Int. J. Nonlinear Anal. Appl. **2** (2011) No.1, 1–10
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

BIFURCATION IN A VARIATIONAL PROBLEM ON A SURFACE WITH A CONSTRAINT

P. VYRIDIS[1]*

Abstract. We describe a variational problem on a surface under a constraint of geometrical character. Necessary and sufficient conditions for the existence of bifurcation points are provided. In local coordinates the problem corresponds to a quasilinear elliptic boundary value problem. The problem can be considered as a physical model for several applications referring to continuum medium and membranes.

1. Bifurcation in Variational Problems

The classical bifurcation problem has the following form:

$$f[u, \lambda] = 0 \tag{1.1}$$

where f is a mapping $f : X \times \mathbb{R} \to \mathbb{R}$, X is a Hilbert space and $\lambda \in \mathbb{R}$ is a real parameter. We always assume that

$$f[0, \lambda] = 0$$

for all $\lambda \in \mathbb{R}$.

Definition 1.1. The number $\lambda_0 \in \mathbb{R}$ is called a bifurcation point for equation (1.1) if and only if in every sufficiently small neighborhood $\mathcal{U} \subset X \times \mathbb{R}$ of $(0, \lambda_0)$ there exists a solution (u, λ) of (1.1) with $u \neq 0$.

The present problem has been investigated for a long time with plenty of interesting results. The problem has been successfully resolved and the necessary and sufficient conditions for the existence of bifurcation points have already been posed.

The bifurcation problem of variational character is of special interest since the integral functionals involved to this problem are models of the deformation energy of the continuum medium [1]. The problem of the form (1.1) is reduced to

$$f[u, \lambda] = G'[u] - \lambda\, F'[u] = 0 \tag{1.2}$$

where G, F are functionals defined on the Hilbert space X. It is proved by I. V. Skrypnik that if the functionals F, G satisfy some specific assumptions, then the necessary conditions for the existence of bifurcation points are also sufficient ones

Date: Received: January 2010; Revised: May 2010.

2000 *Mathematics Subject Classification.* Primary 58E30; Secondary 58E07.

Key words and phrases. Calculus of Variations, Bifurcation points, Critical points, Boundary Value Problem for a PDE with mean curvature.

*: Corresponding author.

[7]. Actually, according to this theory every $\lambda \in \mathbb{R}$, corresponding to a non zero critical point u of the functional

$$I[u,\lambda] = G[u] - \lambda\, F[u],$$

is a bifurcation point for the equation

$$I'[u,\lambda]\, w = G'[\,u\,]\, w - \lambda\, F'[\,u\,]\, w = 0 \tag{1.3}$$

in Hilbert space X if and only if the equation

$$I''[0,\lambda](u,w) = (I''[0,\lambda]\, u,\, w) = 0 \tag{1.4}$$

is satisfied by a non zero solution for all $w \in X$. This statement is valid when the functionals F and G satisfy the following conditions [7]: Let $\mathcal{V}$ a neighborhood of $0 \in X$. We describe the properties of the functionals F, G.
1. Functional G:
The functional G is weakly continuous, differentiable and its differential is Lipschitz continuous with

$$G'[u] = A\, u + N(u)\,, \tag{1.5}$$

where A is a linear self adjoint and compact operator. For the nonlinear part N the following estimate holds:

$$\|N(u)\| \leq c\, \|u\|^p\,, \tag{1.6}$$

where c is a positive constant, $p > 1$ and $u \in \mathcal{V}$.
2. Functional F:
The functional F is differentiable with the property:

$$F'[u] = B\, u + L(u)\,, \tag{1.7}$$

where B is a linear, bounded, self adjoint and positive definite operator. For the nonlinear part L the following estimates hold:

$$\|L(u)\| \leq c\, \|u\|^r\,, \quad \|L(u_1) - L(u_2)\| \leq c\, \left(\|u_1\|^{r-1} - \|u_1\|^{r-1}\right)\, \|u_1 - u_2\| \tag{1.8}$$

where c is a positive constant, $r > 1$ and $u,\, u_1,\, u_2 \in \mathcal{V}$.
Note that under these assumptions equation (1.4) can be rewritten as

$$A\, u - \lambda\, B\, u = 0.$$

In addition to the bifurcation problem of variational character the following problem attracted some special interest [8]

$$f[u,\lambda] = G'[\,u] - \lambda F'[\,u] = 0, \quad \Phi[\,u] = 0 \tag{1.9}$$

where the differentiable mapping $\Phi : X \longrightarrow \mathbb{R}$ satisfies $\Phi[\,0] = 0$. A problem of this type is called a bifurcation problem under the restriction of a constraint. The equation of the constraint

$$\Phi[\,u] = 0 \tag{1.10}$$

restricts the domain of (1.2) to a smaller subspace according to Lyapunov - Schmidt decomposition. We consider that the solutions of equation (1.10) for small values of $\|u\|$ is a coset in a neighborhood of $0 \in X$, i.e.

$$X = X_1 \oplus X_2\,,$$

and there exists a continuous differentiable mapping h from a small neighborhood of $0 \in X_1$ to a small neighborhood of $0 \in X_2$ such that the set of all solutions:

$$u = v + w\,, \;\; v \in X_1\,, \;\; w \in X_2$$

is written in the form:

$$u = v + h(v)\,, \quad v \in X_1 \tag{1.11}$$

with

$$h(0) = 0\,, \quad h'(0) = 0\,. \tag{1.12}$$

According to (1.11) we define the functionals:

$$J[\,v] = G[v + h(v)] = G[\,u]\,, \quad v \in X_1 \tag{1.13}$$

and

$$Q[\,v] = F[v + h(v)] = F[\,u]\,, \quad v \in X_1\,. \tag{1.14}$$

Then the derivatives

$$\mathcal{D}F[\,u] = Q'[\,v]\,, \quad \mathcal{D}G[\,u] = J'[\,v]$$

have the meaning of differentiation of the functionals F and G along the tangential direction of the manifold (1.11). Suppose that $G'[\,0\,] = F'[\,0\,] = 0$ holds. Then for all $\lambda \in \mathbb{R}$ we obtain:

$$\mathcal{D}G[\,0\,] - \lambda\,\mathcal{D}F[\,0\,] = 0\,.$$

Definition 1.2. The number $\lambda_0 \in \mathbb{R}$ is a bifurcation point for equation

$$\mathcal{D}G[\,u\,] - \lambda\,\mathcal{D}F[\,u\,] = 0 \tag{1.15}$$

if in the intersection of a sufficient small neighborhood $\mathcal{U} \subset X \times \mathbb{R}$ of $(0, \lambda_0)$ with the manifold (1.11),there exists a nonzero solution of equation (1.15).

It has been proved [11] that the functionals (1.14) and (1.13) satisfy the properties (1.7), (1.8), (1.5), (1.6) and the appropriate conditions of continuity and differentiability, with the additional condition $r \geq 2$ in a small neighborhood of subspace X_1. This leads to the following result [11]

Theorem 1.3. *Let X be a Hilbert space and the functionals $G[\,u]$, $F[\,u]$, defined in a neighborhood of $0 \in X$, satisfy properties (1.7), (1.8), (1.5), (1.6) and the appropriate conditions of continuity and differentiability for $r \geq 2$. Let $\Phi : X \longrightarrow \mathbb{R}$ be a continuous differentiable functional, which satisfies the conditions:*

$$\Phi[\,0\,] = 0\,, \quad Ker\,\Phi'[\,0\,] = X_1 \neq 0\,.$$

Then the number $\lambda \neq 0$ is a bifurcation point for problem (1.15) if and only if the equation

$$(PAP - \lambda\,PBP)\,u = 0\,, \quad u \in X_1\,,$$

where $P : X \longrightarrow X_1$ the orthogonal projector, has a non zero solution.

It is obvious that bifurcation points exists when $PAP \neq 0$.

2. Description of the problem and constraint

Let M be a smooth surface in $\mathbb{R}^3$ and $\vec{\eta}(x)$, where $x \in \mathbb{R}^3$, a continuously differentiable vector field identified to the normal vector field for every $x \in M$. We assume that the mean curvature of surface M not vanished. Let $U \subset \mathbb{R}^3$ an open set with $\operatorname{diam} U < \delta$, where $\delta > 0$ small enough and $S = M \cap U$ an open and connected set in M, with boundary ∂S consisting of two non-intersecting sufficiently smooth components Γ and Γ_1. We denote by $\vec{\nu}(x)$ a differentiable vector field in $\mathbb{R}^3$, which is the normal vector field of the one dimensional curve ∂S for every $x \in \partial S$, located formally in the tangent plane $T_x M \subset \mathbb{R}^3$.

The ∇_i operator is the i-th component of the tangent differentiation with respect to the surface M [4]:

$$\nabla_i = \frac{\partial}{\partial x^i} - \eta^i(x)\,\eta^j(x)\,\frac{\partial}{\partial x^j}\,, \quad i = 1, 2, 3\,, \quad x \in M. \tag{2.1}$$

δ_i is the i-th component of the tangent directional differentiation along the curve ∂S:

$$\delta_i = \tau^i(x)\,\frac{d}{ds} = \tau^i(x)\,\tau^j(x)\,\frac{\partial}{\partial x^j}\,, \quad i = 1, 2, 3\,, \quad x \in \partial S\,, \tag{2.2}$$

where $\vec{\tau}(x)$ for $x \in \mathbb{R}^3$ is a $\mathcal{C}^\infty$ vector field identified for each $x \in \partial S$ to the unitary tangent vector field of the curve ∂S, and belongs to the tangent plane $T_x M$ for each $x \in \partial S$. For these differential operators the following formulae of integration by parts hold [6]

$$\int_S u \nabla_i v\, dS = \int_{\partial S} u\, v\, \nu^i\, ds - \int_S H n^i u\, v\, dS - \int_S v\, \nabla_i u\, dS\,, \tag{2.3}$$

and [11]

$$\int_{\partial S} u\, \delta_i\, v\, ds = -\int_{\partial S} (K \nu^i + R \eta^i)\, u\, v\, ds - \int_{\partial S} v\, \delta_i\, u\, ds\,, \tag{2.4}$$

where

$$H = -\nabla_i \eta^i$$

is the mean curvature of surface M [4], K is the geodesic curvature and R is the normal curvature of curve ∂S, located in the surface M [2].

Let a vector field $\vec{u} \in H_0(S,\, T_x M)$ where

$$H_0(S,\, T_x M) = \left\{ \vec{u} \in W_2^1(S,\, T_x M)\,,\ \vec{u}|_\Gamma \in W_2^2(\Gamma,\, T_x M)\,.\ \vec{u}|_{\Gamma_1} = \vec{0} \right\}.$$

We denote by $W_2^1(S, T_x M)$ and $W_2^2(\Gamma, T_x M)$ the Sobolev spaces of functions defined on S and Γ with values in $T_x M \subset \mathbb{R}^3$ respectively. For every specific $\vec{u} \in H_0(S,\, T_x M)$ we introduce the following functionals

$$F[\,\vec{u}\,] = \frac{1}{2} \int_S a_{ijkl}(x)\, \nabla_j u^i\, \nabla_l u^k dS + \frac{1}{2} \int_\Gamma |\delta_i \delta_i\, \vec{u}|^2 ds \tag{2.5}$$

$$G[\,\vec{u}\,] = \int_\Gamma q(\vec{u}, x)\, ds \tag{2.6}$$

$$I[\,\vec{u}, \lambda\,] = F[\,\vec{u}\,] - \lambda\, G[\,\vec{u}\,]\,, \quad \lambda \in \mathbb{R}. \tag{2.7}$$

The coefficients $a_{ijkl} \in L_\infty(S)$ satisfy the symmetry properties $a_{ijkl}(x) = a_{klij}(x)$, and they are positive definite, i.e.

$$a_{ijkl}(x)\,\xi^{ij}\xi^{kl} \geq \Lambda\,\xi^{ij}\,\xi^{ij}\,, \qquad \Lambda > 0$$

for all the matrices $(\xi^{ij})_{(i,j)}$. The functional $I[\vec{u},\lambda]$ can be considered as the energy functional of a continuum medium with special characteristics determined by the coefficients a_{ijkl}. The medium is the interior of a shell Γ, which is under the influence of a force density coming from a potential $\lambda\, q(\vec{u},x)$. The medium is fixed up to a part Γ_1 of the boundary ∂S. Hence, the first term of the functional $F[\vec{u}]$ represents the random deformation of the medium while the rest of the expression comes from the deformation of the shell.

The function q is three times differentiable, and satisfies the following properties

$$q(0,x) = 0\,, \quad q_{u^i}(0,x) = 0\,, \quad x \in \Gamma\,, \quad i = 1,2,3. \tag{2.8}$$

The present study focuses on the investigation of the critical points of functional $I[\,\vec{u},\lambda\,]$, i.e. the functions $\vec{u} \in H_0(S,\,T_xM)$, where $x \in S$, such that for all $\vec{w} \in H_0(S,\,T_xM)$

$$I'[\,\vec{u},\lambda\,]\,\vec{w} = 0 \tag{2.9}$$

is valid under the existence of a constraint with the property of leaving the area of the domain S invariant on the surface M. We define the mapping

$$y : \partial S \longrightarrow M \qquad y(x) = x + \vec{u}(x)\,, \tag{2.10}$$

where $\vec{u}(x) \in H_0(S,\,T_xM)$ for small values of $\|\vec{u}\|_{H_0}$. Since M can be considered as a graph of a two times differentiable function f defined on a bounded domain $V \subset \mathbb{R}^2$, we choose a coordinate system, which is transformed from the initial one by an appropriate composition of a translation and rotation. Then

$$S = M \cap U = \left\{(x^1, x^2, f(x^1,x^2)),\ (x^1,x^2) \in V\right\}\,.$$

On such a coordinate system on $\mathbb{R}^3$, we pick up the axes x^1, x^2 from the tangent plane of the surface M at the point $x \in S$, while the axis x^3 comes along the normal vector $\vec{\eta}$ of the surface M at point x. The following properties for components of the normal vector of M at x are valid for this specific system of local coordinates:

$$\eta^3 = \frac{1}{\sqrt{1+|\mathrm{grad} f|^2}}\,, \quad \eta^j = -\eta^3\frac{\partial f}{\partial x^j}\,, \quad j = 1,2\,, \quad dS = \frac{1}{n^3}\,dx^1\,dx^2\,, \tag{2.11}$$

while the components of the tangential differentiation (2.1) in these local coordinates are written as:

$$\nabla_i = (\delta_{i1} - n^i n^1)\frac{\partial}{\partial x^1} + (\delta_{i2} - n^i n^2)\frac{\partial}{\partial x^2}\,, \quad i = 1,2,3\,, \tag{2.12}$$

where δ_{ij} stands for the Kronecker symbol. According to (2.10):

$$y^1 = x^1 + u^1\,, \quad y^2 = x^2 + u^2\,, \quad y^3 = f(y^1,y^2) \tag{2.13}$$

Mapping (2.10) leaves invariant the area of S if and only if

$$\int_V \sqrt{g(x+\vec{u}(x))}\,dx^1dx^2 = \int_V \sqrt{g(x)}\,dx^1dx^2\,, \tag{2.14}$$

where $g(x) = \det(g_{ij}(x))$ and $g_{ij}(x)$ are the components of metric tensor at $x \in M$. The mapping

$$\Phi[\,\vec{u}\,] = \int_V \sqrt{g(x + \vec{u}(x))}\, dx^1 dx^2 - \int_V \sqrt{g(x)}\, dx^1 dx^2 \tag{2.15}$$

is defined on a small neighborhood of $\vec{0} \in H_0(S,\, T_xM)$. It is obvious that (2.14) holds if and only if

$$\Phi[\,\vec{u}\,] = 0. \tag{2.16}$$

The mapping $\Phi : H_0(S,\, T_xM) \longrightarrow \mathbb{R}$ is continuously differentiable in a small neighborhood of $\vec{0} \in H_0(S,\, T_xM)$. Let a vector field $\vec{u} \in H_0(S, T_xM)$ then $\vec{u}|_{\partial S}$ is reduced to the form

$$\vec{u}(x) = \varphi(x)\,\vec{\tau}(x) + \psi(x)\,\vec{\nu}(x)\,, \quad \varphi\,,\, \psi \in W_2^2(\partial S)\,, \quad x \in \partial S\,. \tag{2.17}$$

Proposition 2.1. *The solutions of equation (2.16) are form a coset, i.e.*

$$H_0(S,\, T_xM) = X_1 \oplus X_2\,, \tag{2.18}$$

where

$$\begin{aligned} X_1 = &\ \left\{\vec{v} \in H_0(S,\, T_xM), \quad \vec{v}|_\Gamma = \varphi\,\vec{\tau} + \psi\vec{\nu}, \quad \textstyle\int_\Gamma \psi\, ds = 0\right\}\,, \\ X_2 = &\ \left\{\vec{v} \in H_0(S,\, T_xM), \quad \vec{v}|_\Gamma = \varphi\,\vec{\tau} + C|\Gamma|^{-1}\vec{\nu}, \quad C \in \mathbb{R}\right\}. \end{aligned}$$

Proof. This result is a consequence of Lyapunov - Schmidt decomposition and the implicit function theorem. We consider equation (2.16) for small values of the norm $\|\vec{u}\|_{H_0(S,\,T_xM)}$. Obviously

$$\Phi[\,\vec{0}\,] = 0\,.$$

On a fixed point $x \in \partial S \subset M$ we introduce the above coordinate system. Then $g_{ij}(x) = \delta_{ij}$ and

$$g_{ij}(y) = \frac{\partial\, y^k}{\partial\, x^i}\,\frac{\partial\, y^k}{\partial\, x^j}\,, \quad k = 1,2,3\,, \quad i,j = 1,2\,.$$

Using the coordinate transformation (2.13), we obtain:

$$g_{11}(y) = \beta^2(y)(1 + u^1_{x^1})^2 + \gamma^2(y)\,(u^2_{x^1})^2 + 2\,f_{y^1}\,f_{y^2}\,(1 + u^1_{x^1})\,u^2_{x^1}\,,$$

$$g_{22}(y) = \beta^2(y)(u^1_{x^2})^2 + \gamma^2(y)(1 + u^2_{x^2})^2 + 2\,f_{y^1}\,f_{y^2}\,u^1_{x^2}\,(1 + u^2_{x^2})\,,$$

$$g_{12}(y) = \beta^2(y)(1+u^1_{x^1})\,u^1_{x^2} + \gamma^2(y)\,u^2_{x^1}(1+u^2_{x^2}) + f_{y^1}\,f_{y^2}\left[u^1_{x^2}u^2_{x^1} + (1 + u^1_{x^1})(1 + u^2_{x^2})\right]\,,$$

$$g(y) = g_{11}(y)\,g_{22}(y) - g^2_{12}(y)\,,$$

where

$$\beta(y) = \sqrt{1 + f^2_{y^1}}\,, \quad \gamma(y) = \sqrt{1 + f^2_{y^2}}\,.$$

For every $\varepsilon > 0$ there exists $\delta > 0$ such for $\operatorname{diam} U < \delta$ the inequality $|n^i(x)| < \varepsilon$ holds for $i = 1,2$ and $x \in U$. This means

$$|n^i(x)\,n^j(x)| \to 0\,, \quad i,j = 1,2\,, \quad x \in U\,.$$

Then we derive:

$$\Phi'[\,\vec{0}\,]\,\vec{v} = \int_V \operatorname{grad}\left(\frac{1}{n^3}\right)\,\vec{v}\,dx^1dx^2 + \int_V (v^1_{x^1} + v^2_{x^2})\,\frac{1}{n^3}\,dx^1dx^2\,. \tag{2.19}$$

From (2.11), (2.12) and (2.3) equation (2.19) reduces to:

$$\Phi'[\vec{0}]\,\vec{v} = \int_S \left(\nabla_1 v^1 + \nabla_2 v^2\right)\, dS = \int_\Gamma \vec{v}\,\vec{\nu}\, ds\,.$$

Thus,

$$\mathrm{Ker}\Phi'[\vec{0}] = \left\{\vec{v} \in H_0(S,\, T_xM),\ \vec{v}|_\Gamma = \varphi\,\vec{\tau} + \psi\,\vec{\nu}\,,\ \int_\Gamma \psi\, ds = 0\right\} \neq \{\vec{0}\}. \tag{2.20}$$

We define $X_1 = \mathrm{Ker}\Phi'[\vec{0}]$, $X_2 = X_1^\perp$, i.e.

$$X_2 = \left\{\vec{v} \in H_0(S,\, T_xM),\quad \vec{v}|_\Gamma = \varphi\,\vec{\tau} + C\,|\Gamma|^{-1}\,\vec{\nu},\quad C \in \mathbb{R}\right\}. \tag{2.21}$$

Then the set of the solutions of equation (2.16) is written in the form

$$\vec{u} = \vec{v} + h(\vec{v}),\quad \vec{v} \in X_1\,, \tag{2.22}$$

and there exist $\delta_1, \delta_2 \in \mathbb{R}$ small such that the mapping

$$h : B_{\delta_1}(\vec{0}) \subset X_1 \longrightarrow B_{\delta_2}(\vec{0}) \subset X_2$$

is differentiable and

$$h(\vec{0}) = \vec{0}\,,\quad h'(\vec{0}) = \vec{0} \tag{2.23}$$

are valid. □

Assuming the existence of constraint (2.16) the functionals (2.5),(2.6) and (2.7) reduce to the form

$$F[\,\vec{v}\,] = \frac{1}{2}\int_S a_{ijkl}(x)\,\nabla_j\left(v^i + h(\vec{v})^i\right)\,\nabla_l\left(v^k + h(\vec{v})^k\right)\, dS\, +$$

$$+\frac{1}{2}\int_\Gamma |\delta_i\delta_i\,\left(\vec{v} + h(\vec{v})\right)|^2\, ds\,, \tag{2.24}$$

$$G[\,\vec{v}\,] = \int_\Gamma q\left(\vec{v} + h(\vec{v}),\, x\right)\, ds\,, \tag{2.25}$$

$$I[\,\vec{v}, \lambda\,] = F[\,\vec{v}\,] - \lambda\, G[\,\vec{v}\,]\,,\quad \lambda \in \mathbb{R}\,,\quad \vec{v} \in X_1\,. \tag{2.26}$$

We define the critical points for the functional I on the subspace X_1

Definition 2.2. A critical point for the functional I under the constraint (2.16), for a given $\lambda \in \mathbb{R}$, is the vector field $\vec{v} \in X_1$, which satisfies the relation

$$I'[\,\vec{v}, \lambda\,]\,\vec{w} = 0 \tag{2.27}$$

for each $\vec{w} \in X_1$.

Relation (2.27) is rewritten in the equivalent form

$$\int_S a_{ijkl}(x)\,\nabla_j\left(w^i + h'(\vec{v})^i\vec{w}\right)\,\nabla_l\left(v^k + h(\vec{v})^k\right) dS +$$

$$+\int_\Gamma \delta_i\delta_i\,\left(\vec{v} + h(\vec{v})\right)\delta_j\delta_j\left(\vec{w} + h'(\vec{v})\vec{w}\right) ds - \tag{2.28}$$

$$-\lambda\int_\Gamma q_{u^i}\left(\vec{v} + h(\vec{v}), x\right)\left(w^i + (h'(\vec{v})\vec{w})^i\right) ds = 0.$$

Since the conditions (2.8), (2.23) are valid the vector field $\vec{v} = \vec{0}$ is a critical point of equation (2.27) for all $\lambda \in \mathbb{R}$. We consider the linearised equation

$$I''[\,\vec{0}, \lambda\,]\,(\vec{v}, \vec{w}) = 0\,, \quad \vec{v}, \vec{w} \in X_1\,, \tag{2.29}$$

or equivalently

$$\int_S a_{ijkl}(x) \nabla_j v^i \nabla_l w^k dS + \int_\Gamma \delta_i \delta_i \vec{v}\, \delta_j \delta_j \vec{w}\, ds - \lambda \int_\Gamma q_{u_i u_j}(\vec{0}, x)\, v^i w^j\, ds = 0\,, \tag{2.30}$$

which corresponds to equation (2.27). Using the formulae of integration by parts (2.3) and (2.4) under the additional assumptions $\partial S \in \mathcal{C}^\infty$, $a_{ijkl} \in \mathcal{C}^\infty(\bar{S})$, $q \in \mathcal{C}^\infty(\mathbb{R}^3, \partial S)$, and proposition (2.1), we derive the equivalent to (2.30) boundary value problem ($D = \delta_i \delta_i$)

$$\begin{aligned}
&H\eta^l a_{ijkl}(x) \nabla_j v^i + \nabla_l \left(a_{ijkl}(x) \nabla_j v^i\right) = 0, && x \in S \\
&a_{ijkl}(x) \nu^l \tau^k \nabla_j v^i + [K^2 + R^2 + \delta_j (K\nu^j + R\eta^j)]\, \tau^k D v^k + \\
&\quad + \tau^k D^2 v^k - \lambda q_{u^i u^k}(\vec{0}, x) v^i \tau^k = 0, && x \in \Gamma \\
&a_{ijkl}(x) \nu^l \nu^k \nabla_j v^i + [K^2 + R^2 + \delta_j (K\nu^j + R\eta^j)]\, \nu^k D v^k + \\
&\quad + \nu^k D^2 v^k - \lambda q_{u^i u^k}(\vec{0}, x) v^i \nu^k = C, && x \in \Gamma \\
&\vec{v} = \vec{0}, && x \in \Gamma_1,
\end{aligned} \tag{2.31}$$

where constant C is defined by the relation:

$$C = \frac{1}{|\Gamma|} \int_\Gamma [a_{ijkl}(x) \nu^l \nu^k \nabla_j v^i + \left[K^2 + R^2 + \delta_j (K\nu^j + R\eta^j)\right] \nu^k D v^k + \\ + \nu^k D^2 v^k - \lambda q_{u^i u^k}(\vec{0}, x) v^i \nu^k]\, ds\,.$$

3. Existence of bifurcation points

The existence of bifurcation points for problem (2.27) is based on theorem (1.3). We verify the assumptions of this theorem.

Proposition 3.1. *The functionals (2.5) and (2.6) satisfy the conditions (1.7), (1.8) and (1.5), (1.6), respectively, for all $\vec{u}$ in a neighborhood of $\vec{0}$ in Hilbert space $H_0(S, T_x M)$ with all the appropriate conditions of continuity and differentiability.*

Proof. We can verify [9, 11], using the introduced coordinate system, that the expression

$$\|\vec{u}\| = \left[\int_S a_{ijkl}(x)\, \nabla_j u^i \nabla_l u^k\, dS + \int_\Gamma |\delta_i \delta_i \vec{u}|^2\, ds\right]^{1/2} \tag{3.1}$$

defines a norm on Hilbert space $H_0(S, \mathbb{R}^3)$ equivalent to the standard one. Thus functional (2.5) can be rewritten as:

$$F[\,\vec{u}\,] = \frac{1}{2} \|\vec{u}\|^2_{H_0(S, T_x M)} = \frac{1}{2}\, (\vec{u},\, \vec{u})_{H_0(S, T_x M)}\,, \tag{3.2}$$

where (,) is the corresponding inner product in Hilbert space $H_0(S,\, T_x M)$. Now it is obvious that the functional (2.5) is differentiable and satisfies the conditions

(1.7), (1.8). The functional (2.6) is differentiable due to the smoothness of function q. The fact that the functional (2.6) is weakly continuous, and its differential is local Lipschitz continuous, comes from the Sobolev embedding theorem of the space $W_2^2(\Gamma, \mathbb{R}^3)$ into the space $\mathcal{C}(\Gamma, \mathbb{R}^3)$. Expression (1.5) holds, where

$$(A\,\vec{u}, \vec{w})_{H_0} = \int_\Gamma q_{u^i u^j}(\vec{0}, x)\, u^i w^j\, ds$$

and

$$(N(\vec{u}), \vec{w})_{H_0} = \int_\Gamma \left[q_{u^i}(\vec{u}, x) - q_{u^i u^j}(\vec{0}, x)\, u^j \right]\, w^i\, ds$$

for all $\vec{w} \in H_0(S, T_x M)$. The operator $A : H_0(S, T_x M) \longrightarrow H_0(S, T_x M)$ is linear and symmetric. The same embedding theorem implies that the operator A is also compact. Finally, the estimate (1.6) holds due to the above embedding theorem for $p = 2$. □

We are now ready to formulate the existence of bifurcation points for problem (2.27).

Theorem 3.2. *The number λ_0 is a bifurcation point for problem (2.27) if and only if equation (2.30) has a nonzero solution for all $\vec{w} \in X_1$.*

Proof. This is a straight result of theorem (1.3) since the properties of functionals $F\,, G$ hold from proposition (3.1) and properties of functional Φ from proposition (2.1). The integral equation (2.30) corresponding to (2.29) can be written in the equivalent form:

$$(\vec{v},\, \vec{w}) - \lambda\; (A\,\vec{v},\, \vec{w}) = 0$$

for all $\vec{w} \in X_1$ due to proposition (3.1). This implies that

$$\vec{v} - \lambda\, A\,\vec{v} = 0\,.$$

Obviously, if $q_{u^i u^j}(\vec{0},\, x) \neq 0$, there exist bifurcation points. □

References

1. J. B. Antman, S. Keller, *Bifurcation Theory and Nonlinear Eigenvalue Problems*, W. A. Benjamin (1969)
2. H. Cartan, *Differential Calculus - Differential forms*, Herman Paris (1971)
3. D. Gilbarg, N. S. Trüdinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1977)
4. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Notes on Pure Mathematics 10, publications of Australian National University Canberra (1977)
5. L. A. Lyusternik, L. Schnirelman, *Topological Methods in Variational Problems*, Moscow (1930)
6. V. G. Osmolovskii, *The Variational Problem of Phase Transitions in Mechanics of Continuum Medium*, publications of Saint Petersburg University (2000)
7. I. V. Skrypnik, *Nonlinear Partial Differential Equations of Higher Order*, Kiev (1973)
8. I. Tadjbakhan, F. Odeh, *Equillibrium States of Elastic Rings*, Ann. Appl. Math. 18.N1 59 (1967)
9. P. Vyridis, *Bifurcation in a Quasilinear Variational Problem on a One - Dimensional Manifold*, J. Math. Sci. (N.Y.) 106(3) 2919 (2001)

10. P. Vyridis, *Variational Problem on Equilibrium of an Elasic Medium, Located in an Elasic Shell*, J. Math. Sci. (N.Y.) 112(1) 3992 (2002)
11. P. Vyridis, *A Bifurcation Problem with Integral Constraints*, Ph.D. Thesis, Saint Petersburg University (2003)

[1] Department of Informatics and Telecommunications, Kalamata Technological Educational Institute, Branch of Sparta, 23100 Sparta, Greece.

E-mail address: pvyridis@gmail.com