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Coupled coincidence point and common coupled fixed point theorems in complex valued metric spaces

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Abstract

In this paper, we introduce the concept of a w-compatible mappings and utilize the same to discuss the ideas of coupled coincidence point and coupled point of coincidence for nonlinear contractive mappings in the context of complex valued metric spaces besides proving existence theorems which are following by corresponding unique coupled common fixed point theorems for such mappings. Some illustrative examples are also given to substantiate our newly proved results.

Keywords: Common fixed point; Contractive type mapping; coupled coincidence point; coupled point of coincidence; Complex valued metric space.
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1. Introduction and preliminaries

The axiomatic development of a metric space was essentially carried out by French mathematician Frechet in the year 1906. The utility of metric spaces in the natural growth of Functional Analysis is enormous. Inspired from the impact of this natural idea to mathematics in general and to Functional Analysis in particular, several researchers attempted various generalizations of this notion in the recent past such as: rectangular metric spaces, semi metric spaces, quasi metric spaces, quasi semi metric spaces, pseudo metric spaces, probabilistic metric spaces, 2-metric spaces, D-metric spaces,

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G-metric spaces, , K-metric spaces, Cone metric spaces etc and by now there exists considerable literature on all these generalizations of metric spaces (see [8]-[5]).

Most recently, Azam et al. [1] and Fayyaz et al. [12] studied complex valued metric spaces wherein some fixed point theorems for mappings satisfying a rational inequality were established. Naturally, this new idea can be utilized to define complex valued normed spaces and complex valued inner product spaces which, in turn, offer a lot of scopes for further investigation. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring . However, in complex valued metric spaces, we can study improvements of a host of results of analysis involving divisions.

In this paper we prove common fixed point theorems involving two pairs of weakly compatible mappings satisfying certain rational expressions in complex valued metric space.

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1.1. [1] Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1). $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d_2). $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3). $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Example 1.2. [12]. Let $X = \mathbb{C}$ be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Example 1.3. [14] Let $X = \mathbb{C}$ be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|$$

where $0 \leq k \leq \frac{\pi}{2}$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Definition 1.4. [1] Let (X, d) be a complex valued metric space and $B \subseteq X$

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that

$$N(b,r) \subseteq B$$

where $N(b,r) = \{y \in X : d(b,y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$,

$$N(x,r) \cap (B \backslash X) \neq \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

(v)The family

$$F = \{N(x,r) : x \in X, 0 \prec r\}$$

is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 1.5. [1] Let (X,d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence$\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, . We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) the sequence$\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space (X,d) is a complete complex valued metric space If every Cauchy sequence is convergent.

In [1], Azam et al. established the following two lemmas.

Lemma 1.6. Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 1.7. Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

2. Main results

Bhashkar et al. [2] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. They also discussed an application of their result by investigating the existence and uniqueness of solution for periodic boundary value problem. Recently, Lakshmikantham et al. [10] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces.

Definition 2.1. [2] An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Inspired with Definition 2.1 in the following we introduce the concept of a coupled fixed point of a mapping $F : X \times X \to X$.

Definition 2.2. An element $(x, y) \in X \times X$ is called

($\mathbf{g_1}$) a coupled coincidence point of mapping $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence;

($\mathbf{g_2}$) a common coupled fixed point of mappings $F : X\times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Note that if g is the identity mapping, then Definition 2.2 reduces to Definition 2.1. We introduce the following definition.

Definition 2.3. The mappings $F : X \times X \to X$ and $g : X \to X$ are called $w-$compatible if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Definition 2.4. [10] Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have a mixed g-monotone property if F is monotone g-nondecreasing in its first argument and monotone g-nonincreasing in its second argument, that is, for any $x, y \in X$,

$$\begin{aligned} & x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y) \\ \text{and,} \quad & y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_2) \preceq F(x, y_1). \end{aligned}$$

Now we prove our main result.

Theorem 2.5. *Let (X, d) be a complex valued metric space, $F : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} d(F(x, y), F(u, v)) \precsim\, & a_1 d(gx, gu) + a_2 d(gy, gv) \\ & + \frac{a_3 d(F(x, y), gx) + a_4 d(F(x, y), gu)}{1 + d(u, v)} \\ & + \frac{a_5 d(F(u, v), gu) + a_6 d(F(u, v), gx)}{1 + d(x, y)}, \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, where $a_i, i = 1, 2, \ldots, 6$ are nonnegative real numbers such that $\sum_{i=1}^{6} a_i < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X.

Proof . Let x_0, y_0 be any two arbitrary point in X. Set $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, this can be done because $F(X \times X) \subseteq g(X)$. Continuing this process we obtain two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n).$$

From 2.1, we have

$$
\begin{aligned}
d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\precsim a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n) \\
&\quad + \frac{a_3 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + a_4 d(F(x_{n-1}, y_{n-1}), gx_n)}{1 + d(x_n, y_n)} \\
&\quad + \frac{a_5 d(F(x_n, y_n), gx_n) + a_6 d(F(x_n, y_n), gx_{n-1})}{1 + d(x_{n-1}, y_{n-1})} \\
&= a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n) \\
&\quad + \frac{a_3 d(gx_n, gx_{n-1}) + a_4 d(gx_n, gx_n)}{1 + d(x_n, y_n)} \\
&\quad + \frac{a_5 d(gx_{n+1}, gx_n) + a_6 d(gx_{n+1}, gx_{n-1})}{1 + d(x_{n-1}, y_{n-1})} \\
&\precsim a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n) + \frac{a_3 d(gx_n, gx_{n-1})}{1 + d(x_n, y_n)} \\
&\quad + \frac{a_5 d(gx_{n+1}, gx_n) + a_6 d(gx_{n+1}, gx_n) + a_6 d(gx_n, gx_{n-1})}{1 + d(x_{n-1}, y_{n-1})}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
d(gx_n, gx_{n+1}) &\precsim a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n) + \frac{a_3 d(gx_n, gx_{n-1})}{1 + d(x_n, y_n)} \\
&\quad + \frac{a_5 d(gx_{n+1}, gx_n) + a_6 d(gx_{n+1}, gx_n) + a_6 d(gx_n, gx_{n-1})}{1 + d(x_{n-1}, y_{n-1})}
\end{aligned}
$$

so that

$$
\begin{aligned}
|d(gx_n, gx_{n+1})| &\leq a_1 |d(gx_{n-1}, gx_n)| + a_2 |d(gy_{n-1}, gy_n)| + \frac{a_3 |d(gx_n, gx_{n-1})|}{|1 + d(x_n, y_n)|} \\
&\quad + \frac{a_5 |d(gx_{n+1}, gx_n)| + a_6 |d(gx_{n+1}, gx_n)| + a_6 |d(gx_n, gx_{n-1})|}{|1 + d(x_{n-1}, y_{n-1})|}
\end{aligned}
$$

since $|1 + d(x_n, y_n)| \geq 1$ and $|1 + d(x_{n-1}, y_{n-1})| \geq 1$, therefore

$$
\begin{aligned}
|d(gx_n, gx_{n+1})| &\leq a_1 |d(gx_{n-1}, gx_n)| + a_2 |d(gy_{n-1}, gy_n)| + a_3 |d(gx_n, gx_{n-1})| \\
&\quad + a_5 |d(gx_{n+1}, gx_n)| + a_6 |d(gx_{n+1}, gx_n)| + a_6 |d(gx_n, gx_{n-1})| \\
&= (a_1 + a_3 + a_6) |d(gx_n, gx_{n-1})| + (a_5 + a_6) |d(gx_{n+1}, gx_n)| \\
&\quad + a_2 |d(gy_{n-1}, gy_n)|
\end{aligned}
$$

from which it follows

$$
(1 - a_5 - a_6) |d(gx_n, gx_{n+1})| \leq (a_1 + a_3 + a_6) |d(gx_n, gx_{n-1})| + a_2 |d(gy_{n-1}, gy_n)|. \tag{2.2}
$$

Similarly, one can prove that

$$
(1 - a_5 - a_6) |d(gy_n, gy_{n+1})| \leq (a_1 + a_3 + a_6) |d(gy_n, gy_{n-1})| + a_2 |d(gx_{n-1}, gx_n)|. \tag{2.3}
$$

Because of the symmetry in 2.1,

$$\begin{aligned}
d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
&\precsim a_1 d(gx_n, gx_{n-1}) + a_2 d(gy_n, gy_{n-1}) \\
&\quad + \frac{a_3 d(F(x_n, y_n), gx_n) + a_4 d(F(x_n, y_n), gx_{n-1})}{1 + d(x_{n-1}, y_{n-1})} \\
&\quad + \frac{a_5 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + a_6 d(F(x_{n-1}, y_{n-1}), gx_n)}{1 + d(x_n, y_n)} \\
&= a_1 d(gx_n, gx_{n-1}) + a_2 d(gy_n, gy_{n-1}) \\
&\quad + \frac{a_3 d(gx_{n+1}, gx_n) + a_4 d(gx_{n+1}, gx_{n-1})}{1 + d(x_{n-1}, y_{n-1})} \\
&\quad + \frac{a_5 d(gx_n, gx_{n-1}) + a_6 d(gx_n, gx_n)}{1 + d(x_n, y_n)} \\
&\precsim a_1 d(gx_n, gx_{n-1}) + a_2 d(gy_n, gy_{n-1}) \\
&\quad + \frac{a_3 d(gx_{n+1}, gx_n) + a_4 d(gx_{n+1}, gx_n) + a_4 d(gx_n, gx_{n-1})}{1 + d(x_n, y_n)} \\
&\quad + \frac{a_5 d(gx_n, gx_{n-1})}{1 + d(x_n, y_n)}.
\end{aligned}$$

So

$$\begin{aligned}
|d(gx_{n+1}, gx_n)| &\leq a_1 |d(gx_n, gx_{n-1})| + a_2 |d(gy_n, gy_{n-1})| + a_3 |d(gx_{n+1}, gx_n)| \\
&\quad + a_4 |d(gx_{n+1}, gx_n)| + a_4 |d(gx_n, gx_{n-1})| + a_5 |d(gx_n, gx_{n-1})|.
\end{aligned}$$

From which it follows

$$(1 - a_3 - a_4)|d(gx_{n+1}, gx_n)| \leq (a_1 + a_4 + a_5)|d(gx_{n-1}, gx_n)| + a_2 |d(gy_n, gy_{n-1})|. \tag{2.4}$$

Similarly, we can prove that

$$(1 - a_3 - a_4)|d(gy_{n+1}, gy_n)| \leq (a_1 + a_4 + a_5)|d(gy_{n-1}, gy_n)| + a_2 |d(gx_n, gx_{n-1})|. \tag{2.5}$$

Let $\delta_n = |d(gx_n, gx_{n+1})| + |d(gy_n, gy_{n+1})|$. Now, from 2.2 and 2.3 respectively 2.4 and 2.5 we obtain:

$$(1 - a_5 - a_6)\delta_n \leq (a_1 + a_2 + a_3 + a_6)\delta_{n-1}. \tag{2.6}$$

$$(1 - a_3 - a_4)\delta_n \leq (a_1 + a_2 + a_4 + a_5)\delta_{n-1}. \tag{2.7}$$

Finally, from 2.6 and 2.7 we have

$$(2 - a_3 - a_4 - a_5 - a_6)\delta_n \leq (2a_1 + 2a_2 + a_3 + a_4 + a_5 + a_6)\delta_{n-1},$$

that is,

$$\delta_n \leq \eta \delta_{n-1}, \quad \eta = \frac{2a_1 + 2a_2 + a_3 + a_4 + a_5 + a_6}{2 - a_3 - a_4 - a_5 - a_6} < 1. \tag{2.8}$$

Consequently, we have

$$0 \leq \delta_n \leq \eta \delta_{n-1} \leq \cdots \leq \eta^n \delta_0. \tag{2.9}$$

If $\delta_0 = 0$ then (x_0, y_0) is a coupled coincidence point of F and g. So let $0 < \delta_0$. If $m > n$, we have

$$d(gx_{n+m}, gx_n) \precsim d(gx_{n+m}, gx_{n+m-1}) + d(gx_{n+m-1}, gx_{n+m-2}) + \cdots + d(gx_{n-1}, gx_n),$$

and

$$d(gy_{n+m}, gy_n) \precsim d(gy_{n+m}, gy_{n+m-1}) + d(gy_{n+m-1}, gy_{n+m-2}) + \cdots + d(gy_{n-1}, gy_n),$$

so that

$$|d(gx_{n+m}, gx_n)| \leq |d(gx_{n+m}, gx_{n+m-1})| + |d(gx_{n+m-1}, gx_{n+m-2})| + \cdots + |d(gx_{n-1}, gx_n)|,$$

and

$$|d(gy_{n+m}, gy_n)| \leq |d(gy_{n+m}, gy_{n+m-1})| + |d(gy_{n+m-1}, gy_{n+m-2})| + \cdots + |d(gy_{n-1}, gy_n)|.$$

Therefore,

$$\begin{aligned} |d(gx_{n+m}, gx_n)| + |d(gy_{n+m}, gy_n)| \leq{} & (|d(gx_{n+m}, gx_{n+m-1})| + |d(gy_{n+m}, gy_{n+m-1})|) \\ & + (|d(gx_{n+m-1}, gx_{n+m-2})| + |d(gy_{n+m-1}, gy_{n+m-2})|) \\ & + \cdots + (|d(gx_{n-1}, gx_n)| + |d(gy_{n-1}, gy_n)|), \end{aligned}$$

that is,

$$\begin{aligned} |d(gx_{n+m}, gx_n)| + |d(gy_{n+m}, gy_n)| &\leq \delta_{n+m-1} + \delta_{n+m-2} + \cdots + \delta_n \\ &\leq (\eta^{n+m-1} + \eta^{n+m-2} + \cdots + \eta^n)\delta_0 \\ &= \frac{\eta^n}{1-\eta}\delta_0 \to 0, as \;\; n \to \infty, \end{aligned}$$

hence $|d(gx_{n+m}, gx_n)| \to 0$ and $|d(gy_{n+m}, gy_n)| \to 0$ as $n \to \infty$. Thus by Lemma 1.7, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequence in $g(X)$. Since $g(X)$ is complete subset of X, so there exists x and y in X such that $gx_n \to gx$ and $gy_n \to gy$. Now, we prove that$F(x, y) = gx$ and $F(y, x) = gy$. For that, we have

$$\begin{aligned} d(F(x,y), gx) &\precsim d(F(x,y), gx_{n+1}) + d(gx_{n+1}, gx) \\ &= d(F(x,y), F(x_n, y_n)) + d(gx_{n+1}, gx) \\ &\precsim a_1 d(gx, gx_n) + a_2 d(gy, gy_n) + \frac{a_3 d(F(x,y), gx) + a_4 d(F(x,y), gx_n)}{1 + d(x_n, y_n)} \\ &\quad + \frac{a_5 d(F(x_n, y_n), gx_n) + a_6 d(F(x_n, y_n), gx)}{1 + d(x,y)} + d(gx_{n+1}, gx) \\ &= a_1 d(gx, gx_n) + a_2 d(gy, gy_n) + \frac{a_3 d(F(x,y), gx) + a_4 d(F(x,y), gx_n)}{1 + d(x_n, y_n)} \\ &\quad + \frac{a_5 d(gx_{n+1}, gx_n) + a_6 d(gx_{n+1}, gx)}{1 + d(x,y)} + d(gx_{n+1}, gx) \\ &\precsim a_1 d(gx, gx_n) + a_2 d(gy, gy_n) + \frac{a_3 d(F(x,y), gx) + a_4 d(F(x,y), gx) + a_4 d(gx, gx_n)}{1 + d(x_n, y_n)} \\ &\quad + \frac{a_5 d(gx_{n+1}, gx) + a_5 d(gx, gx_n) + a_6 d(gx_{n+1}, gx)}{1 + d(x,y)} + d(gx_{n+1}, gx), \end{aligned}$$

so that

$$
\begin{aligned}
|d(F(x,y),gx)| \leq\, & a_1|d(gx,gx_n)| + a_2|d(gy,gy_n)| + a_3|d(F(x,y),gx)| \\
& + a_4|d(F(x,y),gx)| + a_4|d(gx,gx_n)| + a_5|d(gx_{n+1},gx)| + a_5|d(gx,gx_n)| \\
& + a_6|d(gx_{n+1},gx)| + |d(gx_{n+1},gx)|,
\end{aligned}
$$

which further implies that,

$$
\begin{aligned}
|d(F(x,y),gx)| \leq\, & \frac{a_1+a_4+a_5}{1-a_3-a_4}|d(gx_n,gx)| + \frac{a_5+a_6}{1-a_3-a_4}|d(gx_{n+1},gx)| \\
& + \frac{a_2}{1-a_3-a_4}|d(gy_n,gy)|.
\end{aligned}
\tag{2.10}
$$

Since $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$, then, $|d(F(x,y),gx)| = 0$, it follows that $d(F(x,y),gx) = 0$, and hence $F(x,y) = gx$. Similarly, we can prove $F(y,x) = gy$. Hence (x,y) is coupled coincidence point of the mappings F and g. □

By setting $a_1 = a_2 = \alpha$, $a_3 = a_4 = \beta$ and $a_5 = a_6 = \gamma$ in Theorem 2.5, we deduce the following corollary.

Corollary 2.6. *Let (X,d) be a complex valued metric space, $F : X \times X \to X$ and $g : X \to X$ be mappings satisfying*

$$
\begin{aligned}
d(F(x,y),F(u,v)) \precsim\, & \alpha[d(gx,gu) + d(gy,gv)] + \beta\frac{d(F(x,y),gx) + d(F(x,y),gu)}{1+d(u,v)} \\
& + \gamma\frac{d(F(u,v),gu) + d(F(u,v),gx)}{1+d(x,y)},
\end{aligned}
\tag{2.11}
$$

for all $x,y,u,v \in X$, where α, β and γ are nonnegative real numbers such that $\alpha+\beta+\gamma < \frac{1}{2}$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X.

Now, we present two examples showing that Theorem 2.5 is a proper extension of known results.

Example 2.7. Let $X = [0,\infty)$. The mapping $d : X \times X \to \mathbb{C}$ defined as:

$$d(x,y) = |x-y|i.$$

Clearly, (X,d) is a complete complex valued metric space. We define the functions $F : X \times X \to X$ and $g : X \to X$ by

$$g(x) = 3x \quad \text{and} \quad F(x,y) = \frac{x}{5} + \frac{y}{5}.$$

It is easy to verify that F and g satisfy all the conditions of Theorem 2.5, taking $a_1 = a_2 = \frac{1}{10}, a_3 = a_4 = \frac{1}{12}$ and $a_5 = a_6 = \frac{1}{14}$. Moreover $(0,0)$, is common coupled coincidence point of F and g.

Example 2.8. Let $X = [0,\infty)$ and $d : X \times X \to \mathbb{C}$ is a mapping defined by: $d(x,y) = |x-y| + |x-y|i$. Clearly, (X,d) is a complete complex valued metric space. We define the functions $F : X \times X \to X$ and $g : X \to X$ by

$$g(x) = 3x \quad \text{and} \quad F(x,y) = x + \frac{|\sin y|}{5}.$$

It is easy to verify that F and g satisfy all the conditions of Theorem 2.5, taking $a_1 = \frac{2}{3}, a_2 = \frac{1}{9}$ and $a_3 = a_4 = a_5 = a_6 = \frac{1}{15}$. Moreover $(0,0)$ is common coupled coincidence point of F and g.

Theorem 2.9. *Let (X,d) be a complex valued metric space, $F : X \times X \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 2.5. If F and g are $w-$compatible, then F and g have unique common coupled fixed point. Moreover, common fixed point of F and g has the form (u,u) for some $u \in X$.*

Proof . First we claim that coupled point of coincidence is unique. suppose that $(x,y),(x^*,y^*) \in X \times X$ with $g(x) = F(x,y)$, $g(y) = F(y,x)$, and $g(x^*) = F(x^*,y^*)$, $g(y^*) = F(y^*,x^*)$. Using 2.1, we get

$$\begin{aligned}
d(gx,gx^*) &= d(F(x,y),F(x^*,y^*)) \\
&\precsim a_1 d(gx,gx^*) + a_2 d(gy,gy^*) \\
&\quad + \frac{a_3 d(F(x,y),gx) + a_4 d(F(x,y),gx^*)}{1+d(x^*,y^*)} \\
&\quad + \frac{a_5 d(F(x^*,y^*),gx^*) + a_6 d(F(x^*,y^*),gx)}{1+d(x,y)} \\
&= a_1 d(gx,gx^*) + a_2 d(gy,gy^*) + \frac{a_4 d(gx,gx^*)}{1+d(x^*,y^*)} + \frac{a_6 d(gx^*,gx)}{1+d(x,y)},
\end{aligned}$$

so that

$$|d(gx,gx^*)| \leq a_1|d(gx,gx^*)| + a_2|d(gy,gy^*)| + |\frac{a_4 d(gx,gx^*)}{1+d(x^*,y^*)}| + |\frac{a_6 d(gx^*,gx)}{1+d(x,y)}|,$$

and

$$|d(gx,gx^*)| \leq a_1|d(gx,gx^*)| + a_2|d(gy,gy^*)| + a_4|d(gx,gx^*)| + a_6|d(gx^*,gx).$$

Therefore

$$|d(gx,gx^*)| \leq (a_1+a_4+a_6)|d(gx,gx^*)| + a_2|d(gy,gy^*)|. \tag{2.12}$$

Similarly, one gets

$$|d(gy,gy^*)| \leq (a_1+a_4+a_6)|d(gy,gy^*)| + a_2|d(gx,gx^*)|. \tag{2.13}$$

Thus

$$|d(gx,gx^*)| + |d(gy,gy^*)| \leq (a_1+a_2+a_4+a_6)(|d(gx,gx^*)| + |d(gy,gy^*)|).$$

Since $a_1+a_2+a_4+a_6 < 1$, therefore we have $|d(gx,gx^*)| + |d(gy,gy^*)| = 0$, which implies that $gx = gx^*$ and $gy = gy^*$. Similarly we can prove that $gx = gy^*$ and $gy = gx^*$. Therefore (gx,gy) is unique coupled point of coincidence of F and g. Now, let $g(x) = u$. Then we have $u = g(x) = F(x,x)$. By $w-$compatibility of F and g, we have

$$g(u) = g(g(x)) = g(F(x,x)) = F(gx,gx) = F(u,u).$$

Then (gu,gu) is coupled point of coincidence of F and g. Consequently $gu = gx$. Therefore $u = gu = F(u,u)$. Hence (u,u) is unique common coupled fixed point of F and g. □

Theorem 2.10. *Let (X,d) be a complex valued metric space, $F : X \times X \to X$ and $g : X \to X$ be $w-$compatible mappings such that*

$$d(F(x,y), F(u,v)) \preceq kU + mV \tag{2.14}$$

for all $x, y, u, v \in X$, where,

$$U, V \in S_{u,v}^{x,y} = \{d(gx, gu), d(gy, gv), \frac{d(F(x,y), gx)}{1 + d(u,v)}, \frac{d(F(x,y), gu)}{1 + d(u,v)}, \frac{d(F(u,v), gu)}{1 + d(x,y)}\},$$

and k, m are nonnegative real numbers such that $k + m < 1$. If $F(X \times X) \subseteq g(X)$ than F and g have unique common coupled fixed point having the form (u,u) for some $u \in X$.

Proof . Following similar arguments to those given in Theorem 2.5, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n).$$

Now, from 2.14, we have

$$d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \preceq kU + mV, \tag{2.15}$$

where $U, V \in S_{x_n,y_n}^{x_{n-1},y_{n-1}}$, and

$$d(gy_n, gy_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \preceq kU + mV, \tag{2.16}$$

where $U, V \in S_{y_n,x_n}^{y_{n-1},x_{n-1}}$.

We have the following 15 cases:

$(i) : U = d(gx_{n-1}, gx_n)$ and $V = d(gx_{n-1}, gx_n)$.

$(ii) : U = d(gx_{n-1}, gx_n)$ and $V = d(gy_{n-1}, gy_n)$.

$(iii) : U = d(gx_{n-1}, gx_n)$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_{n-1})}{1+d(gx_n,gy_n)}$.

$(iv) : U = d(gx_{n-1}, gx_n)$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_n)}{1+d(gx_n,gy_n)}$.

$(v) : U = d(gx_{n-1}, gx_n)$ and $V = \frac{d(F(x_n,y_n),gx_n)}{1+d(gx_{n-1},gy_{n-1})}$.

$(vi) : U = d(gy_{n-1}, gy_n)$ and $V = d(gy_{n-1}, gy_n)$.

$(vii) : U = d(gy_{n-1}, gy_n)$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_{n-1})}{1+d(gx_n,gy_n)}$.

$(viii) : U = d(gy_{n-1}, gy_n)$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_n)}{1+d(gx_n,gy_n)}$.

$(ix) : U = d(gy_{n-1}, gy_n)$ and $V = \frac{d(F(x_n,y_n),gx_n)}{1+d(gx_{n-1},gy_{n-1})}$.

$(x) : U = \frac{d(F(x_{n-1},y_{n-1}),gx_{n-1})}{1+d(gx_n,gy_n)}$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_{n-1})}{1+d(gx_n,gy_n)}$.

$(xi) : U = \frac{d(F(x_{n-1},y_{n-1}),gx_{n-1})}{1+d(gx_n,gy_n)}$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_n)}{1+d(gx_n,gy_n)}$.

$(xii) : U = \frac{d(F(x_{n-1},y_{n-1}),gx_{n-1})}{1+d(gx_n,gy_n)}$ and $V = \frac{d(F(x_n,y_n),gx_n)}{1+d(gx_{n-1},gy_{n-1})}$.

$(xiii) : U = \frac{d(F(x_{n-1},y_{n-1}),gx_n)}{1+d(gx_n,gy_n)}$ and $V = \frac{d(F(x_{n-1},y_{n-1}),gx_n)}{1+d(gx_n,gy_n)}$.

$(xiv) : U = \frac{d(F(x_{n-1},y_{n-1}),gx_n)}{1+d(gx_n,gy_n)}$ and $V = \frac{d(F(x_n,y_n),gx_n)}{1+d(gx_{n-1},gy_{n-1})}$.

$(xv) : U = \frac{d(F(x_n,y_n),gx_n)}{1+d(gx_{n-1},gy_{n-1})}$ and $V = \frac{d(F(x_n,y_n),gx_n)}{1+d(gx_{n-1},gy_{n-1})}$.

In case (i), according to 2.15 and 2.16 we get

$$d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \preceq (k+m)d(gx_{n-1}, gx_n),$$

and

$$d(gy_n, gy_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \preceq (k+m)d(gy_{n-1}, gy_n).$$

Therefore,

$$|d(gx_n, gx_{n+1})| \leq (k+m)|d(gx_{n-1}, gx_n)|,$$

and

$$|d(gy_n, gy_{n+1})| \leq (k+m)|d(gy_{n-1}, gy_n)|.$$

Hence,

$$(|d(gx_n, gx_{n+1})| + |d(gy_n, gy_{n+1})|) \leq (k+m)(|d(gx_{n-1}, gx_n)| + |d(gy_{n-1}, gy_n)|),$$

and also, in the cases $(ii), (iii), (vi), (vii), (x), (xiii)$ and (xv) according to 2.15 and 2.16 we obtain that

$$(|d(gx_n, gx_{n+1})| + |d(gy_n, gy_{n+1})|) \leq (k+m)(|d(gx_{n-1}, gx_n)| + |d(gy_{n-1}, gy_n)|).$$

Similarly, in the cases $(ix), (xii)$ and (xiv), according to 2.15 and 2.16 we again obtain that

$$(|d(gx_n, gx_{n+1})| + |d(gy_n, gy_{n+1})|) \leq \frac{k}{1-m}(|d(gx_{n-1}, gx_n)| + |d(gy_{n-1}, gy_n)|),$$

and, in the cases $(iv), (v), (viii)$ and (xi) we have

$$(|d(gx_n, gx_{n+1})| + |d(gy_n, gy_{n+1})|) \leq k(|d(gx_{n-1}, gx_n)| + |d(gy_{n-1}, gy_n)|).$$

Thus we conclude that

$$(|d(gx_n, gx_{n+1})| + |d(gy_n, gy_{n+1})|) \leq \beta(|d(gx_{n-1}, gx_n)| + |d(gy_{n-1}, gy_n)|),$$

for some $\beta \in \{k, (k+m), \frac{k}{1-m}\}(\beta < 1)$ and for all $n \geq 1$.

Following similar arguments to those given in Theorem 2.5, (x, y) is common coupled coincidence point of F and g, where $gx_n \to gx$ and $gy_n \to gy$. Now we prove that coupled point of coincidence is unique.

Suppose that $(x, y), (x^*, y^*) \in X \times X$ such that $g(x) = F(x, y), g(y) = F(y, x)$ and $g(x^*) = F(x^*, y^*), g(y^*) = F(y^*, x^*)$.

By using 2.14 we get

$$d(gx, gx^*) = d(F(x, y), F(x^*, y^*)) \preceq kU + mV \tag{2.17}$$

where $U, V \in S^{x,y}_{x^*,y^*}$, and

$$d(gy, gy^*) = d(F(y, x), F(y^*, x^*)) \preceq kU + mV, \tag{2.18}$$

where $U, V \in S^{y,x}_{y^*,x^*}$.

Again we have the following 15 cases:

$(i) : U = d(gx, gx^*)$ and $V = d(gx, gx^*)$.

$(ii) : U = d(gx, gx^*)$ and $V = d(gy, gy^*)$.

$(iii) : U = d(gx, gx^*)$ and $V = \frac{d(F(x,y),gx)}{1+d(gx^*,gy^*)}$.

$(iv) : U = d(gx, gx^*)$ and $V = \frac{d(F(x,y),gx^*)}{1+d(gx^*,gy^*)}$.

$(v) : U = d(gx, gx^*)$ and $V = \frac{d(F(x^*,y^*),gx^*)}{1+d(gx,gy)}$.

$(vi) : U = d(gy, gy^*)$ and $V = d(gy, gy^*)$.

$(vii) : U = d(gy, gy^*)$ and $V = \frac{d(F(x,y),gx)}{1+d(gx^*,gy^*)}$.

$(viii) : U = d(gy, gy^*)$ and $V = \frac{d(F(x,y),gx^*)}{1+d(gx^*,gy^*)}$.

$(ix) : U = d(gy, gy^*)$ and $V = \frac{d(F(x^*,y^*),gx^*)}{1+d(gx,gy)}$.

$(x) : U = \frac{d(F(x,y),gx)}{1+d(gx^*,gy^*)}$ and $V = \frac{d(F(x,y),gx)}{1+d(gx^*,gy^*)}$.

$(xi) : U = \frac{d(F(x,y),gx)}{1+d(gx^*,gy^*)}$ and $V = \frac{d(F(x,y),gx^*)}{1+d(gx^*,gy^*)}$.

$(xii) : U = \frac{d(F(x,y),gx)}{1+d(gx^*,gy^*)}$ and $V = \frac{d(F(x^*,y^*),gx^*)}{1+d(gx,gy)}$.

$(xiii) : U = \frac{d(F(x,y),gx^*)}{1+d(gx^*,gy^*)}$ and $V = \frac{d(F(x,y),gx^*)}{1+d(gx^*,gy^*)}$.

$(xiv) : U = \frac{d(F(x,y),gx^*)}{1+d(gx^*,gy^*)}$ and $V = \frac{d(F(x^*,y^*),gx^*)}{1+d(gx,gy)}$.

$(xv) : U = \frac{d(F(x^*,y^*),gx^*)}{1+d(gx,gy)}$ and $V = \frac{d(F(x^*,y^*),gx^*)}{1+d(gx,gy)}$.

In the cases $(i), (ii), (iv), (vi), (viii)$ and $(xiii)$ according to 2.17 and 2.18, we obtain that

$$|d(gx, gx^*)| + |d(gy, gy^*)| \le (k + m)(|d(gx, gx^*)| + |d(gy, gy^*)|). \tag{2.19}$$

Similarly, from 2.17 and 2.18, in the cases $(iii), (v), (vii), (ix), (x), (xi), (xii), (xiv)$ and (xv) we again obtain that

$$|d(gx, gx^*)| + |d(gy, gy^*)| \le \alpha(|d(gx, gx^*)| + |d(gy, gy^*)|), \tag{2.20}$$

where $\alpha \in \{0, k, m\}$ $(\alpha < 1)$. Since $k + m < 1$ and $\alpha < 1$, therefore from 2.19 and 2.20, we have $|d(gx, gx^*)| + |d(gy, gy^*)| = 0$, i.e. $gx = gx^*$ and $gy = gy^*$. That is, (gx, gx) is the unique common coupled point of coincidence. Since F and g are $w-$compatible maps, we have,

$$g(g(x)) = F(gx, gx). \tag{2.21}$$

Let $u = g(x)$. By 2.21, we have $g(u) = F(u, u)$. Therefore (gu, gu) is a coupled point of coincidence of F and g. Consequently, $u = g(u) = F(u, u)$. Hence (u, u) is unique common coupled fixed point of F and g. □

Example 2.11. Let $X = \{(x, 0) : x \in [0, \infty)\} \cup \{(0, x) : x \in [0, \infty)\}$. Define $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|i$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Clearly, (X, d) is a complete complex valued metric space. Consider mappings $F : X \times X \to X$ and $g : X \to X$, given by

$$g(x) = \begin{cases} (0,\text{t}); & x = (t, 0), t \in [0, \infty) \\ (\text{t},0); & x = (0, t), t \in [0, \infty) \end{cases} \quad \text{and} \quad F\big((x_1, x_2), (y_1, y_2)\big) = (\frac{x_1}{2}, \frac{x_2}{2}).$$

Note that F and g satisfy all the conditions of Theorem 2.9, if we take $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = \frac{1}{12}$. Moreover $(0, 0)$ is the unique common coupled fixed point of F and g.

Next, we give a example which supports Theorems 2.5 and 2.9, while a lot of results in partially ordered metric spaces, cone metric spaces, ordered cone metric spaces and complex valued metric spaces for example the results of Nashine et al. [11] do not.

Example 2.12. Let $X = \{ix : x \in [0, 1]\}$. Define $d : X \times X \to \mathbb{C}$ by $d(x, y) = i|x - y|$, where $x, y \in X$. Clearly, (X, d) is a complete complex valued metric space. Consider mappings $F : X \times X \to X$ and $g : X \to X$, given by

$$F(x, y) = i\frac{x^4 + y^4}{16} \quad \text{and} \quad g(x) = i\frac{x^4}{4}.$$

If $y_1 = \frac{3}{4}i$ and $y_2 = \frac{1}{2}i$, then $g(y_2) = i\frac{\frac{1}{16}}{4} \preceq i\frac{\frac{81}{4^4}}{4} = g(y_1)$, but for $x = i$, we get

$$F(x, y_2) = F(i, \frac{1}{2}i) = i\frac{1 + \frac{1}{16}}{16} \preceq i\frac{1 + \frac{81}{4^4}}{16} = F(i, \frac{3}{4}i) = F(x, y_1).$$

So, the mappings F and g do not satisfy the mixed g-monotone property. Therefore, Theorems 3.1 and 3.2 of Nashine et al. [11] cannot be supported to reach this conclusion.

Now, we show that Theorems 2.5 and 2.9 can be used for this case.

$$\begin{aligned} d(F(x, y), F(u, v)) &= i|i\frac{x^4 + y^4}{16} - i\frac{u^4 + v^4}{16}| = \frac{1}{4}i|i\frac{x^4 - u^4}{4} + i\frac{y^4 - v^4}{4}| \\ &\preceq \frac{1}{4}i|i\frac{x^4 - u^4}{4}| + \frac{1}{4}i|i\frac{y^4 - v^4}{4}| = \frac{1}{4}i|\frac{x^4 - u^4}{4}| + \frac{1}{4}i|\frac{y^4 - v^4}{4}| \\ &= \frac{1}{4}d(g(x), g(u)) + \frac{1}{4}d(g(y), g(v)), \end{aligned}$$

where $a_1 = a_2 = \frac{1}{4}$ and $a_3 = a_4 = a_5 = a_6 = 0$. Note that $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$.

$$F(X \times X) = \{ix : x \in [0, \frac{1}{8}]\} \subset g(X) = \{ix : x \in [0, \frac{1}{4}]\},$$

and $g(X)$ is a complete subset of X. Hence, the conditions of Theorem 2.5 are satisfied. Therefore, F and g have a coupled coincidence point in X.
Since, $g(0) = F(0, 0)$ and

$$g(F(0, 0)) = g(0) = F(0, 0) = F(g(0), g(0)),$$

then F and g are W-compatible. Therefore, conditions of Theorem 2.9 are satisfied. Now, we can apply Theorem 2.9 to conclude the existence of a unique common coupled fixed point of F and g that is a point $(0, 0)$.

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