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Study on efficiency of the Adomian decomposition method for stochastic differential equations

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Abstract

Many time-varying phenomena of various fields in science and engineering can be modeled as a stochastic differential equations, so investigation of conditions for existence of solution and obtain the analytical and numerical solutions of them are important. In this paper, the Adomian decomposition method for solution of the stochastic differential equations are improved. Uniqueness and convergence of their adapted solutions are reviewed. The efficiency of the method is demonstrated through the two numerical experiments.

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1. Introduction

Stochastic differential equations (SDEs) are applied in many fields, including economics, theoretical physics, biology, mathematical finance and etc. [3, 13, 15]. Unfortunately, in many cases analytic solutions are not available, thus numerical methods are needed to approximate them [5, 6, 9, 12, 18]. In numerical approaches investigation on properties of the solutions of SDEs is valuable [4, 11, 13, 15, 17, 18].

The suggested method by George Adomian, so-called Adomian decomposition method (ADM), has been developed for solving different kinds of equations in recent years [7, 8, 10, 16]. In [2], the ADM is employed to solve the stochastic problems which have special importance in engineering and sciences. For some analytical results about the ADM and some other applications of this method,

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the interested reader can see [1, 7, 14]. One major advantage of the ADM is to find the solution of linear and nonlinear differential equations without dependence on any small parameter like as perturbation method. In the ADM, the solution is considered as a sum of an infinite series which rapidly converges to an accurate solution.

In this paper, we present a new approach to prove the convergence of the ADM applied to SDEs of the form

$$\begin{cases} dX_t = f(t, X_t)dt + g(t, X_t)dB_t, \quad t \in I = [0, T], \\ X_0 = \mathcal{X}. \end{cases} \tag{1.1}$$

Where $\mathcal{X}$ is the known random variable, f and g are some given functions, B_t is 1-dimensional Brownian motion starting at the origin, and X_t is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also, in this study, we give the computational outcomes via the improved ADM for solving SDEs, to support our theoretical discussion.

The rest of the paper is organized as follows: In next section, we implement the stochastic Adomian decomposition method (SADM) for the problem (1.1). In Section 3, we show convergence of the SADM for a general SDE with an iterative process. In Section 4, the suggested SADM is applied to some numerical experiments. Finally, Section 5 contains a brief conclusion.

2. Basic Idea for SADM and Preliminaries

By considering Eq. (1.1) and applying the usual integration notation we obtain

$$X_t = \mathcal{X} + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dB_s. \tag{2.1}$$

The ADM expresses the unknown function X_t by an infinite series [1],

$$X_t = \sum_{i=0}^{\infty} X_t^{(i)},$$

where the components $X_t^{(i)}$ are usually determined recurrently. Substituting this infinite series in Eq. (2.1) leads to

$$\sum_{i=0}^{\infty} X_t^{(i)} = \mathcal{X} + \int_0^t f(s, \sum_{i=0}^{\infty} X_s^{(i)})ds + \int_0^t g(s, \sum_{i=0}^{\infty} X_s^{(i)})dB_s. \tag{2.2}$$

The Adomian procedure can be presented as the following:

$$\begin{cases} X_t^{(0)} = \mathcal{X}, \\ X_t^{(n+1)} = \int_0^t f(s, X_s^{(n)})ds + \int_0^t g(s, X_s^{(n)})dB_s, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{2.3}$$

Defining the nth partial sum of the generated sequence $\{X_t^{(i)}\}_{i=0}^{\infty}$ as $S_t^{(n)} = \sum_{i=0}^{n} X_t^{(i)}$, consequently, approximate solution of the Eq. (1.1) with the ADM may be obtained by $X_t \approx \lim_{n\to\infty} S_t^{(n)}$.

Lemma 2.1. (The 1-dimensional Itô's formula)(see [13]) Let $h(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ belongs to $C^2([0, \infty) \times \mathbb{R})$, and assume that the stochastic process X_t follows of the SDE (1.1), then we have

$$dh(t, X_t) = \left[\frac{\partial h}{\partial t}(t, X_t) + f(t, X_t)\frac{\partial h}{\partial x}(t, X_t) + \frac{1}{2}g^2(t, X_t)\frac{\partial^2 h}{\partial x^2}(t, X_t)\right] \mathrm{d}t + g(t, X_t)\frac{\partial h}{\partial x}(t, X_t)\mathrm{d}B_t.$$

Lemma 2.2. (Lipschitz condition) For all $t \in I = [0, T]$ and $X, Y \in L_2(I)$, there exists a function $g(t) > 0$ such that

$$\|f(t, X) - f(t, Y)\| \leq g(t)\|X - Y\|,$$

where $\int_0^T g(t)\mathrm{d}t < \infty$.

3. Convergence of the SADM

In this section first uniqueness of solution has been demonstrated under some conditions. Then, convergence of the approximate solution obtained with the SADM to the exact solution of Eq. (1.1) is shown.

Theorem 3.1. *Let $f(t, X_t)$ and $g(t, X_t)$ satisfy in the Lipschitz condition, i.e.*

$$\|f(t, X_t) - f(t, Y_t)\| \leq g_1(t)\|X_t - Y_t\|,$$
$$\|g(t, X_t) - g(t, Y_t)\| \leq g_2(t)\|X_t - Y_t\|.$$

Then, the SDE (1.1) *has a unique solution whenever* $0 \leq \zeta < 1$, *where* $\zeta = M(T + T^{\frac{1}{2}})$, *and* $M = \max_{0 \leq t \leq T}\{g_1(t), g_2(t)\}$.

Proof . Let X_t and X_t^* be two different solutions of the Eq. (1.1) or correspondingly the Eq. (2.1), so

$$\begin{aligned} \| X_t - X_t^* \| &\leq \Big\| \int_0^t [f(s, X_s) - f(s, X_s^*)]ds \Big\| + \Big\| \int_0^t [g(s, X_s) - g(s, X_s^*)]dB_s \Big\| \\ &\leq \|X_t - X_t^*\| \Big(\int_0^t g_1(s)ds + \int_0^t g_2(s)dB_s \Big) \\ &\leq M\|X_t - X_t^*\|(T + \sqrt{T}) = \zeta\|X_t - X_t^*\|. \end{aligned}$$

Namely $(1 - \zeta)\|X_t - X_t^*\| \leq 0$, since $0 \leq \zeta < 1$, then $\|X_t - X_t^*\| = 0$, implies $X_t = X_t^*$ and this completes the proof. □

Theorem 3.2. *If for* $i = 0, 1, 2, \cdots$ *there is* $0 \leq \zeta < 1$ *such that* $\|X_t^{(i+1)}\| \leq \zeta\|X_t^{(i)}\|$, *then* $\sum_{i=0}^{\infty} X_t^{(i)}$ *which is obtained by* (2.3) *as a solution of the Eq.* (1.1), *converges to the exact solution* X_t.

Proof . We have $S_t^0 = \mathcal{X}$ and $S_t^{(n)} = \sum_{i=0}^{n} X_t^{(i)}$, now we show that $\{S_t^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $L_2(0, T)$. For this purpose, one gets

$$\|S_t^{(n+1)} - S_t^{(n)}\| = \|X_t^{(n+1)}\| \leq \zeta\|X_t^{(n)}\| \leq \zeta^2\|X_t^{(n-1)}\| \leq \cdots \leq \zeta^{n+1}\|X_t^{(0)}\|.$$

Also for $n, m \in N$, $m \geq n$ we have

$$\begin{aligned} \|S_t^{(m)} - S_t^{(n)}\| &\leq \|S_t^{(m)} - S_t^{(m-1)}\| + \|S_t^{(m-1)} - S_t^{(m-2)}\| + \cdots + \|S_t^{(n+1)} - S_t^{(n)}\| \\ &\leq \big(\zeta^m + \zeta^{m-1} + \cdots + \zeta^{n+1}\big)\|X_t^{(0)}\| \\ &= \frac{\zeta^{n+1} - \zeta^{m+1}}{1 - \zeta}\|X_t^{(0)}\|, \end{aligned}$$

hence, $\lim_{m,n\to+\infty}\|S_t^{(m)}-S_t^{(n)}\|=0$, i.e., $\{S_t^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence and it implies that

$$\exists S_t\in L_2(0,T),\ \lim_{n\to+\infty}S_t^{(n)}=S_t.$$

On the other hand $S_t=\lim_{n\to+\infty}\sum_{i=0}^{n}X_t^{(i)}$, therefore S_t is the desired solution for the Eq. (1.1) which SADM converges to it, and according to the ADM process, S_t satisfies in the following equation

$$S_t=\mathcal{X}+\int_0^t f(\tau,S_\tau)d\tau+\int_0^t g(\tau,S_\tau)dB_\tau.$$

□

4. Illustrative examples

In order to demonstrate the efficiency and accuracy of the presented method in this study, here we apply the SADM to solve some SDEs.

Example 4.1. Consider the following SDE with initial condition [9],

$$\begin{cases}\dot{X}_t=aX_t+bX_tB_t,\\ X_0=\mathcal{X},\end{cases}\tag{4.1}$$

where the coefficients a and b are constants. The corresponding integral equation to the above equation is

$$X_t=\mathcal{X}+a\int_0^t X_s ds+b\int_0^t X_s dB_s.$$

Implementation of the SADM for this equation yields,

$$\begin{cases}X_t^{(0)}=\mathcal{X},\\ X_t^{(n+1)}=a\int_0^t X_s^{(n)}ds+b\int_0^t X_s^{(n)}dB_s,\ n=0,1,2,\ldots.\end{cases}\tag{4.2}$$

From the recursive relation (4.2) and Lemma 2.1 we derive

$$\begin{aligned}
X_t^{(1)}=&\mathcal{X}\Big(at+bB_t\Big),\\
X_t^{(2)}=&\mathcal{X}\left(a^2\frac{t^2}{2}+abtB_t+\frac{b^2}{2}\Big(B_t^2-t\Big)\right),\\
X_t^{(3)}=&\mathcal{X}\left(a^3\frac{t^3}{3!}+a^2b\frac{t^2}{2}B_t+\frac{ab^2}{2}\Big(tB_t^2-t^2\Big)+\frac{b^3}{6}\Big(B_t^3-3tB_t\Big)\right),\\
X_t^{(4)}=&\mathcal{X}\left(a^4\frac{t^4}{4!}-a^2b^2\frac{t^3}{4}+b^4\frac{t^2}{8}+a^3b\frac{t^3}{3!}B_t+a^2b^2\frac{t^2}{4}B_t^2+\frac{ab^3}{6}\Big(tB_t^3-3t^2B_t\Big)\right.\\
&\left.+\frac{b^4}{24}(B_t^4-6tB_t^2)\right),
\end{aligned}$$

$$X_t^{(5)} = \mathcal{X}\Bigg(a^5\frac{t^5}{5!} - a^3b^2\frac{t^4}{12} + ab^4\frac{t^3}{8} + a^4b\frac{t^4}{4!}B_t - a^2b^3\frac{t^3}{4}B_t + b^5\frac{t^2}{8}B_t + a^3b^2\frac{t^3}{12}B_t^2 + a^2b^3\frac{t^2}{12}B_t^3 + \frac{ab^4}{2}\Big(\frac{t}{12}B_t^4 - \frac{t^2}{2}B_t^2\Big) + \frac{b^5}{6}\Big(\frac{B_t^5}{20} - \frac{t}{2}B_t^3\Big)\Bigg),$$

$$X_t^{(6)} = \mathcal{X}\Bigg(a^6\frac{t^6}{6!} - a^4b^2\frac{t^5}{48} + a^2b^4\frac{t^4}{16} - b^6\frac{t^3}{48} + a^5b\frac{t^5}{5!}B_t - a^3b^3\frac{t^4}{12}B_t + ab^5\frac{t^3}{8}B_t + a^4b^2\frac{t^4}{48}B_t^2 - a^2b^4\frac{t^3}{8}B_t^2 + b^6\frac{t^2}{16}B_t^2 + a^3b^3\frac{t^3}{36}B_t^3 + a^2b^4\frac{t^2}{48}B_t^4 + \frac{ab^5}{2}\left(\frac{t}{60}B_t^5 - \frac{t^2}{6}B_t^3\right) + \frac{b^6}{6}\left(\frac{B_t^6}{120} - \frac{t}{8}B_t^4\right)\Bigg),$$

$$\vdots$$

By continuing this process $X_t^{(i)}$ and consequently $S_t^{(i)}, i = 1, 2, 3, \cdots$, can be calculated. Namely

$$S_t^{(2)} = \mathcal{X}\left(1 + \left((a - \frac{b^2}{2})t + bB_t\right) + \frac{a^2b^2}{2} + \xi_t^{(2)}\right),$$

$$S_t^{(4)} = \mathcal{X}\left(1 + \left((a - \frac{b^2}{2})t + bB_t\right) + \frac{\left((a - \frac{b^2}{2})t + bB_t\right)^2}{2!} + \xi_t^{(4)}\right),$$

$$S_t^{(6)} = \mathcal{X}\left(1 + \left((a - \frac{b^2}{2})t + bB_t\right) + \frac{\left((a - \frac{b^2}{2})t + bB_t\right)^2}{2!} + \frac{\left((a - \frac{b^2}{2})t + bB_t\right)^3}{3!} + \xi_t^{(6)}\right),$$

where

$$\xi_t^{(2)} = abB_t + \frac{b^2B_t^2}{2},$$

$$\xi_t^{(4)} = -a^2b^2\frac{t^3}{4} + a^4\frac{t^4}{4!} + a^3\frac{t^3}{3!}\Big(1 + bB_t\Big) + ab^2\frac{t^2}{2}B_t\Big(a - b\Big) + b^2\frac{t}{4}B_t^2\Big(2a - b^2 + a^2t\Big) + \frac{b^3B_t^3}{3!}\Big(1 + at\Big) + \frac{b^4}{4!}B_t^4,$$

$$\xi_t^{(6)} = \Big((a^2 - 2ab^2)^2 + a^4 - a^2b^2\Big)\frac{t^4}{48} + ab\Big((a - 3b^2)^2 - 3(a^2 + 2b^4)\Big)\frac{t^3}{3!}B_t + (2ab - b^3)^2\frac{t^2}{16}B_t^2 + b^3(2a - b^2)\frac{t}{12}B_t^3 + b^4(2 + a^2t^2)\frac{B_t^4}{48} + a^5\frac{t^5}{5!}(1 + bB_t) + a^3b(a - 2b^2)\frac{t^4}{4!}B_t + a^2b^2(2a + 3b^2)\frac{t^3}{4!}B_t^2 + ab^3(a - b^2)\frac{t^2}{12}B_t^3 + b^4(2a - b^2)\frac{t}{48}B_t^4 + \frac{B_t^5b^5}{5!}(1 + at) + a^6\frac{t^6}{6!} - a^4b^2\frac{t^5}{48} + a^4b^2\frac{t^4}{48}B_t^2 + a^3b^3\frac{t^3}{36}B_t^3 + \frac{b^6}{6!}B_t^6,$$

are the noise terms. By computing $S_t^{(i)}$ for sufficiently large value of i, it seems to be reasonable for approximate the exact solution,

$$X_t = \mathcal{X}\exp\left((a - \frac{1}{2}b^2)t + bB_t\right).$$

Example 4.2. Let us consider the following SDE,

$$\begin{cases} \dot{X}_t = 2(t+1)X_t + 2X_t B_t, \\ X_0 = \mathcal{X}. \end{cases} \tag{4.3}$$

Iterative process of the SADM for Eq. (4.3) is

$$\begin{cases} X_t^{(0)} = \mathcal{X}, \\ X_t^{(n+1)} = 2\int_0^t (s+1)X_s^{(n)} ds + 2\int_0^t X_s^{(n)} dB_s, \ n = 0, 1, 2, \ldots. \end{cases}$$

Using above approach and the 1-dimensional Itô's formula provide the following successive approximations,

$$\begin{aligned}
X_t^{(1)} =& \mathcal{X}\Big(t^2 + 2t + 2B_t\Big), \\
X_t^{(2)} =& \mathcal{X}\left(\frac{t^4}{2} + 2t^3 + 2t^2 - 2t + B_t\Big(2t^2 + 4t\Big) + 2B_t^2\right), \\
X_t^{(3)} =& \mathcal{X}\left(\frac{t^6}{6} + t^5 + 2t^4 - \frac{2t^3}{3} - 4t^2 + B_t\Big(t^4 + 4t^3 + 4t^2 - 4t\Big) + B_t^2\Big(2t^2 + 4t\Big) + \frac{4}{3}B_t^3\right), \\
X_t^{(4)} =& \mathcal{X}\left(\frac{t^8}{4!} + \frac{t^7}{3} + 4t^6 + \frac{t^5}{3} - \frac{10t^4}{3} - 4t^3 + 2t^2 + B_t\Big(\frac{t^6}{3} + 2t^5 + 4t^4 - \frac{4t^3}{3} - 8t^2\Big)\right. \\
&\left. + B_t^2\Big(t^4 + 4t^3 + 4t^2 - 4t\Big) + B_t^3\Big(\frac{4t^2}{3} + \frac{8t}{3}\Big) + \frac{2}{3}B_t^4\right), \\
&\vdots
\end{aligned}$$

Thus,

$$\begin{aligned}
S_t^{(1)} =& \mathcal{X}\left(1 + \Big(t^2 + 2B_t\Big) + 2t\right), \\
S_t^{(2)} =& \mathcal{X}\left(1 + \Big(t^2 + 2B_t\Big) + \frac{\Big(t^2 + 2B_t\Big)^2}{2!} + 2t^2(t+1) + 4tB_t\right), \\
S_t^{(3)} =& \mathcal{X}\left(1 + \Big(t^2 + 2B_t\Big) + \frac{\Big(t^2 + 2B_t\Big)^2}{2!} + \frac{\Big(t^2 + 2B_t\Big)^3}{3!} + 2t^4 + \frac{4}{3}t^3 - 2t^2\right. \\
&\left. + B_t(4t^3 + 4t^2) + B_t^2\Big(2t^2 + 4t\Big) + \frac{4}{3}B_t^3\right),
\end{aligned}$$

$$
\begin{aligned}
S_t^{(4)} =&\mathcal{X}\Bigg(1+\Big(t^2+2B_t\Big)+\frac{\Big(t^2+2B_t\Big)^2}{2!}+\frac{\Big(t^2+2B_t\Big)^3}{3!}+\frac{\Big(t^2+2B_t\Big)^4}{4!}\\
&+\frac{t^7}{3}+4t^6+\frac{t^5}{3}-\frac{4t^4}{3}-\frac{8t^3}{3}+B_t\Big(2t^5+4t^4+\frac{8t^3}{3}-4t^2\Big)+B_t^2\Big(4t^3+6t^2\Big)\\
&+B_t^3\Big(\frac{8t}{3}+\frac{4}{3}\Big)+\frac{2}{3}B_t^4\Bigg),\\
&\vdots
\end{aligned}
$$

From above results, it is observed that $S_t^{(i)},\ i=1,2,3,\dots,$ have a proper solidarity with the exact solution $X_t=\mathcal{X}\exp\left(t^2+2B_t\right)$ of Eq. (4.3) as a part of Maclurin expansion.

5. Conclusion

The Adomian decomposition method has been known to be a powerful scheme for solving many functional equations. Here, we used a stochastic version of this method for solving some stochastic differential equations. Also, the uniqueness and convergence of solution established in a special case of stochastic differential equations under some appropriate assumptions. Finally, two examples are given to demonstrate the powerfulness of the proposed method.

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