_{m \to \infty} \lim \sup_{n \to \infty} \langle -z_m, j(y_n - z_m) \rangle \]

\[ \leq 0 \]

Finally, we prove that \( x_n \to p \) as \( n \to \infty \). Since \( \delta(1 - \beta_n)v_n \to 0 \) and \( \delta(1 - \beta_n)v_n/\alpha_n \to 0 \) as \( n \to \infty \), if we denote by \( w_n \) the value of \( 2v_n + v_n^2 \), it implies that \( \delta(1 - \beta_n)w_n \to 0 \) and \( \delta(1 - \beta_n)w_n/\alpha_n \to 0 \) as \( n \to \infty \), then there exists \( n_0 \in \mathbb{N} \) such that \( \delta(1 - \beta_n)w_n < 1/2 \) and \( \delta(1 - \beta_n)(w_n/\alpha_n) < 1/2 \), for
all $n \geq n_0$. From recursion formula (3.1), we obtain
\[ ||x_{n+1} - p||^2 = ||[1 - \delta(1 - \beta_n)](y_n - p) + \delta \sum_{i \geq 1} \sigma_i \sigma_i(T_i^n y_n - p)||^2 \]
\[ \leq [1 - \delta(1 - \beta_n)]||y_n - p||^2 + \delta \sum_{i \geq 1} \sigma_i ||T_i^n y_n - p||^2 \]
\[ \leq [1 - \delta(1 - \beta_n)]||y_n - p||^2 + \delta(1 - \beta_n)(1 + v_n)^2||y_n - p||^2 \]
\[ = [1 - \delta(1 - \beta_n) + \delta(1 - \beta_n)(1 + w_n)]||y_n - p||^2 \]
\[ = [1 + \delta(1 - \beta_n)w_n||(1 - \alpha_n)(x_n - p) - \alpha_n p||^2 \]
\[ \leq [1 + \delta(1 - \beta_n)w_n](1 - \alpha_n)||x_n - p||^2 \]
\[ + 2\alpha_n \langle -p, j(y_n - p) \rangle \]
\[ = [1 + \delta(1 - \beta_n)w_n](1 - \alpha_n)||x_n - p||^2 \]
\[ + 2\alpha_n [1 + \delta(1 - \beta_n)w_n] - p, j(y_n - p) \]
\[ = [1 - \alpha_n + \delta(1 - \alpha_n)(1 - \beta_n)w_n] \]
\[ + 2\alpha_n[1 + \delta(1 - \beta_n)w_n] - p, j(y_n - p) \]
\[ \leq \left[ 1 - \alpha_n \left( 1 - \delta(1 - \beta_n)w_n/\alpha_n \right) \right] \]
\[ + \alpha_n \left( 1 - \delta(1 - \beta_n)w_n/\alpha_n \right) \]
\[ \times \frac{2[1 + \delta(1 - \beta_n)w_n] - p, j(y_n - p)}{1 - \delta(1 - \beta_n)w_n/\alpha_n}. \]

Observe that $\sum \alpha_n(1 - \delta(1 - \beta_n)w_n/\alpha_n) = \infty$ and
\[ \limsup_{n \to \infty} \left( \frac{2[1 + \delta(1 - \beta_n)w_n] - p, j(y_n - p)}{1 - \delta(1 - \beta_n)w_n/\alpha_n} \right) \leq 0. \]

Applying Lemma 2.3, we obtain $||x_n - p|| \to 0$ as $n \to \infty$. This completes the proof. □

**Remark 3.3.** By Gossez and Lami [16], we know that if $E$ satisfies Opial’s condition, then $E$ has a weakly continuous duality mapping. Thus, Theorem 3.2 hold in uniformly convex and uniformly smooth Banach spaces which satisfies Opial’s condition and also hold in real Hilbert spaces.

### 4. Numerical example

In this section, we discuss the direct application of Theorem 3.2 with a typical example on real line. Letting $T : C \subseteq E \to C$, then we consider the following:

\[ E = \mathbb{R}, C = [0, 1], Tx = x, \alpha_n = \frac{1}{n + 1}, \beta_n = \frac{1}{2n^2 + 1}, \delta = \frac{1}{2}, \forall n \geq 1 \]

$T$ here is nonexpansive which is particular case of our Theorem. Thus the scheme can be simplified as

\[ x_{n+1} = \left( \frac{n(n^2 + 1)}{(n + 1)(2n^2 + 1)} + \frac{n^{n+2}}{(2n^2 + 1)(n + 1)^n} \right)x_n, \quad n \geq 1. \quad (4.1) \]

Take the initial point $x_1 = 0.5$, the numerical experiment result using MATLAB is given in Figure 1, which shows the iteration process of the sequence $\{x_n\}$ converges to 0.
Figure 1: $x_1 = 0.5$, the convergence process of the sequence $\{x_n\}$ generated by (4.1).

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References


