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The solutions to the operator equation $TXS - SX^*T^* = A$ in Hilbert C*-modules

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Abstract

In this paper, we find explicit solution to the operator equation $TXS^* - SX^*T^* = A$ in the general setting of the adjointable operators between Hilbert C*-modules, when T, S have closed ranges and S is a self adjoint operator.

Keywords: Operator equation; Moore-Penrose inverse; Hilbert C*-module.
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1. Introduction

The equation $TXS^* - SX^*T^* = A$ was studied by Yuan [8] for finite matrices and Xu et al. [6] generalized the results to Hilbert C*-modules, under the condition that ran(S) is contained in ran(T). When T equals an identity matrix or identity operator, this equation reduces to $XS^* - SX^* = A$, which was studied by Braden [1] for finite matrices, and Djordjevic [2] for the Hilbert space operators. In this paper, we find explicit solution to the operator equation $TXS^* - SX^*T^* = A$ in the general setting of the adjointable operators between Hilbert C*-modules, when T, S have closed ranges and S is a self adjoint operator.

Throughout this paper, $\mathcal{A}$ is a C*-algebra. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules, and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of the adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the range and the null space of T are denoted by ran(T) and ker(T) respectively. In case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{X})$ which we

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abbreviate to $\mathcal{L}(\mathcal{X})$, is a C*-algebra. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity. Let $\mathcal{M}$ be closed submodule of a Hilbert $\mathcal{A}$-module $\mathcal{X}$, then $P_{\mathcal{M}}$ is orthogonal projection onto $\mathcal{M}$, in the sense that $P_{\mathcal{M}}$ is self adjoint idempotent operator.

Theorem 1.1. [4, Theorem 3.2] Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then

- $\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with complement ran(T*).
- ran(T) is orthogonally complemented in $\mathcal{Y}$, with complement $\ker(T^*)$.
- The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.

Xu and Sheng [7] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ satisfying the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

It is well-known that $T^\dagger$ exists if and only if ran(T) is closed, and in this case $(T^\dagger)^* = (T^*)^\dagger$. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed range, then $TT^\dagger$ is the orthogonal projection from $\mathcal{Y}$ onto ran(T) and $T^\dagger T$ is the orthogonal projection from $\mathcal{X}$ onto ran(T*). Projection, in the sense that they are self adjoint idempotent operators.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. Note that $P_{\mathcal{M}}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}}, \quad T_2 = P_{\mathcal{N}} T(1 - P_{\mathcal{M}})$, $T_3 = (1 - P_{\mathcal{N}}) T P_{\mathcal{M}}$ and $T_4 = (1 - P_{\mathcal{N}}) T (1 - P_{\mathcal{M}})$.

The proof of the following Lemma can be found in [5, Corollary 1.2.] or [3, Lemma 1.1.].

Lemma 1.2. Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

The proof of the following lemma is the same as in the matrix case.

Lemma 1.3. Suppose that $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ have closed ranges, $A \in \mathcal{L}(\mathcal{Y})$.Then the equation

$$TXS = A \quad , \quad X \in \mathcal{L}(\mathcal{Z}, \mathcal{X}) \tag{1.1}$$

has a solution if and only if

$$TT^{\dagger}AS^{\dagger}S = A.$$

In which case, any solution X to equation (1.1) is of the form

$$X \;=\; T^{\dagger}AS^{\dagger} + V - T^{\dagger}TVSS^{\dagger},$$

where $V \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ is arbitrary.

2. Main results

In this section, we find explicit solution to the operator equation

$$TXS^{*} - SX^{*}T^{*} = A, \tag{2.1}$$

in the general setting of the adjointable operators between Hilbert C*-modules, when T, S have closed ranges and S is self adjoint operator. Hence equation (2.1) get into

$$TXS - SX^{*}T^{*} = A. \tag{2.2}$$

Lemma 2.1. *Suppose that $\mathcal{Y}, \mathcal{Z}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator, $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:*
(a) There exists a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ to the operator equation $TX - X^{*}T^{*} = A$.
(b) $A = -A^{*}$
If (a) or (b) is satisfied, then any solution to equation

$$TX - X^{*}T^{*} = A \quad , \quad X \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \tag{2.3}$$

has the form

$$X = \frac{1}{2}T^{-1}A + T^{-1}Z, \tag{2.4}$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfying $Z^{*} = Z$.

Proof .$(a) \Rightarrow (b)$: Obvious.
$(b) \Rightarrow (a)$: Note that, if $A = -A^{*}$ then $X = \frac{1}{2}T^{-1}A + T^{-1}Z$ is a solution to equation (2.3). The following sentences state this claim

$$\begin{aligned}
& T(\frac{1}{2}T^{-1}A + T^{-1}Z) - (\frac{1}{2}A^{*}(T^{*})^{-1} + Z^{*}(T^{*})^{-1})T^{*} \\
= \;& \frac{1}{2}(TT^{-1}A - A^{*}(T^{*})^{-1}T^{*}) + TT^{-1}Z - Z^{*}(T^{*})^{-1}T^{*} \\
= \;& A + Z - Z^{*} = A.
\end{aligned}$$

On the other hand, let X be any solution to equation (2.3). Then $X = T^{-1}A + T^{-1}X^*T^*$. We have

$$\begin{aligned} X &= T^{-1}A + T^{-1}X^*T^* \\ &= \frac{1}{2}T^{-1}A + \frac{1}{2}T^{-1}A + T^{-1}X^*T^* \\ &= \frac{1}{2}T^{-1}A + T^{-1}(\frac{1}{2}A + X^*T^*). \end{aligned}$$

Taking $Z = \frac{1}{2}A + X^*T^*$, we get $Z^* = Z$. $\square$

Theorem 2.2. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $A, S \in \mathcal{L}(\mathcal{X})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that S is self adjoint, both S and T have closed ranges, and $AS^\dagger S = A$ and $T^\dagger S^\dagger S = T^\dagger$. Then the following statements are equivalent:*
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ to equation (2.2).
(b) $A = -A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.

If (a) or (b) is satisfied, then any solution to equation (2.2) *has the form*

$$X = T^\dagger AS^\dagger - \frac{1}{2}T^\dagger ATT^\dagger S^\dagger + T^\dagger ZTT^\dagger S^\dagger + V - T^\dagger TVSS^\dagger, \tag{2.5}$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z - Z^*)T = 0$ and $V \in \mathcal{L}(\mathcal{X})$ is arbitrary.

Proof . $(a) \Rightarrow (b)$: Obviously, $A = -A^*$. Also,

$$\begin{aligned} &(1 - TT^\dagger)A(1 - TT^\dagger) = (1 - TT^\dagger)(TXS^* - SX^*T^*)(1 - TT^\dagger) \\ = \ &(T - TT^\dagger T)XS^*(1 - TT^\dagger) - (1 - TT^\dagger)SX^*(T^* - T^*TT^\dagger) = 0. \end{aligned}$$

$(b) \Rightarrow (a)$: Note that the condition $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$ is equivalent to $A = ATT^\dagger + TT^\dagger A - TT^\dagger ATT^\dagger$. On the other hand, since $T^*(Z - Z^*)T = 0$, then $(Z - Z^*)T \in \ker(T^*) = \ker(T^\dagger)$. Therefore $T^\dagger(Z - Z^*)T = 0$ or equivalently $TT^\dagger ZTT^\dagger - TT^\dagger Z^*(T^\dagger)^*T^* = 0$. Hence we have

$$\begin{aligned} &TT^\dagger AS^\dagger S - \frac{1}{2}TT^\dagger ATT^\dagger S^\dagger S + TT^\dagger ZTT^\dagger S^\dagger S + T(V - T^\dagger TVSS^\dagger)S \\ - \ &SS^\dagger A^*(T^\dagger)^*T^* + \frac{1}{2}SS^\dagger TT^\dagger A^*(T^\dagger)^*T^* + S^\dagger TT^\dagger Z^*(T^\dagger)^*T^* + S^\dagger(V^* - SS^\dagger V^*T^\dagger T)T^* \\ = \ &ATT^\dagger + TT^\dagger A - TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger - TT^\dagger Z^*(T^\dagger)^*T^* = A. \end{aligned}$$

That is, any operator X of the form (2.5) is a solution to equation (2.2).

Since T has closed range, we have $\mathcal{Z} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$. Now, T has the matrix form

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. On the other hand, $A = -A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$ imply that A has the form

$$A = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where $A_1 = -A_1^*$. Since T has closed range, so $\mathcal{X} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$ and $\mathcal{Y} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$, and hence operator X has the following matrix form

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix}.$$

Now by using matrix form for operators T, X and A, we have

$$\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_4 \end{bmatrix} - \begin{bmatrix} S_1 & S_2^* \\ S_2 & S_4 \end{bmatrix}\begin{bmatrix} X_1^* & X_3^* \\ X_2^* & X_4^* \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} T_1X_1S_1 + T_1X_2S_2^* - S_1X_1^*T_1^* - S_2X_2^*T_1^* & T_1X_1S_2 + T_1X_2S_4 \\ -S_2^*X_1^*T_1^* - S_4X_2^*T_1^* & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix}.$$

Therefore

$$T_1(X_1S_1 + X_2S_2^*) - (S_1X_1^* + S_2X_2^*)T_1^* = A_1, \tag{2.6}$$

$$T_1X_1S_2 + T_1X_2S_4 = A_2. \tag{2.7}$$

By Lemma 1.2, T_1 is invertible. Hence, Lemma 2.1 implies that

$$X_1S_1 + X_2S_2^* = \frac{1}{2}T_1^{-1}A_1 + T_1^{-1}Z_1, \tag{2.8}$$

where $Z_1 \in \mathcal{L}(\text{ran(T)})$ satisfy $Z_1^* = Z_1$. Now, multiplying T_1^{-1} from the left to equation (2.7), we get

$$X_1S_2 + X_2S_4 = T_1^{-1}A_2. \tag{2.9}$$

Now, by applying equations (2.8) and (2.9), we have

$$\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1 & S_2 \\ S_2^* & S_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}T_1^{-1}A_1 + T_1^{-1}Z_1 & T_1^{-1}A_2 \\ 0 & 0 \end{bmatrix}. \tag{2.10}$$

On the other hand, since

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} : \begin{bmatrix} \text{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \text{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

then

$$\frac{1}{2}T^\dagger ATT^\dagger = \begin{bmatrix} \frac{1}{2}T_1^{-1}A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T^\dagger ZTT^\dagger = \begin{bmatrix} T_1^{-1}Z_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$T^\dagger A(1 - TT^\dagger) = \begin{bmatrix} 0 & T^{-1}A_2 \\ 0 & 0 \end{bmatrix}$$

and

$$T^\dagger TX = \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}.$$

Consequently equation (2.10) gets into

$$\begin{aligned} T^\dagger TXS &= \frac{1}{2}T^\dagger ATT^\dagger + T^\dagger ZTT^\dagger + T^\dagger A(1 - TT^\dagger) \\ &= T^\dagger A - \frac{1}{2}T^\dagger ATT^\dagger + T^\dagger ZTT^\dagger \end{aligned} \tag{2.11}$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z - Z^*)T = 0$. By multiplication $S^\dagger S$ on the right and $T^\dagger T$ on the left to equation (2.11) and by these facts that $AS^\dagger S = A$, $T^\dagger S^\dagger S = T^\dagger$ and Lemma 1.3 implies that

$$X = T^\dagger AS^\dagger - \frac{1}{2}T^\dagger ATT^\dagger S^\dagger + T^\dagger ZTT^\dagger S^\dagger + V - T^\dagger TVSS^\dagger,$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z - Z^*)T = 0$ and $V \in \mathcal{L}(\mathcal{X})$ is arbitrary. □

References

[1] H. Braden, *The equations* $A^TX \pm X^TA = B$, SIAM J. Matrix Anal. Appl. 20 (1998) 295–302.

[2] D.S. Djordjevic, *Explicit solution of the operator equation* $A^*X + X^*A = B$, J. Comput. Appl. Math. 200 (2007) 701–704.

[3] D.S. Djordjevic and N.C. Dincic,*Reverse order law for the Moore-Penrose inverse,* J. Math. Anal. Appl. 361 (2010) 252–261.

[4] E.C. Lance, *Hilbert C*-Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.

[5] M. Mohammadzadeh Karizaki, M. Hassani, M. Amyari and M. Khosravi, *Operator matrix of Moore-Penrose inverse operators on Hilbert C*-modules*, Colloq. Math. 140 (2015) 171–182.

[6] Q. Xu, L. Sheng, Y. Gu, *The solutions to some operator equations*, Linear Algebra Appl. 429 (2008) 1997–2024.

[7] Q. Xu and L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C*-modules*, Linear Algebra Appl. 428 (2008) 992–1000.

[8] Y. Yuan, *Solvability for a class of matrix equation and its applications*, J. Nanjing Univ. (Math. Biquart.) 18 (2001) 221–227.