Int. J. Nonlinear Anal. Appl. **2** (2011) No.2, 1–6
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

HYERS-ULAM AND HYERS-ULAM-RASSIAS STABILITY OF NONLINEAR INTEGRAL EQUATIONS WITH DELAY

JOSÉ R. MORALES[1] AND EDIXON M. ROJAS[2*]

ABSTRACT. In this paper we are going to study the Hyers–Ulam–Rassias types of stability for nonlinear, nonhomogeneous Volterra integral equations with delay on finite intervals.

1. INTRODUCTION

Volterra integral equations have been extensively studied since its appearance in 1896. Part of this interest arises from the wide range of applications where this kind of equations appears, for instance in semiconductors, fluid flow, chemical reactions, elasticity and population dynamic among others (see [2, 5, 9, 12]). An important subject related to the applications is the stability of the equations, where a functional equation is *stable* if for every approximate solution, there exists an exact solution near it. The stability problem of functional equations originated from a question of Ulam concerning the stability of group homomorphisms [14]: *given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \longrightarrow G'$ satisfies*

$$\rho(f(xy), f(x)f(y)) < \delta \quad \textit{for all } x, y \in G,$$

then a homomorphism $h : G \longrightarrow G'$ exists with $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$?. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces, he proved that each solution of the inequality $\|f(x + y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function (*Hyer-Ulam stability*). Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. More precisely, he attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows: $\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$ and proved the Hyers theorem (*Hyers-Ulam-Rassias stability*). The terminologies Hyers-Ulam stability and Hyers-Ulam-Rassias stability can also be applied to the case of other functional equations, differential equations, and of various integral equations. The paper of Rassias has provided a lot of influence in the development of what is called *generalized Hyers-Ulam-Rassias stability of functional equations* (see [6]).

Date: Received: April 2011; Revised: Jun 2011.
2000 *Mathematics Subject Classification.* Primary 45D05, 26D10; Secondary 34K20,47H10.
Key words and phrases. Hyers–Ulam–Rassias stability.
*: Corresponding author.

2. The nonhomogeneous nonlinear Volterra integral equations with delay

In this paper we are going to consider the following class of nonhomogeneous nonlinear integral equations with a delay:

$$u(x) = f(x) + \Psi\left(\int_a^x \Phi(x,t,u(t),u(\alpha(t)))dt\right) \equiv (Tu)(x), \tag{2.1}$$

where $-\infty < a \leq x \leq b < +\infty$ with a, b fixed. $f(x)$ is a complex-valued continuous function on $[a,b]$ and $\Phi(x,t,u(t),u(\alpha(t)))$ is continuous with respect to the three variables x, t and u on $[a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C}$ satisfying the Lipschitz condition

$$|\Phi(x,t,u(t),u(\alpha(t))) - \Phi(x,t,v(t),v(\alpha(t)))| \leq L|u(t)-v(t)|.$$

Here, $\alpha : [a,b] \longrightarrow [a,b]$ is a continuous delay function which therefore fulfill $\alpha(t) \leq x$ for all $t \in [a,b]$. The above integral equation (2.1) will be considered on the complete metric space of complex-valued continuous functions on the interval $[a,b]$, $X := (C[a,b], d)$ where, as usual, $d(f,g) = \sup\{|f(x)-g(x)| \ : \ x \in [a,b]\}$ and furthermore we will assume that Ψ is a Banach contraction mapping on X. I.e.,

$$d(\Psi(f),\Psi(g)) \leq Kd(f,g), \quad 0 \leq K < 1.$$

The formal definitions of the above-mentioned two types of stability for the case of equation (2.1) can be defined as follows. If for each function u satisfying

$$\left| u(x) - f(x) - \Psi\left(\int_a^x \Phi(x,t,u(t),u(\alpha(t)))dt\right)\right| \leq \sigma(x)$$

(where σ is a nonnegative function), there is a solution u_0 of the nonlinear Volterra integral equation (2.1) and a constant $C_1 > 0$ independent of u and u_0 such that

$$|u(x) - u_0(x)| \leq C_1\sigma(x),$$

for all x, then we say that the nonlinear integral equation with delay (2.1) has the *Hyers-Ulam-Rassias stability*. In the case where σ takes the form of a constant function, we say that the integral equation (2.1) has the *Hyers-Ulam stability*.

Despite the large amount of works on Volterra integral equations, the interest on this kind of stability of these integral equations is quite recent, see [3, 4, 7, 8, 11].

2.1. On the existence of the solution of the Nonlinear Volterra integral equations with delay.

First of all, we are going to prove the existence and uniqueness of the solution of the integral equations (2.1).

Theorem 2.1. *Let a, b fixed real numbers $-\infty < a < b < +\infty$, $K \in [0,1)$ and $L \in (0,+\infty)$ such that $KL < \frac{1}{b-a}$. Let $\alpha : [a,b] \longrightarrow [a,b]$ be a continuous function such that*

$$\alpha(x) \leq x \quad \textit{for all } x \in [a,b]$$

and $f : [a,b] \longrightarrow \mathbb{C}$ be a continuous function. Assume furthermore that $\Phi : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|\Phi(x,t,u(t),u(\alpha(t))) - \Phi(x,t,v(t),v(\alpha(t)))| \leq L|u(t)-v(t)|,$$

and that Ψ is a Banach contraction on X with contraction constant K. Then there is one and only one solution u of

$$u(x) = f(x) + \Psi\left(\int_a^x \Phi(x,t,u(t),u(\alpha(t)))dt\right)$$

on $[a,b]$.

Proof. Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \Psi\left(\int_a^x \Phi(x,t,u_n(x),u_n(\alpha(x)))dt\right) \equiv (Tu_n)(x), \quad n = 1,2,\dots. \tag{2.2}$$

Since Ψ is a Banach contraction mapping on X and $\Phi(x,t,u(x),u(\alpha(x)))$ is assumed Lipschitz on u, then we have

$$\begin{aligned}
&|u_{n+1}(x) - u_n(x)| = \\
&\left|\Psi\left(\int_a^x \Phi(x,t,u_n(t),u_n(\alpha(t)))dt\right) - \Psi\left(\int_a^x \Phi(x,t,u_{n-1}(t),u_{n-1}(\alpha(t)))dt\right)\right| \\
&\le K \sup_{x\in[a,b]} \left|\int_a^x \Phi(x,t,u_n(t),u_n(\alpha(t)))dt - \int_a^x \Phi(x,t,u_{n-1}(t),u_{n-1}(\alpha(t)))dt\right| \\
&\le KL \sup_{x\in[a,b]} \int_a^x |u_n(t) - u_{n-1}(t)|dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|u_{n+1}(x) - u_n(x)| \le KL \sup_{x\in[a,b]} \int_a^x |u_n(t_1) - u_{n-1}(t_1)|dt_1 \\
&\le (KL)^2 \sup_{x\in[a,b]} \int_a^x \sup_{t_1\in[a,b]} \int_a^{t_1} |u_{n-1}(t_2) - u_{n-2}(t_2)|dt_2 \\
&\vdots \\
&\le (KL)^n \sup_{x\in[a,b]} \int_a^x \dots \sup_{t_{n-2}\in[a,b]} \int_a^{t_{n-2}} |u_2(t_{n-1}) - u_1(t_{n-1})|dt_{n-1}\dots dt_1.
\end{aligned}$$

Therefore, $|u_{n+1}(x) - u_n(x)| \le (KL)^n (b-a)^n d(Tu_1, u_1)$. Since X is a complete metric space, and $KL < \frac{1}{b-a}$, then we conclude by using the Weierstrass M-test that

$$\sum_{n=1}^{+\infty} (u_{n+1}(x) - u_n(x))$$

is absolutely and uniformly convergent on $[a,b]$. Due to the fact that $u_n(x)$ can be written as

$$u_n(x) = u_1(x) + \sum_{k=1}^{n-1} (u_{k+1}(x) - u_k(x)),$$

so there exists a unique solution $u \in X$ such that $\lim_{n\to+\infty} u_n = u$. Taking limit of both sides of (2.2), we obtain

$$\begin{aligned} u(x) = \lim_{n\to+\infty} u_{n+1}(x) = & \lim_{n\to+\infty}\left(f(x)+\Psi\left(\int_a^x \Phi(x,t,u_n(t),u_n(\alpha(t)))dt\right)\right) \\ =& f(x)+\Psi\left(\int_a^x \Phi(x,t,\lim_{n\to+\infty} u_n(t),\lim_{n\to+\infty} u_n(\alpha(t)))dt\right) \\ =& f(x)+\Psi\left(\int_a^x \Phi(x,t,u(t),u(\alpha(t)))dt\right). \end{aligned}$$

Therefore, the limit function u is the unique solution $u \in X$ such that $Tu = u$. □

3. Hyers-Ulam and Hyers-Ulam-Rassias stability for the nonhomogeneous Volterra equations with delay

In section we are going to prove that under the conditions of Theorem 2.1, the class of Volterra integral equations with delay (2.1) has both the Hyers-Ulam and the Hyers-Ulam-Rassias stability.

Theorem 3.1. *Under the assumptions of Theorem 2.1, the equation $Tu = u$, where T is defined by (2.1), has the Hyers-Ulam stability; that is, for every $\varphi \in X$ and $\epsilon > 0$ with*

$$|T\varphi - \varphi| \leq \epsilon$$

there exists a unique $u \in X$ such that

$$Tu = u,$$

$$|\varphi - u| \leq C\epsilon$$

for some $C \geq 0$.

Proof. Let $\varphi \in X$, $\epsilon > 0$ and $|T\varphi - \varphi| \leq \epsilon$. As was proved in Theorem 2.1,

$$u(t) = \lim_{n\to+\infty}(T^n\varphi)(t)$$

is an exact solution of the equation $Tx = x$. Since $T^n\varphi$ converges uniformly to u as $n \to +\infty$, then there is a natural number N such that $|T^N\varphi - u| \leq \epsilon$. Thus,

$$\begin{aligned} |\varphi - u| \leq& |\varphi - T^N\varphi| + |T^N\varphi - u| \\ \leq& |\varphi - T\varphi| + |T\varphi - T^2\varphi| + \cdots + |T^{N-1}\varphi - T^N\varphi| + |T^N\varphi - u| \\ \leq& d(\varphi,T\varphi) + \kappa d(\varphi,T\varphi) + \cdots + \kappa^{n-1}d(\varphi,T\varphi) + \epsilon \\ \leq& (1+\kappa+\kappa^2+\cdots+\kappa^{N-1})\epsilon + \epsilon \\ \leq& \frac{\epsilon}{1-\kappa} + \epsilon = \left(\frac{2-\kappa}{1-\kappa}\right)\epsilon \end{aligned}$$

where $\kappa = LK(b-a)$. This complete the proof. □

Corollary 3.2. *Theorem 3.1 holds for every finite interval $[a,b]$, $[a,b)$, $(a,b]$ and (a,b) when $-\infty < a < b < +\infty$.*

Theorem 3.3. *Under the assumptions of Theorem 2.1, the equation $Tu = u$, where T is defined by (2.1), has the hyers-Ulam-Rassias stability; that is, for every $\varphi \in X$ and $\sigma(x) > 0$ for all $x \in [a, b]$ with*

$$|T\varphi - \varphi| \leq \sigma(x),$$

there exists a unique $u \in X$ such that

$$Tu = u,$$

$$|\varphi - u| \leq C_1\sigma(x)$$

for some $C_1 > 0$.

Proof. Let $\varphi \in X$ and σ a nonnegative function on $[a, b]$ such that

$$|T\varphi - \varphi| \leq \sigma(x).$$

In addition, let $u \in X$ the unique solution of the Volterra equation with delay (2.1) on X. Then, we have

$$\begin{aligned} |\varphi - u| \leq& |\varphi - T\varphi| + |T\varphi - u| \\ \leq& \sigma(x) + |T\varphi - u|. \end{aligned} \tag{3.1}$$

On the other hand, notice that

$$\begin{aligned} &|T\varphi - Tu| = |T\varphi - u| = \\ &\left|\Psi\left(\int_a^x \Phi(x, t, \varphi(t), \varphi(\alpha(t))dt\right) - \Psi\left(\int_a^x \Phi(x, t, u(t), u(\alpha(t))dt\right)\right| \\ &\leq K\left|\int_a^x \Phi(x, t, \varphi(t)\varphi(\alpha(t))dt - \int_a^x \Phi(x, t, u(t), u(\alpha(t))dt\right| \\ &\leq KL\int_a^x |\varphi(t) - u(t)|dt. \end{aligned}$$

Thus, we obtain that

$$|T\varphi - u| \leq KL(b - a)d(\varphi, u). \tag{3.2}$$

Therefore, from inequalities (3.1) and (3.2) we conclude that

$$|\varphi - u| \leq d(\varphi, u) \leq \sigma(x) + KL(b - a)d(\varphi, u)$$

which implies then

$$|\varphi - u| \leq d(\varphi, u) \leq C_1\sigma(x)$$

with $C_1 = \frac{1}{1-KL(b-a)}$. I.e., the equation (2.1) has the Hyers-Ulam-Rassias stability. □

Corollary 3.4. *Theorem 3.3 holds for every finite interval $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) when $-\infty < a < b < +\infty$.*

Proposition 3.5. *For infinite intervals, Theorems 3.1 and 3.3 are not necessarily true.*

Proof. Let us consider the function $\Phi(x,t,u(t),u(\alpha(t))) = 2(u(t)+u(\alpha(t)))$ with the delay function $\alpha(t) = -t$ for $t \in [0,+\infty)$. Let be $f(x) = e^{-x}$ and $\Psi : X \longrightarrow X$ the mapping defined by $\Psi(g) = \frac{1}{2}g$. The exact solution of the equation

$$u(x) = e^{-x} + \frac{1}{2}\int_0^x 2(u(t)+u(\alpha(t)))dt, \quad \text{on } [0,+\infty)$$

is the function $u(x) = e^x$. Moreover, notice that the functions above satisfy the conditions of Theorem 2.1. However, by choosing $\epsilon = 1$ and $y(x) = 0$, we get $T(y) = e^{-x}$, so $d(Ty,y) \le \epsilon = 1$ but on the other hand $d(y,u) = +\infty$, therefore there exists no Hyers-Ulam stability constant $C \ge 0$ such that $d(y,u) \le C\epsilon$ is true. Notice that the same argument can be applied for the case of the study the Hyers-Ulam-Rassias stability. □

References

1. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, vol. 2, 64–66, 1950. 1
2. T.A. Burton, *Volterra Integral and Differential Equations* (2nd ed.), Mathematics in Science and Engineering 202, Elseiver, Amsterdam, 2005. 1
3. L.P. Castro and A. Ramos, *Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations*, Banach J. Math. Anal., **3**, No 1, 36–43, 2009 2
4. L.P. Castro and A. Ramos, *Hyers-Ulam and Hyers-Ulam-Rassias stability of Volterra integral equations with delay*. Integral methods in science and engineering. Vol. 1, 85–94, Birkhäuser Boston, Inc., Boston, MA, 2010 2
5. C. Corduneanu, *Principles of Differential and Integral Equations* (2nd ed.), Chelsea, New York, 1988. 1
6. G.L. Fortis, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50**, 143–190, 1995. 1
7. M. Gachpazan and O. Baghani, *Hyers-Ulam stability of nonlinear integral equation*, Fixed Point Theory and Applications, Vol 2010, Article ID 927640, 6 pages, 2010. 2
8. M. Gachpazan and O. Baghani, *Hyers-Ulam stability of Volterra integral equation*, J. Nonlinear Anal. Appl., No 2, 19–25, 2010. 2
9. G. Gripenberg, S.O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990. 1
10. D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Aca. Sci., vol. 27, 222–224, 1941. 1
11. S.M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications, Vol 2007, Article ID 57064, 9 pages, 2007. 2
12. V. Lakshmikantham and M.R.M. Rao, *Thoery of Integro-differential Equations*, Stability and Control: Theory, Methods and Applications 1, Gordon and Breach Publ. Philadelphia, 1995. 1
13. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., vol. 72, no. 2, 297–300, 1978. 1
14. S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960. 1

[1] Departamento de Matemáticas, Universidad de Los Andes, Mérida, Venezuela.
E-mail address: moralesj@ula.ve

[2] Departamento de Matemáticas, Pontificia Universidad Javeriana, Bogotá, Colombia.
E-mail address: edixon.rojas@javeriana.edu.co