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Some inequalities in connection to relative orders of entire functions of several complex variables

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Abstract

Let f, g and h be all entire functions of several complex variables. In this paper we would like to establish some inequalities on the basis of relative order and relative lower order of f with respect to g when the relative orders and relative lower orders of both f and g with respect to h are given.

Keywords: entire function; several complex variables; relative order; relative lower order.
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1. Introduction and preliminaries

Let f be an entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i,\ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i,\ i = 1, 2\}$. Then in view of maximum principal and Hartogs theorem {[7], p. 2, p. 51}, $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2.

The following definition is well known:

Definition 1.1. {[7], p. 339, (see also [1])} The order ${}_{v_2}\rho_f$ and the lower order ${}_{v_2}\lambda_f$ of an entire function f of two complex variables are defined as

$${}_{v_2}\rho_f = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)} \text{ and } {}_{v_2}\lambda_f = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)},$$

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where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.

If we consider the above definiton for single variable, then the definition coincides with the classical definition of order (see [14]) which is as follows:

Definition 1.2. ([14]) The order ρ_f and the lower order λ_f of an entire function f are defined in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r},$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$.

If f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f{}^{-1}(s) = \infty$. Bernal {[2], [3]} introduced the definition of relative order of g with respect to f, denoted by $\rho_f(g)$ as follows :

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [14] if $g(z) = \exp z$.

During the past decades, several authors (see [5],[9],[10],[11],[12],[13]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

Definition 1.3. ([4]) The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:

$$\begin{aligned} {}_{v_2}\rho_g(f) &= \inf\{\mu > 0 : M_f(r_1, r_2) < M_g(r_1^\mu, r_2^\mu) ; r_1 \geq R(\mu), r_2 \geq R(\mu)\} \\ &= \limsup_{r_1, r_2\to\infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \end{aligned}$$

where f and g are entire functions holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1.1 {see [4]} if $g(z) = \exp(z_1 z_2)$.

Extending this notion, Dutta [6] introduced the idea of relative order of entire functions of several complex variables in the following way:

Definition 1.4. ([6]) Let $f(z_1, z_2, \ldots, z_n)$ and $g(z_1, z_2, \ldots, z_n)$ be any two entire functions of n complex variables $z_1, z_2, \ldots, z_n$ with maximum modulus functions $M_f(r_1, r_2, \ldots, r_n)$ and $M_g(r_1, r_2, \ldots, r_n)$ respectively then the relative order of f with respect to g, denoted by ${}_{v_n}\rho_g(f)$ is defined by

$${}_{v_n}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2, \ldots, r_n) < M_g(r_1^\mu, r_2^\mu, \ldots, r_n^\mu) ; \text{for } r_i \geq R(\mu), i = 1, 2, \ldots, n\}.$$

The above definition can equivalently be written as

$$_{v_n}\rho_g(f) = \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log M_g^{-1} M_f(r_1,r_2,\ldots,r_n)}{\log(r_1 r_2 \ldots r_n)}.$$

Similarly, one can define the relative lower order of f with respect to g denoted by $_{v_n}\lambda_g(f)$ as follows:

$$_{v_n}\lambda_g(f) = \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log M_g^{-1} M_f(r_1,r_2,\ldots,r_n)}{\log(r_1 r_2 \ldots r_n)}.$$

If we consider $g(z_1, z_2, \ldots, z_n) = \exp(z_1 z_2 \ldots z_n)$, then Definition 1.4 reduces to the following classical definition of order and lower order in connection with several complex variables:

Definition 1.5. The order $_{v_n}\rho_f$ and the lower order $_{v_n}\lambda_f$ of an entire function f of two complex variables are defined as

$$\begin{aligned} _{v_n}\rho_f &= \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[2]} M_f(r_1,r_2,\ldots,r_n)}{\log(r_1 r_2 \ldots r_n)} \text{ and} \\ _{v_n}\lambda_f &= \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[2]} M_f(r_1,r_2,\ldots,r_n)}{\log(r_1 r_2 \ldots r_n)}. \end{aligned}$$

Also an entire function of several complex variables for which order and lower order are the same is said to be of regular growth. The function $\exp(z_1 z_2 \ldots z_n)$ is an example of regular growth of entire function of several complex variables. Further the functions which are not of regular growth are said to be of irregular growth. Similarly for an entire function of several complex variables for which relative order and relative lower order with respect to another entire function of several complex variables are the same is said to be of regular relative growth with respect to that entire function. Also the functions which are not of regular relative growth with respect to entire functions are said to be of irregular relative growth with respect to respective entire functions.

Now a question may arise about relative order (relative lower order) of f with respect to another entire function g when relative order (relative lower oreder) of f and g with respect to another entire function h are respectively given. In this paper we intend to provide this answer. Interested researchers may also think over to establish such type of results on the coupled systems of equations with entire and polynomial functions {cf. [8]} . We do not explain the standard definitions and notations in the theory of entire function of two complex variables as those are available in [7].

2. The main results

In this section we present the main results of the paper.

Theorem 2.1. *Let f, g and h be any three entire functions of several complex variables such that relative order (relative lower order) of f with respect to h and relative order (relative lower order) of f with respect to h are $_{v_n}\rho_h(f)$ $(_{v_n}\lambda_h(f))$ and $_{v_n}\rho_h(g)$ $(_{v_n}\lambda_h(g))$ respectively. Then*

$$\begin{aligned} \frac{_{v_n}\lambda_h(f)}{_{v_n}\rho_h(g)} \le {_{v_n}\lambda_g(f)} &\le \min\left\{\frac{_{v_n}\lambda_h(f)}{_{v_n}\lambda_h(g)}, \frac{_{v_n}\rho_h(f)}{_{v_n}\rho_h(g)}\right\} \\ &\le \max\left\{\frac{_{v_n}\lambda_h(f)}{_{v_n}\lambda_h(g)}, \frac{_{v_n}\rho_h(f)}{_{v_n}\rho_h(g)}\right\} \le {_{v_n}\rho_g(f)} \le \frac{_{v_n}\rho_h(f)}{_{v_n}\lambda_h(g)}. \end{aligned}$$

Proof . From the definitions of ${}_{v_n}\rho_h(f)$ and ${}_{v_n}\lambda_h(f)$, we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\begin{aligned} M_h^{-1}M_f(r_1,r_2,...,r_n) &\leq \exp\{({}_{v_n}\rho_h(f)+\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_f(r_1,r_2,...,r_n) &\leq M_h[\exp\{({}_{v_n}\rho_h(f)+\varepsilon)\log(r_1r_2...r_n)\}]\ , \end{aligned} \tag{2.1}$$

$$\begin{aligned} M_h^{-1}M_f(r_1,r_2,...,r_n) &\geq \exp\{({}_{v_n}\lambda_h(f)-\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_f(r_1,r_2,...,r_n) &\geq M_h[\exp\{({}_{v_n}\lambda_h(f)-\varepsilon)\log(r_1r_2...r_n)\}]\ , \end{aligned} \tag{2.2}$$

and also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity we get that

$$\begin{aligned} M_h^{-1}M_f(r_1,r_2,...,r_n) &\geq \exp\{({}_{v_n}\rho_h(f)-\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_f(r_1,r_2,...,r_n) &\geq M_h[\exp\{({}_{v_n}\rho_h(f)-\varepsilon)\log(r_1r_2...r_n)\}]\ , \end{aligned} \tag{2.3}$$

$$\begin{aligned} M_h^{-1}M_f(r_1,r_2,...,r_n) &\leq \exp\{({}_{v_n}\lambda_h(f)+\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_f(r_1,r_2,...,r_n) &\leq M_h[\exp\{({}_{v_n}\lambda_h(f)+\varepsilon)\log(r_1r_2...r_n)\}]\,. \end{aligned} \tag{2.4}$$

Similarly from the definitions of ${}_{v_n}\rho_h(g)$ and ${}_{v_n}\lambda_h(g)$, it follows for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} M_h^{-1}M_g(r_1,r_2,...,r_n) &\leq \exp\{({}_{v_n}\rho_h(g)+\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_g(r_1,r_2,...,r_n) &\leq M_h[\exp\{({}_{v_n}\rho_h(g)+\varepsilon)\log(r_1r_2...r_n)\}] \\ i.e.,\ M_h(r_1,r_2,...,r_n) &\geq M_g\left[\exp\left[\frac{\log(r_1r_2...r_n)}{({}_{v_n}\rho_h(g)+\varepsilon)}\right]\right]. \end{aligned} \tag{2.5}$$

$$\begin{aligned} M_h^{-1}M_g(r_1,r_2,...,r_n) &\geq \exp\{({}_{v_n}\lambda_h(g)-\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_g(r_1,r_2,...,r_n) &\geq M_h[\exp\{({}_{v_n}\lambda_h(g)-\varepsilon)\log(r_1r_2...r_n)\}] \\ i.e.,\ M_h(r_1,r_2,...,r_n) &\leq M_g\left[\exp\left[\frac{\log(r_1r_2...r_n)}{({}_{v_n}\lambda_h(g)-\varepsilon)}\right]\right] \end{aligned} \tag{2.6}$$

and for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity we obtain that

$$\begin{aligned} M_h^{-1}M_g(r_1,r_2,...,r_n) &\geq \exp\{({}_{v_n}\rho_h(g)-\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_g(r_1,r_2,...,r_n) &\geq M_h[\exp\{({}_{v_n}\rho_h(g)-\varepsilon)\log(r_1r_2...r_n)\}] \\ i.e.,\ M_h(r_1,r_2,...,r_n) &\leq M_g\left[\exp\left[\frac{\log(r_1r_2...r_n)}{({}_{v_n}\rho_h(g)-\varepsilon)}\right]\right]. \end{aligned} \tag{2.7}$$

$$\begin{aligned} M_h^{-1}M_g(r_1,r_2,...,r_n) &\leq \exp\{({}_{v_n}\lambda_h(g)+\varepsilon)\log(r_1r_2...r_n)\} \\ i.e.,\ M_g(r_1,r_2,...,r_n) &\leq M_h[\exp\{({}_{v_n}\lambda_h(g)+\varepsilon)\log(r_1r_2...r_n)\}] \\ i.e.,\ M_h(r_1,r_2,...,r_n) &\geq M_g\left[\exp\left[\frac{\log(r_1r_2...r_n)}{({}_{v_n}\lambda_h(g)+\varepsilon)}\right]\right]. \end{aligned} \tag{2.8}$$

Now from (2.3) and in view of (2.5), we get for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\log M_g^{-1}M_f(r_1,r_2,...,r_n) \geq \log M_g^{-1}M_h[\exp\{({}_{v_n}\rho_h(f)-\varepsilon)\log(r_1r_2...r_n)\}]$$

$$\begin{aligned} & i.e.,\ \log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \\ \geq\ & \log M_g^{-1} M_g\left[\exp\left[\frac{\log \exp\left\{\left({ }_{v_n} \rho_h(f)-\varepsilon\right) \log \left(r_1 r_2 \ldots r_n\right)\right\}}{\left({ }_{v_n} \rho_h(g)+\varepsilon\right)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e.,\ \log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) & \geq \frac{\left({ }_{v_n} \rho_h(f)-\varepsilon\right)}{\left({ }_{v_n} \rho_h(g)+\varepsilon\right)} \log \left(r_1 r_2 \ldots r_n\right) \\ i.e.,\ \frac{\log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right)}{\log \left(r_1 r_2 \ldots r_n\right)} & \geq \frac{\left({ }_{v_n} \rho_h(f)-\varepsilon\right)}{\left({ }_{v_n} \rho_h(g)+\varepsilon\right)}. \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\begin{aligned} \limsup_{r_1, r_2 \rightarrow \infty} \frac{\log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right)}{\log \left(r_1 r_2 \ldots r_n\right)} & \geq \frac{{ }_{v_n} \rho_h(f)}{{ }_{v_n} \rho_h(g)} \\ i.e.,\ { }_{v_n} \rho_g(f) & \geq \frac{{ }_{v_n} \rho_h(f)}{{ }_{v_n} \rho_h(g)}. \end{aligned} \tag{2.9}$$

Analogously, from (2.2) and in view of (2.8) it follows for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity that

$$\log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \geq \log M_g^{-1} M_h\left[\exp \left\{\left({ }_{v_n} \lambda_h(f)-\varepsilon\right) \log \left(r_1 r_2 \ldots r_n\right)\right\}\right]$$

$$\begin{aligned} & i.e.,\ \log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \\ \geq\ & \log M_g^{-1} M_g\left[\exp\left[\frac{\log \exp\left\{\left({ }_{v_n} \lambda_h(f)-\varepsilon\right) \log \left(r_1 r_2 \ldots r_n\right)\right\}}{\left({ }_{v_n} \lambda_h(g)+\varepsilon\right)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e.,\ \log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) & \geq \frac{\left({ }_{v_n} \lambda_h(f)-\varepsilon\right)}{\left({ }_{v_n} \lambda_h(g)+\varepsilon\right)} \log \left(r_1 r_2 \ldots r_n\right) \\ i.e.,\ \frac{\log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right)}{\log \left(r_1 r_2 \ldots r_n\right)} & \geq \frac{\left({ }_{v_n} \lambda_h(f)-\varepsilon\right)}{\left({ }_{v_n} \lambda_h(g)+\varepsilon\right)}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\begin{aligned} \limsup_{r_1, r_2 \rightarrow \infty} \frac{\log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right)}{\log \left(r_1 r_2 \ldots r_n\right)} & \geq \frac{{ }_{v_n} \lambda_h(f)}{{ }_{v_n} \lambda_h(g)} \\ i.e.,\ { }_{v_n} \rho_g(f) & \geq \frac{{ }_{v_n} \lambda_h(f)}{{ }_{v_n} \lambda_h(g)}. \end{aligned} \tag{2.10}$$

Again in view of (2.6), we have from (2.1) for all sufficiently large values of $r_1, r_2, \ldots, r_n$ that

$$\log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \leq \log M_g^{-1} M_h\left[\exp \left\{\left({ }_{v_n} \rho_h(f)+\varepsilon\right) \log \left(r_1 r_2 \ldots r_n\right)\right\}\right]$$

$$\begin{aligned} & i.e.,\ \log M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \\ \leq\ & \log M_g^{-1} M_g\left[\exp\left[\frac{\log \exp\left\{\left({ }_{v_n} \rho_h(f)+\varepsilon\right) \log \left(r_1 r_2 \ldots r_n\right)\right\}}{\left({ }_{v_n} \lambda_h(g)-\varepsilon\right)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) &\leq \frac{({}_{v_n}\rho_h(f) + \varepsilon)}{({}_{v_n}\lambda_h(g) - \varepsilon)} \log(r_1 r_2 ... r_n) \\ i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\leq \frac{({}_{v_n}\rho_h(f) + \varepsilon)}{({}_{v_n}\lambda_h(g) - \varepsilon)}. \end{aligned}$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\begin{aligned} \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\leq \frac{{}_{v_n}\rho_h(f)}{{}_{v_n}\lambda_h(g)} \\ i.e., \ {}_{v_n}\rho_g(f) &\leq \frac{{}_{v_n}\rho_h(f)}{{}_{v_n}\lambda_h(g)}. \end{aligned} \tag{2.11}$$

Again from (2.2) and in view of (2.5), we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log M_g^{-1} M_h[\exp\{({}_{v_n}\lambda_h(f) - \varepsilon) \log(r_1 r_2 ... r_n)\}]$$

$$\begin{aligned} & i.e., \ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \\ \geq & \ \log M_g^{-1} M_g \left[\exp\left[\frac{\log \exp\{({}_{v_n}\lambda_h(f) - \varepsilon) \log(r_1 r_2 ... r_n)\}}{({}_{v_n}\rho_h(g) + \varepsilon)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) &\geq \frac{({}_{v_n}\lambda_h(f) - \varepsilon)}{({}_{v_n}\rho_h(g) + \varepsilon)} \log(r_1 r_2 ... r_n) \\ i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\geq \frac{({}_{v_n}\lambda_h(f) - \varepsilon)}{({}_{v_n}\rho_h(g) + \varepsilon)}. \end{aligned}$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\begin{aligned} \liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\geq \frac{{}_{v_n}\lambda_h(f)}{{}_{v_n}\rho_h(g)} \\ i.e., \ {}_{v_n}\lambda_g(f) &\geq \frac{{}_{v_n}\lambda_h(f)}{{}_{v_n}\rho_h(g)}. \end{aligned} \tag{2.12}$$

Also in view of (2.7), we get from (2.1) for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \leq \log M_g^{-1} M_h[\exp\{({}_{v_n}\rho_h(f) + \varepsilon) \log(r_1 r_2 ... r_n)\}]$$

$$\begin{aligned} & i.e., \ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \\ \leq & \ \log M_g^{-1} M_g \left[\exp\left[\frac{\log \exp\{({}_{v_n}\rho_h(f) + \varepsilon) \log(r_1 r_2 ... r_n)\}}{({}_{v_n}\rho_h(g) - \varepsilon)}\right]\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) &\leq \frac{({}_{v_n}\rho_h(f) + \varepsilon)}{({}_{v_n}\rho_h(g) - \varepsilon)} \log(r_1 r_2 ... r_n) \\ i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\leq \frac{({}_{v_n}\rho_h(f) + \varepsilon)}{({}_{v_n}\rho_h(g) - \varepsilon)}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\begin{aligned}\liminf_{r_1,r_2\to\infty}\frac{\log M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\log\left(r_1r_2...r_n\right)} &\leq \frac{{}_{v_n}\rho_h\left(f\right)}{{}_{v_n}\rho_h\left(g\right)}\\ i.e.,\ {}_{v_n}\lambda_g\left(f\right) &\leq \frac{{}_{v_n}\rho_h\left(f\right)}{{}_{v_n}\rho_h\left(g\right)}. \qquad (2.13)\end{aligned}$$

Similarly from (2.4) and in view of (2.6), it follows for a sequence of values of $r_1,r_2,...,r_n$ tending to infinity that

$$\log M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\leq \log M_g^{-1}M_h\left[\exp\left\{\left({}_{v_n}\lambda_h\left(f\right)+\varepsilon\right)\log\left(r_1r_2...r_n\right)\right\}\right]$$

$$\begin{aligned}& i.e.,\ \log M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)\\ \leq\ & \log M_g^{-1}M_g\left[\exp\left[\frac{\log\exp\left\{\left({}_{v_n}\lambda_h\left(f\right)+\varepsilon\right)\log\left(r_1r_2...r_n\right)\right\}}{\left({}_{v_n}\lambda_h\left(g\right)-\varepsilon\right)}\right]\right]\end{aligned}$$

$$\begin{aligned} i.e.,\ \log M_g^{-1}M_f\left(r_1,r_2,...,r_n\right) &\leq \frac{\left({}_{v_n}\lambda_h\left(f\right)+\varepsilon\right)}{\left({}_{v_n}\lambda_h\left(g\right)-\varepsilon\right)}\log\left(r_1r_2...r_n\right)\\ i.e.,\ \frac{\log M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\log\left(r_1r_2...r_n\right)} &\leq \frac{\left({}_{v_n}\lambda_h\left(f\right)+\varepsilon\right)}{\left({}_{v_n}\lambda_h\left(g\right)-\varepsilon\right)}.\end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\begin{aligned}\liminf_{r_1,r_2\to\infty}\frac{\log M_g^{-1}M_f\left(r_1,r_2,...,r_n\right)}{\log\left(r_1r_{2_{r_1r_2...r_n}}\right)} &\leq \frac{{}_{v_n}\lambda_h\left(f\right)}{{}_{v_n}\lambda_h\left(g\right)}\\ i.e.,\ {}_{v_n}\lambda_g\left(f\right) &\leq \frac{{}_{v_n}\lambda_h\left(f\right)}{{}_{v_n}\lambda_h\left(g\right)}. \qquad (2.14)\end{aligned}$$

Thus the theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14). □

In view of Theorem 2.1, one can easily verify the following corollaries:

Corollary 2.2. Let f be an entire function of several complex variables with regular relative growth with respect to an entire function h of several complex variables and g be entire another entire function of several complex variables. Then

$${}_{v_n}\lambda_g\left(f\right)=\frac{{}_{v_n}\rho_h\left(f\right)}{{}_{v_n}\rho_h\left(g\right)} \quad \text{and} \quad {}_{v_n}\rho_g\left(f\right)=\frac{{}_{v_n}\rho_h\left(f\right)}{{}_{v_n}\lambda_h\left(g\right)}.$$

In addition, if ${}_{v_n}\rho_h\left(f\right)={}_{v_n}\rho_h\left(g\right)$, then

$${}_{v_n}\lambda_g\left(f\right)={}_{v_n}\rho_f\left(g\right)=1.$$

Corollary 2.3. Let f, g, h be three entire functions of several complex variables such that g is of regular relative growth with respect to an entire function h. Then

$${}_{v_n}\lambda_g\left(f\right)=\frac{{}_{v_n}\lambda_h\left(f\right)}{{}_{v_n}\rho_h\left(g\right)} \quad \text{and} \quad {}_{v_n}\rho_g\left(f\right)=\frac{{}_{v_n}\rho_h\left(f\right)}{{}_{v_n}\rho_h\left(g\right)}.$$

In addition, if ${}_{v_n}\rho_h\left(f\right)={}_{v_n}\rho_h\left(g\right)$ then

$${}_{v_n}\rho_g\left(f\right)={}_{v_n}\lambda_f\left(g\right)=1.$$

Corollary 2.4. Let f and g be any two entire functions of several complex variables with regular relative growth with respect to another entire function h of several complex variables respectively. Then

$$_{v_n}\lambda_g(f) = {}_{v_n}\rho_g(f) = \frac{_{v_n}\rho_h(f)}{_{v_n}\rho_h(g)}.$$

Corollary 2.5. Let f and g be any two entire functions of several complex variables with regular relative growth and regular relative growth with respect to another entire function h of several complex variables respectively. Also suppose that $_{v_n}\rho_h(f) = {}_{v_n}\rho_h(g)$. Then

$$_{v_n}\lambda_g(f) = {}_{v_n}\rho_g(f) = {}_{v_n}\lambda_f(g) = {}_{v_n}\rho_f(g) = 1.$$

Corollary 2.6. Let f, g and h be any three entire functions of several complex variables such that either f is not of regular relative growth or g is not of regular relative growth with respect to h. Then

$$_{v_n}\rho_g(f) \cdot {}_{v_n}\rho_f(g) \geq 1$$

when f and g are both of regular relative growth with respect to h respectively, then

$$_{v_n}\rho_g(f) \cdot {}_{v_n}\rho_f(g) = 1.$$

Corollary 2.7. Let f, g and h be any three entire functions of several complex variables such that either f is not of regular relative growth or g is not of regular relative growth with respect to h. Then

$$_{v_n}\lambda_g(f) \cdot {}_{v_n}\lambda_f(g) \leq 1.$$

when f and g are both of regular relative growth with respect to h respectively, then

$$_{v_n}\lambda_g(f) \cdot {}_{v_n}\lambda_f(g) = 1.$$

Corollary 2.8. Let f and g be any two entire functions of several complex variables . Then

$$\begin{aligned}
(i)\ {}_{v_n}\lambda_g(f) &= \infty \text{ when } {}_{v_n}\rho_h(g) = 0,\\
(ii)\ {}_{v_n}\rho_g(f) &= \infty \text{ when } {}_{v_n}\lambda_h(g) = 0,\\
(iii)\ {}_{v_n}\lambda_g(f) &= 0 \text{ when } {}_{v_n}\rho_h(g) = \infty\\
&\text{and}\\
(iv)\ {}_{v_n}\rho_g(f) &= 0 \text{ when } {}_{v_n}\lambda_h(g) = \infty.
\end{aligned}$$

Corollary 2.9. Let f and g be any two entire functions of several complex variables . Then

$$\begin{aligned}
(i)\ {}_{v_n}\rho_g(f) &= 0 \text{ when } {}_{v_n}\rho_h(f) = 0,\\
(ii)\ {}_{v_n}\lambda_g(f) &= 0 \text{ when } {}_{v_n}\lambda_h(f) = 0,\\
(iii)\ {}_{v_n}\rho_g(f) &= \infty \text{ when } {}_{v_n}\rho_h(f) = \infty\\
&\text{and}\\
(iv)\ {}_{v_n}\lambda_g(f) &= \infty \text{ when } {}_{v_n}\lambda_h(f) = \infty.
\end{aligned}$$

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