Int. J. Nonlinear Anal. Appl. 3 (2012) No. 1, 55-60
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

An Analog of Titchmarsh's Theorem for the Dunkl Transform in the Space $L^2_\alpha(\mathbb{R})$

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(Communicated by A. Ebadian)

Abstract

In this paper, using a generalized Dunkl translation operator, we obtain an analog of Titchmarsh's Theorem for the Dunkl transform for functions satisfying the Lipschitz-Dunkl condition in $L_{2,\alpha} = L^2_\alpha(\mathbb{R}) = L^2(\mathbb{R}, |x|^{2\alpha+1}dx), \alpha > \frac{-1}{2}$.

Keywords: Dunkl Operator, Dunkl Transform, Generalized Dunkl Translation.
2010 MSC: 44A15, 46E30.

1. Introduction and Preliminaries

Dunkl operators are differential-difference operators introduced in 1989, by Dunkl [5]. On the real line, these operators, which are denoted by $D = D_\alpha$, depend on a real parameter $\alpha > \frac{-1}{2}$ and they associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. For more details about these operators see [4, 5, 6, 7, 10] and the references therein.
Recently in mathematical papers a new class of generalized translations was described and put into use, namely, the generalized Dunkl translations. The generalized Dunkl translations are constructed on the base of certain differential-difference operators which are widely used in mathematical physics (e.g., [5, 9]).

Titchmarsh's [11, Therem 85] characterized the set of functions in $L_{2,\alpha}$ satisfying the cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, we have

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Received: January 2012 *Revised:* February 2012

Theorem 1.1. *[11] Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:*

(1) $\|f(t+h) - f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$

(2) $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})\quad r \longrightarrow \infty$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we obtain an analog of Theorem 1.1 in the Dunkl operator setting by means of generalized Dunkl translation. We point out that similar results have been established in the context of noncompact rank 1 Riemannian symmetric spaces [8].

The Dunkl operator is differential-difference operator D which satisfies the condition

$$\mathrm{D}f(x) = \frac{df}{dx}(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x},\ f \in \mathrm{L}_{2,\alpha}\,.$$

Let $j_\alpha(x)$ be a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha}\,,$$

where $J_\alpha(x)$ is a Bessel function of the first kind ([2], chap 7). The function $j_\alpha(x)$ is infinitely differentiable and even.

We understand a generalized exponential function as the function

$$e_\alpha(x) = j_\alpha(x) + ic_\alpha x j_{\alpha+1}(x), \tag{1.1}$$

where $c_\alpha = (2\alpha+2)^{-1}$.

The function $y = e_\alpha(x)$ satisfies the equation $\mathrm{D}y = y$.

Using the correlation

$$j'_\alpha(x) = -\frac{x j_{\alpha+1}(x)}{2(\alpha+1)}\,.$$

we conclude that the function $e_\alpha(x)$ admits the representation

$$e_\alpha(x) = j_\alpha(x) - ij'_\alpha(x) \tag{1.2}$$

Let $\mathrm{L}_{2,\alpha}$ is the Hilbert space consists of measurable functions $f(x)$ defined on $\mathbb{R}$ with the norm

$$\|f\| = \|f\|_{2,\alpha} = \left(\int_{-\infty}^{+\infty} |f(x)|^2 |x|^{2\alpha+1} dx\right)^{1/2}\,.$$

The Dunkl transform is defined by

$$\widehat{f}(\lambda) = \int_{-\infty}^{+\infty} f(x) e_\alpha(\lambda x) |x|^{2\alpha+1} dx,\ \lambda \in \mathbb{R}\,.$$

The inverse Dunkl transform is defined by the formula

$$f(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-2}\int_{-\infty}^{+\infty}\widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1}d\lambda\,.$$

The Dunkl transform satisfies the Parseval equality

$$\int_{-\infty}^{+\infty}|f(x)|^2|x|^{2\alpha+1}dx = (2^{\alpha+1}\Gamma(\alpha+1))^{-2}\int_{-\infty}^{+\infty}|\widehat{f}(\lambda)||\lambda|^{2\alpha+1}d\lambda\,.$$

In $\mathrm{L}_{2,\alpha}$, we define the operator of the generalized Dunkl translation (see [3])

$$\mathrm{T}_h f(x) = C(\int_0^\pi f_e(G(x,h,\varphi))h^e(x,h,\varphi)sin^{2\alpha}\varphi d\varphi \quad + \quad \int_0^\pi f_0(G(x,h,\varphi))h^0(x,h,\varphi)sin^{2\alpha}\varphi d\varphi)$$

where

$$C = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})},\quad G(x,h,\varphi) = \sqrt{x^2+h^2-2|xh|cos\varphi}$$

$$h^e(x,h,\varphi) = 1 - sgn(xh)cos\varphi\,.$$

and

$$h^0(x,h,\varphi) = \frac{(x+h)h^e(x,h,\varphi)}{G(x,h,\varphi)} \text{ for } (x,h)\neq(0,0)$$

$$h^0(x,h,\varphi) = 0 \text{ for } (x,h)=(0,0)$$

$$f_e(x) = \frac{1}{2}(f(x)+f(-x)),\quad f_0(x) = \frac{1}{2}(f(x)-f(-x))$$

From formula (1.1), we have

$$|1-j_\alpha(x)| \leq |1-e_\alpha(x)|\,.$$

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(p+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\frac{x}{2})^{2n} \tag{1.3}$$

Moreover, from (1.3) we see that

$$\lim_{x\longrightarrow 0}\frac{j_p(x)-1}{x^2} \neq 0\,,$$

by consequence, there exist $c>0$ and $\eta>0$ satisfying

$$|x|\leq\eta \Longrightarrow |j_\alpha(x)-1| \geq c|x|^2 \tag{1.4}$$

In [3], we obtain

$$(\widehat{\mathrm{T}_h f})(\lambda) = e_\alpha(\lambda h)\widehat{f}(\lambda) \tag{1.5}$$

2. Main Result

In this section we give the main result of this paper, We need first to define the Dunkl-Lipschitz class.

Definition 2.1. *Let $\beta \in (0,1)$. A function $f \in \mathrm{L}_{2,\alpha}$ is said to be in the Dunkl-Lipschitz class, denote by $Lip(\beta,2,\alpha)$, if*

$$\|\mathrm{T}_h f(x) + \mathrm{T}_{-h} f(x) - 2f(x)\| = O(h^{\beta}), \;\; as\ h \longrightarrow 0\,.$$

Theorem 2.2. *Let $f \in \mathrm{L}_{2,\alpha}$. Then the following are equivalents*

(1) $f \in Lip(\beta,2,\alpha)$.

(2) $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda = O(r^{-2\beta})$ *as* $r \longrightarrow +\infty$.

Proof . $1 \Longrightarrow 2$: Assume that $f \in Lip(\beta,2,\alpha)$. Then we have

$$\|\mathrm{T}_h f(x) + \mathrm{T}_{-h} f(x) - 2f(x)\| = O(h^{\beta}), \quad as\ h \longrightarrow 0\,.$$

From formulas (1.2) and (1.5), we have the Dunkl transform of $\mathrm{T}_h f(x) + \mathrm{T}_{-h} f(x) - 2f(x)$ is $2(j_\alpha(\lambda h) - 1)$.

Parseval's identity gives

$$\|\mathrm{T}_h f(x) + \mathrm{T}_{-h} f(x) - 2f(x)\| = \int_{-\infty}^{+\infty} |2(1 - j_\alpha(\lambda h))|^2 |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda\,.$$

From (1.4), we have

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \geq \frac{c^2\eta^4}{16} \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda\,.$$

There exists then a positive constant C such that

$$\begin{aligned} \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda &\leq C \int_{-\infty}^{+\infty} |1 - j_\alpha(\lambda h)|^2 |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \\ &\leq Ch^{2\beta} \end{aligned}$$

For all $h > 0$, we have

$$\int_{r\leq|\lambda|\leq 2r} |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \leq Cr^{-2\beta}, \text{ for all } r > 0\,.$$

Furthermore, we obtain

$$\begin{aligned}\int_{|\lambda|\geq r}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= \sum_{i=0}^{\infty}\int_{2^i s\leq|\lambda|\leq 2^{i+1}s}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\ &\leq C\sum_{i=0}^{\infty}(2^i r)^{-2\beta}\\ &\leq Cr^{-2\beta}\end{aligned}$$

This prove that

$$\int_{|\lambda|\geq r}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O(r^{-2\beta}) \ \ as\ r\longrightarrow +\infty\,.$$

$2 \Longrightarrow 1$: Suppose now that

$$\int_{|\lambda|\geq r}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O(r^{-2\beta}) \ \ as\ r\longrightarrow +\infty\,.$$

We write

$$\begin{aligned}\int_{-\infty}^{+\infty}|1-j_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= \int_{|\lambda|<\frac{1}{h}}|1-j_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\ &+ \int_{|\lambda|\geq\frac{1}{h}}|1-j_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\end{aligned}$$

Firstly, we use the formula $|j_\alpha(x)|\leq 1$

$$\int_{|\lambda|\geq\frac{1}{h}}|1-j_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda \leq 4\int_{|\lambda|\geq\frac{1}{h}}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\,.$$

Then

$$\int_{|\lambda|\geq\frac{1}{h}}|1-j_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O(h^{2\beta})\ \text{as}\ h\longrightarrow 0\,.$$

Set

$$\phi(x)=\int_x^{\infty}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\,.$$

An integration by parts, we obtain

$$\begin{aligned}\int_0^x \lambda^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= \int_0^x -\lambda^2\phi^{'}(\lambda)d\lambda = -x^2\phi(x)+2\int_0^x\lambda\phi(\lambda)d\lambda\\ &\leq 2\int_0^x O(\lambda^{1-2\beta})d\lambda\\ &= O(x^{2-2\beta}).\end{aligned}$$

We use the formula $1-j_\alpha(x)=O(x^2),\ \ 0\leq x\leq 1$ see [1]. Then

$$\begin{aligned}\int_{-\infty}^{+\infty}|1-j_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= O(h^2\int_{|\lambda|<\frac{1}{h}}\lambda^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda+O(h^{2\beta})\\ &= O(h^2h^{-2+2\beta})+O(h^{2\beta})\\ &= O(h^{2\beta})+O(h^{2\beta})\\ &= O(h^{2\beta}).\end{aligned}$$

and this end the proof. □

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