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A generalization of Martindale's theorem to (α, β)-homomorphism

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Abstract

Martindale proved that under some conditions every multiplicative isomorphism between two rings is additive. In this paper, we extend this theorem to a larger class of mappings and conclude that every multiplicative $(\alpha, \beta)-$derivation is additive.

Keywords: (α, β)-multiplicative mapping; (α, β)-multiplicative isomorphism; (α, β)-additive mapping; multiplicative $(\alpha, \beta)-$derivations.
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1. Introduction and preliminaries

The question that when a multiplicative isomorphism is additive has been considered by Rickart [8] and Johnson [6]. In 1968 Martindale [7] proved an extension of Rickart's theorem [8]. He proved that under some conditions on a ring $\mathcal{R}$, every multiplicative isomorphism from $\mathcal{R}$ into another ring $\mathcal{S}$ is additive. In addition, the question that when a multiplicative derivation is additive has been investigated by Daif [2]. The authors of [5] extended Daif's theorem to multiplicative $(\alpha, \beta)-$derivations. we give the definition of (α, β)-homomorphism for the first time to extened the concept of homomorphism to a larger class of mappings and then similar to the generalized Daif theorem by Hou, Zhang and Meng [5], we extend Martindale's theorem to $(\alpha, \beta)-$isomorphism and then as a special case we deduce Hou, Zhang and Meng theorem with similar conditions.

Throughout this paper, $\mathcal{R}$ and $\mathcal{S}$ are arbitrary associative rings (not necessarily with identity element). A mapping $\sigma : \mathcal{R} \to \mathcal{S}$ is called multiplicative, if $\sigma(xy) = \sigma(x)\sigma(y)$, for each $x, y \in \mathcal{R}$.

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It is called as a multiplicative isomorphism if in addition it is one to one and onto. A mapping $d:\mathcal{R}\to\mathcal{R}$ is a multiplicative derivation, if for each $x,y\in\mathcal{R}$, we have $d(xy)=d(x)y+xd(y)$. If α and β are automorphisms of $\mathcal{R}$, then a multiplicative $(\alpha,\beta)-$ *derivation* from $\mathcal{R}$ into itself is a mapping $d:\mathcal{R}\to\mathcal{R}$ such that $d(xy)=d(x)\alpha(y)+\beta(x)d(y)$, for each $x,y\in\mathcal{R}$. For further results see ([3, 4]).
In the rest section, first of all note that since the definition of $(\alpha,\beta)-$homomorphism for the first time this section is new and without history so we not provided any reference in it.

Definition 1.1. Suppose that $\alpha:\mathcal{R}\to\mathcal{R}$ and $\beta:\mathcal{S}\to\mathcal{S}$ are arbitrary mappings. Mapping $\sigma:\mathcal{R}\to\mathcal{S}$ is called an $(\alpha,\beta)-$multiplicative mapping, if $\sigma(xy)=\beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))$, for each $x,y\in\mathcal{R}$. In addition, if it is one to one and onto, then is called an $(\alpha,\beta)-$multiplicative isomorphism. It will be called an $(\alpha,\beta)-$additive mapping if $\beta(\sigma(\alpha(x+y)))=\beta(\sigma(\alpha(x)))+\beta(\sigma(\alpha(y)))$, for each $x,y\in\mathcal{R}$.

Remark 1.2. (i) If $\mathcal{R}$ and $\mathcal{S}$ are unital and α,β,σ are unitary (i.e, $\alpha(1_{\mathcal{R}})=1_{\mathcal{R}}$ and $\sigma(1_{\mathcal{R}})=\beta(1_{\mathcal{S}})=1_{\mathcal{S}}$), then every $(\alpha,\beta)-$multiplicative mapping $\sigma:\mathcal{R}\to\mathcal{S}$ is a multiplicative mapping of $\mathcal{R}$ into $\mathcal{S}$. In fact, by putting $y=1$ in the definition of $(\alpha,\beta)-$multiplicative mapping, we have $\sigma(x)=\beta(\sigma(\alpha(x)))$ for each $x\in\mathcal{R}$. Hence $\sigma(xy)=\beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))=\sigma(x)\sigma(y)$.

(ii) If $\mathcal{R}$ and $\mathcal{S}$ are unital and $\beta(1_{\mathcal{S}})=\sigma(1_{\mathcal{R}})=1_{\mathcal{S}}$, and σ is onto, then $\alpha=I_{\mathcal{R}}$, implies that $\beta=I_{\mathcal{S}}$. In fact by putting $y=1$ in the definition of $(\alpha,\beta)-$multiplicative mapping we have $\beta(\sigma(x))=\sigma(x)$ for each $x\in\mathcal{R}$. So $\beta(z)=z$ for each $z\in\mathcal{S}$.

(iii) If $\mathcal{R}$ and $\mathcal{S}$ are unital and σ, α are unitary, σ is one to one and $\beta=I_{\mathcal{S}}$, then $\alpha=I_{\mathcal{R}}$. Putting $y=1$ in the definition of $(\alpha,\beta)-$multiplicative mapping. Then $\sigma(\alpha(x))=\sigma(x)$, which implies that $\alpha(x)=x$ for each $x\in\mathcal{R}$.

(iv) If α and β are multiplicative and idempotents ($\alpha^2=\alpha,\quad \beta^2=\beta$) and σ is an $(\alpha,\beta)-$multiplicative, then $\sigma'=\beta o\sigma o\alpha$, is multiplicative. Since

$$\begin{aligned}\sigma'(xy)=\beta(\sigma(\alpha(xy))) &= \beta(\sigma(\alpha(x)\alpha(y)))\\ &= \beta[\beta(\sigma(\alpha(\alpha(x))))\beta(\sigma(\alpha(\alpha(y))))]\\ &= \beta(\beta(\sigma(\alpha(\alpha(x)))))\beta(\beta(\sigma(\alpha(\alpha(y)))))\\ &= \beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))=\sigma'(x)\sigma'(y).\end{aligned}$$

Note that in this case we have $\sigma'(xy)=\sigma'(x)\sigma'(y)=\beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))=\sigma(xy)$.
If $\mathcal{R}$ is a Banach algebra with a bounded left approximate identity, in particular if $\mathcal{R}$ is a C^*-algebra, then by Cohen's factorization theorem $\mathcal{R}^2=\mathcal{R}$ [1]. So we have $\sigma'=\sigma$.

(v) If $\alpha:\mathcal{R}\to\mathcal{R}$ and $\beta:\mathcal{S}\to\mathcal{S}$ and $\sigma:\mathcal{R}\to\mathcal{S}$ are multiplicative and in addition σ is an $(\alpha,\beta)-$multiplicative, then $\sigma(xyz)=\beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))\beta(\sigma(\alpha(z)))$. In fact

$$\begin{aligned}\sigma(xyz)=\sigma((xy)z) &= \beta(\sigma(\alpha(xy)))\beta(\sigma(\alpha(z)))\\ &= \beta(\sigma(\alpha(x)\alpha(y)))\beta(\sigma(\alpha(z)))\\ &= \beta(\sigma(\alpha(x)\sigma(\alpha(y))))\beta(\sigma(\alpha(z)))\\ &= \beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))\beta(\sigma(\alpha(z))).\end{aligned}$$

(vi) If $\mathcal{R}=\mathcal{S}$ and $\sigma=I$, then obviously in each of the following cases, we have a $(\alpha,\beta)-$ *multiplicative* mapping.

(a) $\alpha=\beta=I$. (b) $\beta=\alpha^{-1}$ or $\beta=-\alpha^{-1}$.

(vii) If σ is a multiplicative mapping, then obviously in each of the following cases we have a $(\alpha,\beta)-$ *multiplicative* mapping.

(a) $\alpha=\sigma^{-1},\quad \beta=\sigma$. (b) $\alpha=\sigma,\quad \beta=\sigma^{-1}$ or $\beta=-\sigma^{-1}$.

Example 1.3. (i) If $\mathbb{R}$ is the set of real numbers and $\sigma, \alpha, \beta : \mathbb{R} \to \mathbb{R}$ are mappings, then in each of the following cases we have a $(\alpha, \beta) - multiplicative$ mapping.
(a) $\sigma(x) = 4x, \quad \alpha(x) = 3x, \quad \beta(x) = \frac{1}{6}x$. In this case we have
$\sigma(xy) = 4xy = (2x)(2y) = \frac{4(3x)}{6}.\frac{4(3y)}{6} = \beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))$
(b) $\sigma(x) = x, \quad \alpha(x) = sinx, \quad \beta(x) = sin^{-1}x.$
(c) $\sigma(x) = x^2, \quad \alpha(x) = \sqrt{|x|}, \quad \beta(x) = x^2.$
(ii) Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ be the ring of integer numbers module 5 and $\sigma : \mathbb{Z}_5 \longrightarrow \mathbb{Z}_5$, $\alpha : \mathbb{Z}_5 \to \mathbb{Z}_5$, $\beta : \mathbb{Z}_5 \to \mathbb{Z}_5$ defined by $\sigma(x) = 4x, \quad \alpha(x) = 3x, \quad \beta(x) = 4x$, then σ is a $(\alpha, \beta) - multiplicative$ mapping. In fact $\sigma(xy) = 4xy = (3x)(3y) = \beta(\sigma(\alpha(x)))\beta(\sigma(\alpha(y)))$.

2. The main results

In this section, we generalized the Martindale's theorem [7] to (α, β)-isomorphisms.

Theorem 2.1. Suppose that $\mathcal{R}$ is a ring containing a family $\{e_\alpha\}_{\alpha \in A}$ of idempotents, such that for each $x \in \mathcal{R}$ satisfies the following conditions:

$$(i) \ \ x\mathcal{R} = 0 \quad \text{implies} \quad x = 0;$$

$$(ii) \ \ e_\alpha \mathcal{R} x = 0 \quad \text{for each} \quad \alpha \in A \quad \text{implies} \quad x = 0;$$

$$(iii) \ \ e_\alpha x e_\alpha R(1 - e_\alpha) = 0 \quad \text{implies} \quad e_\alpha x e_\alpha = 0, \quad \text{for every} \quad \alpha \in A.$$

Suppose that $\mathcal{S}$ is an arbitrary ring, $\alpha : \mathcal{R} \to \mathcal{R}$ and $\beta : \mathcal{S} \to \mathcal{S}$ are bijective and $\alpha_0 \in A$. If $(\alpha, \beta)-$multiplicative isomorphism $\sigma : \mathcal{R} \to \mathcal{S}$ satisfies the following conditions, then it is $(\alpha, \beta)-$additive.

$$(iv) \ \ \beta o \sigma o \alpha(e_{\alpha_0} xy) = \beta o \sigma o \alpha(e_{\alpha_0} x) \beta o \sigma o \alpha(y);$$

$$(v) \ \ \beta o \sigma o \alpha(xy e_{\alpha_0}) = \beta o \sigma o \alpha(x) \beta o \sigma o \alpha(y e_{\alpha_0});$$

$$(vi) \ \ \sigma(xz) + \sigma(yz) = \sigma((x + y)z);$$

$$(vii) \ \ \sigma(zx) + \sigma(zy) = \sigma(z(x + y));$$

for each $x, y, z \in \mathcal{R}$.

Note that under condition 2.2.(iv) ,(iv) holds and (iv) implies that :

$$\beta o \sigma o \alpha(e_{\alpha_0} x) = \beta o \sigma o \alpha(e_{\alpha_0} e_{\alpha_0} x) = \beta o \sigma o \alpha(e_{\alpha_0} e_{\alpha_0}) \beta o \sigma o \alpha(x) = \beta o \sigma o \alpha(e_{\alpha_0}) \beta o \sigma o \alpha(x).$$

Similarly $\beta o \sigma o \alpha(x e_{\alpha_0}) = \beta o \sigma o \alpha(x) \beta o \sigma o \alpha(e_{\alpha_0})$.

The proof of the theorem will be organized in a series of lemmas. We assume that the hypothesis of theorem as needed during the proof. First we begin with the trivial lemma.

Lemma 2.2. $\sigma(0) = 0$.

Proof . Since $\beta o\sigma o\alpha$ is onto, there is $x \in \mathcal{R}$ such that $\beta o\sigma o\alpha(x) = 0$. Hence
$\sigma(0) = \sigma(0.x) = \beta(\sigma(\alpha(0)))\beta(\sigma(\alpha(x))) = \beta(\sigma(\alpha(0))).0 = 0$.
□

For the rest lemma, fix $\alpha_0 \in A$ and set $e_{\alpha_0} = e_1, \quad e_2 = 1 - e_1$.
We will use e_2x, in place of $x - e_1x$.
Take $\mathcal{R}_{ij} = e_i\mathcal{R}e_j, \quad (i, j = 1, 2)$, then we may write $\mathcal{R}$ in the following decomposition
$\mathcal{R} = \mathcal{R}_{11} \bigoplus \mathcal{R}_{12} \bigoplus \mathcal{R}_{21} \bigoplus \mathcal{R}_{22}$.
In fact, $x = (e_1 + (1 - e_1))x(e_1 + (1 - e_1)) = e_1xe_1 + e_1x(1 - e_1) + (1 - e_1)xe_1 + (1 - e_1)x(1 - e_1))$, shows that $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$, and the later sum is a direct sum. Because for instance $z \in \mathcal{R}_{11} \cap \mathcal{R}_{12}$, we have $z = e_1xe_1 = e_1y(1 - e_1)$, for some $x, y \in \mathcal{R}$. Therefore

$$e_1xe_1 = e_1y - e_1ye_1$$

$$e_1(e_1xe_1)e_1 = e_1(e_1y - e_1ye_1)e_1$$

$$e_1xe_1 = e_1ye_1 - e_1ye_1 = 0$$

So $z = e_1xe_1 = 0$.
We denote an element of $\mathcal{R}_{ij}$ by x_{ij}. Since $e_1e_2 = e_1(1 - e_1) = e_1 - e_1^2 = 0$, we have $e_jx_{kl} = 0$ and $x_{ij}x_{kl} = 0, \quad (i, j, k, l = 1, 2, j \neq k)$.

Lemma 2.3. *$\beta(\sigma(\alpha(x_{ii} + x_{jk}))) = \beta(\sigma(\alpha(x_{ii}))) + \beta(\sigma(\alpha(x_{jk})))$ for each $j \neq k$.*

Proof . First assume that $i = j = 1$ and $k = 2$. Since α, β and σ are onto there exist an element z of $\mathcal{R}$ such that $\beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(x_{12})))$. For $a_{1j} \in R_{1j}$, by (vi) we have

$$\begin{aligned} \sigma(za_{1j}) &= \beta(\sigma(\alpha(z)))\beta(\sigma(\alpha(a_{1j}))) \\ &= (\beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(x_{12}))))\beta(\sigma(\alpha(a_{1j}))) \\ &= \beta(\sigma(\alpha(x_{11})))\beta(\sigma(\alpha(a_{1j}))) + \beta(\sigma(\alpha(x_{12})))\beta(\sigma(\alpha(a_{1j}))) \\ &= \sigma((x_{11} + x_{12})a_{1j}). \end{aligned}$$

Therefore $za_{1j} = (x_{11} + x_{12})a_{1j}$, since σ is one to one. Similarly for $a_{2j} \in \mathcal{R}_{2j}$, we have

$$za_{2j} = (x_{11} + x_{12})a_{2j}.$$

Therefore

$$(z - (x_{11} + x_{12}))a_{1j} = 0$$

$$(z - (x_{11} + x_{12}))a_{2j} = 0$$

Hence $[z - (x_{11} + x_{12})][\mathcal{R}_{11} + \mathcal{R}_{12} + R_{21} + \mathcal{R}_{22}] = 0$, or $[z - (x_{11} + x_{12})]\mathcal{R} = 0$.
By (i) we have $z = x_{11} + x_{12}$. It means that $\beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(x_{12}))) = \beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{11} + x_{12})))$.
Similarly for $i = k = 1$ and $j = 2$ and applying (ii) we have $\beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(x_{21}))) = \beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{11} + x_{21})))$. □

Lemma 2.4. *σ is $(\alpha, \beta) -$ additive on $\mathcal{R}_{12}$.*

Proof . Let $x_{12}, y_{12} \in \mathcal{R}_{12}$, and choose $z \in \mathcal{R}$ such that $\beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{12}))) + \beta(\sigma(\alpha(y_{12})))$.
For $a_{1j} \in \mathcal{R}_{1j}$, we have

$$\begin{aligned}\sigma(za_{1j}) &= \beta(\sigma(\alpha(z)))\beta(\sigma(\alpha(a_{1j}))) \\ &= (\beta(\sigma(\alpha(x_{12}))) + \beta(\sigma(\alpha(y_{12}))))\beta(\sigma(\alpha(a_{1j}))) \\ &= \beta(\sigma(\alpha(x_{12})))\beta(\sigma(\alpha(a_{1j}))) + \beta(\sigma(\alpha(y_{12})))\beta(\sigma(\alpha(a_{1j}))) \\ &= \sigma(x_{12}a_{1j}) + \sigma(y_{12}a_{1j}) \\ &= \sigma(0) + \sigma(0) = 0\end{aligned}$$

whence $za_{1j} = 0$. since σ is one to one.
Similarly for $a_{2j} \in \mathcal{R}_{2j}$ we have

$$\sigma(za_{2j}) = (\beta(\sigma(\alpha(x_{12}))) + \beta(\sigma(\alpha(y_{12}))))\beta(\sigma(\alpha(a_{2j}))), \tag{2.1}$$

$$\beta(\sigma(\alpha(e_1)))\beta(\sigma(\alpha(a_{2j}))) = \sigma(e_1a_{2j}) = \sigma(0) = 0, \tag{2.2}$$

$$\beta(\sigma(\alpha(x_{12})))\beta(\sigma(\alpha(y_{12}))) = \sigma(x_{12}y_{12}) = \sigma(0) = 0 \tag{2.3}$$

and by (iv)

$$\beta(\sigma(\alpha(e_1)))\beta(\sigma(\alpha(y_{12}))) = \beta(\sigma(\alpha(e_1y_{12}))) = \beta(\sigma(\alpha(y_{12}))), \tag{2.4}$$

From the above relations,

$$\sigma(za_{2j}) = [\beta(\sigma(\alpha(e_1))) + \beta(\sigma(\alpha(x_{12})))][\beta(\sigma(\alpha(a_{2j}))) + \beta(\sigma(\alpha(y_{12})))\beta(\sigma(\alpha(a_{2j})))], \tag{2.5}$$

Now by (iv) we have
$\beta(\sigma(\alpha(y_{12})))\beta(\sigma(\alpha(a_{2j}))) = \beta(\sigma(\alpha(y_{12}a_{2j})))$,
Since $e_1 = e_1e_1e_1$, by applying Lemma 2.3 we have

$$\beta(\sigma(\alpha(e_1))) + \beta(\sigma(\alpha(x_{12}))) = \beta(\sigma(\alpha(e_1 + x_{12}))), \tag{2.6}$$

Again in each of cases $j = 1$ or $j = 2$ in another term of right hand (3.5) we can apply Lemma 2.3 and obtain that

$$\beta(\sigma(\alpha(a_{2j}))) + \beta(\sigma(\alpha(y_{12}a_{2j}))) = \beta(\sigma(\alpha(a_{2j} + y_{12}a_{2j}))), \tag{2.7}$$

Now from (3.5), (3.6), (3.7) we see that

$$\begin{aligned}\sigma(za_{2j}) &= \beta(\sigma(\alpha(e_1 + x_{12})))\beta(\sigma(\alpha(a_{2j} + y_{12}a_{2j}))) \\ &= \sigma((e_1 + x_{12}))(a_{2j} + y_{12}a_{2j})) \\ &= \sigma(e_1a_{2j} + e_1y_{12}a_{2j} + x_{12}a_{2j} + x_{12}y_{12}a_{2j}) \\ &= \sigma(0 + y_{12}a_{2j} + x_{12}a_{2j} + 0a_{2j}) \\ &= \sigma((y_{12} + x_{12})a_{2j}).\end{aligned}$$

Whence $za_{2j} = (y_{12} + x_{12})a_{2j}$, since σ is one to one. Now $[z - (x_{12} + y_{12})]a_{2j} = 0$.
Then

$$\begin{aligned}[z - (x_{12} + y_{12})]\mathcal{R} &= [z - (x_{12} + y_{12})](\mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}) \\ &= z\mathcal{R}_{11} + z\mathcal{R}_{12} - (x_{12} + y_{12})(\mathcal{R}_{11} + \mathcal{R}_{12}) + [z - (x_{12} + y_{12})](\mathcal{R}_{21} + \mathcal{R}_{22}) \\ &= 0 + 0 + 0 + 0 = 0.\end{aligned}$$

So by (i), $z = x_{11} + y_{12}$. That is
$\beta(\sigma(\alpha(x_{12}))) + \beta(\sigma(\alpha(y_{12}))) = \beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{12} + y_{12})))$.
□

Lemma 2.5. *σ is $(\alpha,\beta)-$ additive on $\mathcal{R}_{11}$.*

Proof . Let $x_{11}, y_{11} \in \mathcal{R}_{11}$. There exist $z \in \mathcal{R}$ such that $\beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(y_{11})))$. Since $\beta o\sigma o\alpha$ is onto. By using (v) and Lemma 2.4 we see that

$$\begin{aligned} \beta o\sigma o\alpha(za_{12}) &= (\beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(y_{11}))))(\beta(\sigma(\alpha(a_{12})))) \\ &= (\beta(\sigma(\alpha(x_{11}a_{12})))) + \beta(\sigma(\alpha(y_{11}a_{12}))) \\ &= \beta(\sigma(\alpha(x_{11}a_{12} + y_{11}a_{12}))). \end{aligned}$$

Therefore $za_{12} = x_{11}a_{12} + y_{11}a_{12}$. Since σ is one to one and consequently
$[z-(x_{11}+y_{11})]a_{12} = 0$. So $[z-(x_{11}+y_{11})]R_{12} = 0$.
Now we write z in terms of its components $z = z_{11} + z_{12} + z_{21} + z_{22}$, and by applying (iv) we have

$$\begin{aligned} \beta(\sigma(\alpha(z))) &= \beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(y_{11}))) \\ &= \beta(\sigma(\alpha(e_1x_{11}))) + \beta(\sigma(\alpha(e_1y_{11}))) \\ &= \beta(\sigma(\alpha(e_1))\beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(e_1)))(\beta(\sigma(\alpha(y_{11})))) \\ &= \beta(\sigma(\alpha(e_1)))(\beta(\sigma(\alpha(x_{11})))) + \beta(\sigma(\alpha(y_{11}))) \\ &= \beta(\sigma(\alpha(e_1)))\beta(\sigma(\alpha(z))) \\ &= \beta(\sigma(\alpha(e_1)))\beta(\sigma(\alpha(z_{11}+z_{12}+z_{21}+z_{22}))) \\ &= \beta(\sigma(\alpha(e_1(z_{11}+z_{12}+z_{21}+z_{22})))) \\ &= \beta(\sigma(\alpha(z_{11}+z_{12}))). \end{aligned}$$

Therefore $z = z_{11} + z_{12}$. Since $\beta o\sigma o\alpha$ is one to one and hence $z_{11} = z_{12} = 0$, by uniqueness of direct sum.
Next using (v) and repeating the above argument with e_1 multiplied on the right, one finds that $z_{12} = 0$, thus yielding $z = z_{11} \in \mathcal{R}_{11}$. Therefore $z-(x_{11}+y_{11}) \in \mathcal{R}_{11}$ and consequently $(z-(x_{11}+y_{11}))\mathcal{R}_{12} = 0$. For some $x \in \mathcal{R}$, we have $e_1xe_1e_1Re_2 = 0$. Hence $e_1xe_1\mathcal{R}(1-e_1) = 0$. This implies that $x = 0$ by condition (iii). Then $0 = e_1xe_1 = z-(x_{11}+y_{11})$. So $z = x_{11}+y_{11}$. Therefore
$\beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(y_{11}))) = \beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x_{11}+y_{11})))$.
□

Lemma 2.6. *σ is $(\alpha,\beta)-$ additive on $e_1\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12}$.*

Proof . Let $x_{11}, y_{11} \in \mathcal{R}_{11}$ and let $x_{12}, y_{12} \in \mathcal{R}_{12}$. By Lemmas 2.3 and 2.4 and 2.5 to see that

$$\begin{aligned} \beta(\sigma(\alpha((x_{11}+x_{12})+(y_{11}+y_{12})))) &= \beta(\sigma(\alpha((x_{11}+y_{11})+(x_{12}+y_{12})))) \\ &= \beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(y_{11}))) + \beta(\sigma(\alpha(x_{12}))) + \beta(\sigma(\alpha(y_{12}))) \\ &= \beta(\sigma(\alpha(x_{11}))) + \beta(\sigma(\alpha(x_{12}))) + \beta(\sigma(\alpha(y_{11}))) + \beta(\sigma(\alpha(y_{12}))) \\ &= \beta(\sigma(\alpha(x_{11}+x_{12}))) + \beta(\sigma(\alpha(y_{11}+y_{12}))) \end{aligned}$$

□

Now we are ready to state the proof of Theorem 2.1.

Proof . Let $x, y \in \mathcal{R}$, there exists $z \in \mathcal{R}$ such that $\beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x))) + \beta(\sigma(\alpha(y)))$. Choose $t_\alpha \in e_\alpha\mathcal{R}$. By Lemma 2.6, σ is $(\alpha, \beta) - \textit{additive}$ on $e_\alpha\mathcal{R}$and by (iv) we have,

$$\begin{aligned}
\beta(\sigma(\alpha(t_\alpha z))) &= (\beta(\sigma(\alpha(t_\alpha)))(\beta(\sigma(\alpha(z))) \\
&= \beta(\sigma(\alpha(t_\alpha)))[\beta(\sigma(\alpha(x)))) + \beta(\sigma(\alpha(y)))] \\
&= (\beta(\sigma(\alpha(t_\alpha x))) + \beta(\sigma(\alpha(t_\alpha y))) \\
&= (\beta(\sigma(\alpha(t_\alpha x + t_\alpha y)))
\end{aligned}$$

So $t_\alpha z = t_\alpha x + t_\alpha y$, since $\beta o \sigma o \alpha$ is one to one. Hence $t_\alpha[z - (x + y)] = 0$. So $e_\alpha\mathcal{R}[z - (x + y)] = 0$. By conditio(ii), $z - (x + y) = 0$ or $z = x + y$. Then
$\beta(\sigma(\alpha(x))) + \beta(\sigma(\alpha(y))) = \beta(\sigma(\alpha(z))) = \beta(\sigma(\alpha(x + y)))$.
□

Corollary 2.7. *If α and β are multiplicative under the conditions of Theorem 2.1, then every multiplicative isomorphism $\sigma : \mathcal{R} \to \mathcal{S}$ is additive.*

Proof . Under the condition of theorem σ is a (α, β)-additive mapping. In other words $\beta o \sigma o \alpha$ is an additive mapping, furthermore since α and β are onto multiplicative isomorphisms. We conclude that also α^{-1} and β^{-1} are multiplicative isomorphisms. In fact,

$$\begin{aligned}
\alpha(xy) &= \alpha(x)\alpha(y) \\
\alpha^{-1}(\alpha(x)\alpha(y)) &= \alpha^{-1}(\alpha(xy)) = xy = \alpha^{-1}(\alpha(x))\alpha^{-1}(\alpha(y)).
\end{aligned}$$

Surjectivity α implies the multiplication of α^{-1}. By the main theorem of [7] α^{-1} and β^{-1} are additive and consequently $\beta^{-1}o(\beta o \sigma o \alpha) o \alpha^{-1} = \sigma$ is an additive mapping. □

Now we recall the following theorem and we state a similar to one.

Theorem 2.8. (see [5, Theorem 1]) Suppose that $\mathcal{R}$ is a ring (not necessarily with an identity) and α and β are ring automorphisms on $\mathcal{R}$. Also assume that there exists an idempotent $e(e \neq 0, e \neq 1)$ such that the following conditions hold:
(a) $\tilde{e}\mathcal{R}x = 0$ implies $x = 0$;
(b) $\tilde{e}xe\mathcal{R}(1 - e) = 0$ implies $\tilde{e}xe = 0$;
(c) $x\mathcal{R} = 0$ implies $x = 0$,
where $\tilde{e} = \beta\alpha^{-1}(e)$. Then every multiplicative $(\alpha, \beta) - \textit{derivation}$ of $\mathcal{R}$ is additive.

As a special case of Theorem 2.1, we conclude the following theorem:

Theorem 2.9. *Suppose that $\mathcal{R}$ is a ring containing a family $\{e_\alpha\}_{\alpha \in A}$ of idempotents, such that for each $\alpha \in A$ and $x \in \mathcal{R}$ satisfies the following conditions:*
(i) $x\mathcal{R} = 0$ implies $x = 0$;
(ii) $e_\alpha\mathcal{R}x = 0$ implies $x = 0$;
(iii) If $e_\alpha x e_\alpha R(1 - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.
If α and β are ring homomorphisms on $\mathcal{R}$ and at least one of α and β is one to one then every multiplicative $(\alpha, \beta) - derivation$ of $\mathcal{R}$ is additive.

Proof . Let $d : \mathcal{R} \to \mathcal{R}$ be a multiplicative $(\alpha, \beta) - \textit{derivation}$, and let
$\mathcal{S} = \left\{ \begin{pmatrix} \beta(x) & d(x) \\ 0 & \alpha(x) \end{pmatrix} | x \in \mathcal{R} \right\}$. Obviously $\mathcal{S}$ is a ring. Define $\sigma : \mathcal{R} \to \mathcal{S}$ by

$\sigma(x) = \begin{pmatrix} \beta(x) & d(x) \\ 0 & \alpha(x) \end{pmatrix}$, for each $x \in \mathcal{R}$. Then σ is onto and one to one, since one of α and β is one to one.
For every $x, y \in \mathcal{R}$, we have

$$\begin{aligned} \sigma(xy) &= \begin{pmatrix} \beta(xy) & d(xy) \\ 0 & \alpha(xy) \end{pmatrix} \\ &= \begin{pmatrix} \beta(x)\beta(y) & d(x)\alpha(y) + \beta(x)d(y) \\ 0 & \alpha(x)\alpha(y) \end{pmatrix} \\ &= \begin{pmatrix} \beta(x) & d(x) \\ 0 & \alpha(x) \end{pmatrix} \\ &\times \begin{pmatrix} \beta(y) & d(y) \\ 0 & \alpha(y) \end{pmatrix} \\ &= \sigma(x)\sigma(y). \end{aligned}$$

Then σ is multiplicative. Hence it is an isomorphism and by Theorem 2.1, it is additive.

$$\begin{aligned} \sigma(x+y) &= \begin{pmatrix} \beta(x+b) & d(x+y) \\ 0 & \alpha(x+y) \end{pmatrix} \\ &= \sigma(x) + \sigma(y) \\ &= \begin{pmatrix} \beta(x) + \beta(y) & d(x) + d(y) \\ 0 & \alpha(x) + \alpha(y) \end{pmatrix}. \end{aligned}$$

Hence d is additive. □

Note that by Theorem 2.9 every derivation on a prime ring $\mathcal{R}$ ($x\mathcal{R}y = 0$ implies that $x = 0$ or $y = 0$), with a nontrivial idempotents e ($e \neq 0, 1$) is additive.

Example 2.10. Suppose that $M_n(\mathbb{C})$ denotes the algebra of all the $n \times n$ complex matrices.
Set $e_k = [a_{ij}]_{n \times n}, \quad (k = 1, 2, ..., n)$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

In other words e_k is the diagonal matrix in which the only nonzero element is the k^{th} element on its diagonal. Theorem 2.9 implies that every multiplicative $(\alpha, \beta)-$derivation on $M_n(\mathbb{C})$ in which one of α and β is one to one is additive.

Example 2.11. Let $\mathcal{X}$ be a Banach space such that $dim(\mathcal{X}) \geq 2$ and let $F(\mathcal{X})$ the algebra of all finite rank operators on $\mathcal{X}$. Using by Hahn Banach theorem, one can show that $F(\mathcal{X})$ is a prime ring. Choose a nonzero idempotent P of $F(\mathcal{X})$. If $T \in F(\mathcal{X})$, then
(i) $TF(\mathcal{X}) = 0$ implies $T = 0$.
(ii) $PF(\mathcal{X})T = 0$ implies $T = 0$.
(iii) $PTPF(\mathcal{X})(I - P) = 0$ implies $PTP = 0$.
Hence if one of the mapping α and β is one to one, every multiplicative derivation on $F(\mathcal{X})$ is additive.

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