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Application of He's homotopy Perturbation Method for Solving Sivashinsky Equation

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Abstract

In this paper, the solution of the evolutionary fourth-order in space, Sivashinsky equation is obtained by means of homotopy perturbation method (**HPM**). The results reveal that the method is very effective, convenient and quite accurate to systems of nonlinear partial differential equations.

Keywords: Homotopy Perturbation Method, Sivashinsky Equation.
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1. Introduction and Preliminaries

In the recent years, the application of the homotopy perturbation method (**HPM**) [1, 7] in nonlinear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easy to solve. The homotopy perturbation method [6], proposed first by He in 1998 and was further developed and improved by He [7, 8, 11]. The method yields a very rapid convergence of the solution series in the most cases. Usually, one iteration leads to high accuracy of the solution. Although goal of He's homotopy perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. A review of recently developed nonlinear analysis methods can be found in [9]. Recently, the applications of homotopy perturbation theory among scientists were appeared [1-6], which has become a powerful mathematical tool, when it is successfully coupled with the perturbation theory [7, 10, 11].

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In the solidification of dilute binary alloy, a planar solid-liquid interface is often found to be unstable, spontaneously assuming a cellular structure. This situation enables one to derive on asymptotic nonlinear partial fourth-order differential equation (PDE) which directly describes the dynamics of the onset and stabilization of cellular structure as below [13]:

$$u_t + u_{xxxx} + \alpha u + ((2-u)u_x)_x = 0, \quad t \in (0,1), \tag{1.1}$$

where $\alpha > 0$ and $t > 0$.
This is called the Sivashinsky equation, see [14, 15]. The exact solution of this equation is not obtainable so, only a numerical scheme has been proposed for the solution of the Sivashinsky equation, see [13, 16, 17]. The authors make their investigations on a finite interval $\Omega = [0,1]$ and they add some initial and boundary conditions in order to obtain the approximate solutions.
In this paper, we apply **HPM** to Sivashinsky equation (Eq. 1.1) and compare our results with Adomian decomposing method (ADM) [13]

2. Basic Idea of Homotopy Perturbation Theory

To illustrate **HPM** consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with boundary conditions:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω.
The operator A can be generally divided into two parts F and N, where F is linear, whereas N is nonlinear. Therefore, Eq. (2.1) can be rewritten as follows:

$$F(u) + N(u) - f(r) = 0. \tag{2.3}$$

He [12] constructed a homotopy $v : \Omega \times [0,1] \longrightarrow \mathrm{R}$ which satisfies:

$$H(v,p) = (1-p)[F(v) - F(v_0)] + p[A(v) - f(r)] = 0, \tag{2.4}$$

or

$$H(v,p) = F(v) - F(v_0) + pF(v_0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $r \in \Omega$, $p \in [0,1]$ that is called homotopy parameter, and v_0 is an initial approximation of (2.1). Hence, it is obvious that:

$$H(v,0) = F(v) - F(v_0) = 0, \qquad H(v,1) = A(v) - f(r) = 0, \tag{2.6}$$

and the changing process of p from 0 to 1, is just that of $H(v,p)$ from $F(v) - F(v_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $F(v) - F(v_0)$ and $A(v) - f(r)$ are called homotopic. Applying the perturbation technique [18], due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of (2.4) or (2.5) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots, \tag{2.7}$$

when $p \to 1$, (2.4) or (2.5) corresponds to (2.3) and becomes the approximate solution of (2.3), i.e.,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots. \tag{2.8}$$

The series (2.8) is convergent for most cases, and the rate of convergence depends on $A(v)$, [6].

3. Application of Homotopy Herturbation Method

To approach the solution of the Sivashinsky equation (Eq. 1.1) by the means of the **HPM**, we consider the Eq. (1.1) in an operator form:

$$Lu = -(u_{xxxx} + \alpha u + ((2-u)u_x)_x), \quad t \in (0,1), \tag{3.1}$$

where the notations $L = \frac{\partial}{\partial t}$ symbolize the linear differential operators. Assuming the inverse of the operator L^{-1} exists and it can conveniently be taken as the definite integral with respect to t from 0 to t, i.e., $L^{-1} = \int_0^t (.)dt$.
Operating on both sides of Eq. (3.1) with the inverse operator of L^{-1}, yields:

$$u(x,t) - u(x,0) = -L^{-1}(u_{xxxx}(x,t) + \alpha u(x,t) + ((2-u(x,t))u_x(x,t))_x). \tag{3.2}$$

For solving this equation by **HPM**, let $F(u) = u(x,t) - u(x,0) = 0$. Hence, we may choose a convex homotopy such that:

$$H(v,p) = v(x,t) - u(x,0) + pL^{-1}(v_{xxxx}(x,t) + \alpha v(x,t) + ((2-v(x,t))v_x(x,t))_x) = 0. \tag{3.3}$$

Substituting (2.7) into (3.3) and equating the terms with identical powers of p, we have:

$p^0 : v_0(x,t) = u(x,0),$

$p^1 : v_1(x,t) = -L^{-1}((v_0(x,t))_{xxxx} + \alpha v_0(x,t) + ((2-v_0(x,t))(v_0(x,t))_x)_x),$

$p^2 : v_2(x,t) = -L^{-1}((v_1(x,t))_{xxxx} + \alpha v_1(x,t) + ((2-v_1(x,t))(v_1(x,t))_x)_x),$

$p^3 : v_3(x,t) = -L^{-1}((v_2(x,t))_{xxxx} + \alpha v_2(x,t) + ((2-v_2(x,t))(v_2(x,t))_x)_x),$
$\vdots$
So we can calculate the terms of $u = \sum_{n=0}^{\infty} v_n$, term by term, otherwise by computing some terms say k, $u \approx \Phi_k = \sum_{n=0}^{k} v_n$, where $u = \lim_{k\to\infty} \Phi_k$ an approximation to the solution would be achieved.

4. Test Examples

For purposes of illustration of the decomposition method for solving the Sivashinsky equation. The computer application program *Maple* was used to execute the algorithm that was used with the numerical examples.

Example 1. Consider the Sivashinsky equation [13]:

$$u_t + u_{xxxx} + \alpha u + ((2-u)u_x)_x = 0, \quad t \in (0,1), \tag{4.1}$$

Subject to the initial condition as:

$$u(x,0) = \operatorname{sech}^2(0.25x), \tag{4.2}$$

with $\alpha = 0.5$.
A homotopy can be readily constructed as follows:

$$u(x,t) - u(x,0) + pL^{-1}(u_{xxxx}(x,t) + \alpha u(x,t) + ((2-u(x,t))u_x(x,t))_x) = 0. \tag{4.3}$$

Substituting (2.7) into (4.3), and equating the terms with identical powers of p, we have:

$p^0 : v_0(x,t) = u(x,0) \Rightarrow \; v_0(x,t) = \text{sech}^2(0.25x)$,

$p^1 : v_1(x,t) = -L^{-1}((v_0(x,t))_{xxxx} + \alpha v_0(x,t) + ((2 - v_0(x,t))(v_0(x,t))_x)_x) \Rightarrow$

$$v_1(x,t) = -0.0625\text{sech}^2(0.25x)\tanh^4(0.25x)t + 1.375\text{sech}^2(0.25x)\tanh^2(0.25x)(0.25 - 0.25\tanh^2(0.25x))t - \text{sech}^2(0.25x)(0.25 - 0.25\tanh^2(0.25x))^2 t - 0.5\text{sech}^2(0.25x)t + 0.25\text{sech}^4(0.25x)\tanh^2(0.25x)t - 0.25\text{sech}^2(0.25x)\tanh^2(0.25x)(2 - \text{sech}^2(0.25x))t,$$

$\vdots$

Continuing this process the complete solution $u(x,t) = \lim_{k\to\infty} \Phi_k(x,t)$ found by means of k-term approximation $\Phi_k(x,t) = \sum_{n=0}^{k-1} v_n(x,t)$.

Tables 1 and 2 show the comparison between results of **HPM** and ADM [1] respectively for $t = 0.001$ and $t = 0.01$. Table.1 as well as Table.2 is obtained $\alpha = 0.5$. These tables confirm that the numerical result of approximate solution for discussed problem from **HPM** is in good agreed well with those obtained by ADM.

Example 2. Let us consider the Sivashinsky equation [13]:

$$u_t + u_{xxxx} + \alpha u = (f(u))_{xx}, \quad t \in (0,1), \quad x \in \Omega, \tag{4.4}$$

Subject to the initial condition as:

$$u(x,0) = \cos(\frac{x}{2}), \tag{4.5}$$

Table 1: Comparison of the results of HPM (3-term) and ADM (5-term) at $t = 0.001$.

x	α	HPM	ADM
0	0.5	0.9995625975	0.986
0.1	0.5	0.9989384228	0.9866433580
0.2	0.5	0.9970690078	0.9885762404
0.3	0.5	0.9939636480	0.9918071125
0.4	0.5	0.9896377276	0.9963502238
0.5	0.5	0.9841125580	1.002225834

Table 2: Comparison of the results of HPM (3-term) and ADM (5-term) at $t = 0.01$.

x	α	HPM	ADM
0	0.5	0.9956345772	0.86
0.1	0.5	0.9950154322	0.8608062394
0.2	0.5	0.9931610181	0.8632248556
0.3	0.5	0.9900803701	0.8672556733
0.4	0.5	0.9857884450	0.8728988254
0.5	0.5	0.9803059754	0.8801554096

with $\alpha = 0.5$ and $f(u) = 0.5u^2$. Following the same procedure as Example 3.1, we have:

$p^0 : v_0(x,t) = \cos(0.5x),$

$p^1 : v_1(x,t) = -0.5625\cos(0.5x)t - 0.25\sin^2(0.5x)t + 0.25\cos^2(0.5x)t,$

$$
\begin{aligned}
p^2 : v_2(x,t) = {} & -0.5(-0.31640625\cos(0.5x) + 0.375\cos^2(0.5x) - 0.375\sin^2(0.5x) - \\
& 0.25\cos(0.5x)(-0.5625\cos(0.5x) - 0.25\sin^2(0.5x) + 0.25\cos^2(0.5x)) - \\
& \sin(0.5x)(0.28125\sin(0.5x) - 0.5\sin(0.5x)\cos(0.5x)) + \\
& \cos(0.5x)(0.140625\cos(0.5x) - 0.25\cos(0.5x) - \\
& 0.25\cos^2(0.5x) + 0.25\sin^2(0.5x)))t^2,
\end{aligned}
$$

$\vdots$

Continuing this process the complete solution $u(x,t) = \lim_{k\to\infty}\Phi_k(x,t)$ found by means of k-term approximation $\Phi_k(x,t) = \sum_{n=0}^{k-1} v_n(x,t)$.

Tables 3 and 4 show the comparison between results of **HPM** and ADM [1] respectively for $t = 0.001$ and $t = 0.01$. Table.3 as well as Table.4 is obtained $\alpha = 0.5$. These tables

Table 3: Comparison of the results of HPM (3-term) and ADM (3-term) at $t = 0.001$.

x	α	HPM	ADM
0	0.5	0.9996874863	0.9996876117
0.1	0.5	0.9984372006	0.9984391979
0.2	0.5	0.9946894778	0.9946970672
0.3	0.5	0.9884537132	0.9884705451
0.4	0.5	0.9797455398	0.9797751490
0.5	0.5	0.9685867875	0.9686325483
0.6	0.5	0.9550054288	0.9550705125
0.7	0.5	0.9390355084	0.9391228424
0.8	0.5	0.9207170564	0.9208292860
0.9	0.5	0.9000959866	0.9002354400
1.0	0.5	0.8772239825	0.8773926378

Table 4: Comparison of the results of HPM (3-term) and ADM (3-term) at $t = 0.01$.

x	α	HPM	ADM
0	0.5	0.9968734902	0.9968861329
0.1	0.5	0.9956182764	0.9956495273
0.2	0.5	0.9918558688	0.9919427107
0.3	0.5	0.9855959559	0.9857746773
0.4	0.5	0.9768546577	0.9771603954
0.5	0.5	0.9656544772	0.9661207740
0.6	0.5	0.9520242386	0.9526826169
0.7	0.5	0.9359990074	0.9368785627
0.8	0.5	0.9176199914	0.9187470131
0.9	0.5	0.8969344251	0.8983320468
1.0	0.5	0.8739954402	0.8756833219

confirm that the numerical result of approximate solution for discussed problem from **HPM** is in good agreed well with those obtained by ADM.

5. Conclusion

In this work, our objective has been to show that approximate solutions of the Sivashinsky equation can be obtained by homotopy perturbation method.
The results obtained from proposed method (**HPM**) have been compared and verified with that obtained by Adomians decomposition method. The results revealed that homotopy perturbation method is powerful mathematical tool for solutions of nonlinear partial differential equations in terms of accuracy and efficiency.

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