TWO COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS

A. RAZANI and M. YAZDI∗

Abstract. Recently, Zhang and Song [Q. Zhang, Y. Song, Fixed point theory for generalized ϕ-weak contractions, Appl. Math. Lett. 22(2009) 75-78] proved a common fixed point theorem for two maps satisfying generalized ϕ-weak contractions. In this paper, we prove a common fixed point theorem for a family of compatible maps. In fact, a new generalization of Zhang and Song’s theorem is given.

1. Introduction and preliminaries

Let X be a metric space. A map T : X → X is a contraction if there exists a constant k ∈ (0,1) such that d(Tx, Ty) ≤ kd(x, y), for all x, y ∈ X.
A map T : X → X is a ϕ-weak contraction if there exists a function ϕ : [0, +∞) → [0, +∞) such that ϕ is positive on (0, +∞), ϕ(0) = 0 and
\[ d(Tx, Ty) ≤ d(x, y) - ϕ(d(x, y)). \] (1.1)
The concept of the weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997. Actually in [1], the authors defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed points. Rhoades [20] showed that most results of [1] are still true for any Banach spaces. Also, Rhoades [20] proved an interesting fixed point theorem which is one of generalizations of the Banach contraction principle because it contains contractions as special cases (ϕ(t) = (1−k)t).

Theorem 1.1. [20] Let (X, d) be a complete metric space and A be a ϕ− weak contraction on X. If ϕ is continuous and nondecreasing function, then A has a unique fixed point.

In fact, the weak contractions are also closely related to maps of Boyd and Wong’s type [4] and Reich’s type [19]. Namely, if ϕ is a lower semi-continuous function from the right, then ψ(t) = t − ϕ(t) is an upper semi-continuous function from the right and moreover, (1.1) turns into d(Tx, Ty) ≤ ψ(d(x, y)). Therefore, the ϕ-weak contraction with a function ϕ is of Boyd and Wong [4]. if we define K(t) = ϕ(t)/t for

Date: Received: March 2011; Revised: July 2011.
2000 Mathematics Subject Classification. 47H10.
Key words and phrases. Common fixed point, Compatible mappings, Weakly Compatible mappings, ϕ-weak contraction, Complete metric space.
∗: Corresponding author.
When $t > 0$ and $K(0) = 0$, then (1.1) is replaced by $d(Tx, Ty) \leq K(d(x, y))d(x, y)$. Thus the φ-weak contraction becomes a Reich type one.

During the last few decades, a number of hybrid contractive mapping results have been obtained by many mathematical researchers. For example, Song [25, 26], Al-Thagafi and Shahzad [2], Shahzad [21] and Hussain and Junck [11] obtained the common fixed point theorems of $f$-contraction ($T(d(Tx, Ty) \leq kd(fx, fy))$), generalized $f$-contraction $$(T(d(Tx, Ty) \leq k \max\{d(fx, fy), d(Tx, fx), d(Ty, fy), \frac{1}{2}[d(fx, Ty) + d(Tx, fy)]\})$$ and generalized $(f,g)$-contraction $$(T(d(Tx, Ty) \leq k \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), \frac{1}{2}[d(fx, Ty) + d(Tx, gy)]\}),$$ respectively.

Song [24] extended the above results to $f$-weak contraction ($d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy))$).

Recently, Zhang and Song [30] proved the following theorem.

**Theorem 1.2.** [30] Let $(X, d)$ be a complete metric space and $T, S : X \rightarrow X$ two mappings such that for all $x, y \in X$, $$d(Tx, Sy) \leq M(x, y) - \phi(M(x, y)),$$ where $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for $t > 0$, $\phi(0) = 0$ and $$M(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\}.$$ Then, there exists a unique point $u \in X$ such that $Tu = Su = u$.

The object of this paper is to prove a common fixed point theorem for a family of compatible maps in a metric space.

2. Main result

In this section, we shall prove a common fixed point theorem for any even number of compatible maps in a complete metric space. In fact, it is a generalization of Zhang and Song’s common fixed point theorem (Theorem 1.2).

Let $(X, d)$ be a metric space and $T$ a self-mapping on $X$. In [7], Ćirić introduced and investigated a class of self-mappings on $X$ satisfying the following condition: $$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \quad (c)$$ where $0 < k < 1$. In [8] Ćirić proved the following common fixed point theorem.

**Theorem 2.1.** Let $(X, d)$ be a complete metric space and let $\{T_\alpha\}_{\alpha \in \mathcal{J}}$ be a family of self-mappings on $X$. If there exists a fixed $\beta \in \mathcal{J}$ such that for each $\alpha \in \mathcal{J}$ and all $x, y \in X$ $$d(T_\alpha x, T_\beta y) \leq \lambda \max\{d(x, y), d(x, T_\alpha x), d(y, T_\beta y), \frac{1}{2}[d(x, T_\beta y) + d(y, T_\alpha x)]\},$$ where $\lambda = \lambda(\alpha) \in (0, 1)$, then all $T_\alpha$ have a unique common fixed point in $X$. 
The class of mappings satisfying the contractive definition of type of (c), as well as its generalization, has proved useful in fixed and common fixed point theory (see [3, 18, 23]).

**Definition 2.2.** [13] Self-maps $A$ and $S$ of a metric space $(X, d)$ are said to be compatible if $d(ASP_n, SAp_n) \to 0$ whenever $\{p_n\}$ is a sequence in $X$ such that $Ap_n, Sp_n \to u$, for some $u \in X$, as $n \to \infty$.

**Definition 2.3.** [15] Self-maps $A$ and $S$ of a metric space $(X, d)$ are said to be weakly compatible if they commute at their coincidence points; i.e. if $Ap = Sp$ for some $p \in X$, then $ASP = SAp$.

This concept is most general among all the commutativity concepts in this field, as every pair of weakly commuting self-maps is compatible and each pair of compatible self-maps is weakly compatible, but the reverse is not true always. Many authors have proved common fixed point theorems for a variety of commuting self-mappings on usual metric, as well as on different kinds of generalized metric spaces([3, 5, 6, 8],[9]-[17], [22, 23],[27]-[29]).

**Theorem 2.4.** [22] Let $A,B,S,T,L$ and $M$ be self-maps of a complete metric space $(X, d)$, satisfying the conditions:

1. $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$;
2. $AB = BA, ST = TS, LB = BL, MT = TM$;
3. For all $x, y \in X$ and for some $k \in (0, 1)$,

$$d(Lx, My) \leq k \max\{d(Lx, ABx), d(My, STy), d(ABx, STy), \frac{1}{2}[d(Lx, STy) + d(My, ABx)]\};$$

4. The pair $(L, AB)$ is compatible and the pair $(M, ST)$ is weakly compatible;
5. Either $AB$ or $L$ is continuous.

Then, $A, B, S, T, L$ and $M$ have a unique common fixed point.

Define $\Phi = \{\varphi : [0, +\infty) \to [0, +\infty)\}$ where each $\varphi \in \Phi$ satisfies the following conditions:

(a) $\varphi$ is lower semi-continuous on $[0, +\infty)$,
(b) $\varphi$ is non-decreasing,
(c) $\varphi(0) = 0$, and
(d) $\varphi(t) > 0$ for each $t > 0$.

Now, we prove our main result.

**Theorem 2.5.** Let $P_1, P_2, \cdots, P_{2n}, Q_0$ and $Q_1$ be self-maps on a complete metric space $(X, d)$, satisfying conditions:

1. $Q_0(X) \subseteq P_1P_3, \cdots, P_{2n-1}(X), Q_1(X) \subseteq P_2P_4, \cdots, P_{2n}(X)$;
\( P_2(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})P_2, \)
\( P_2 P_4(P_6 \cdots P_{2n}) = (P_6 \cdots P_{2n})P_2 P_4, \)
\[
\vdots
\]
\( P_2 \cdots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \cdots P_{2n-2}, \)
\( Q_0(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})Q_0, \)
\( Q_0(P_6 \cdots P_{2n}) = (P_6 \cdots P_{2n})Q_0, \)
\[
\vdots
\]
\( Q_0 P_{2n} = P_{2n} Q_0, \)
\( P_1(P_3 \cdots P_{2n-1}) = (P_3 \cdots P_{2n-1})P_1, \)
\( P_1 P_3(P_5 \cdots P_{2n-1}) = (P_5 \cdots P_{2n-1})P_1 P_3, \)
\[
\vdots
\]
\( P_1 \cdots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \cdots P_{2n-3}, \)
\( Q_1(P_3 \cdots P_{2n-1}) = (P_3 \cdots P_{2n-1})Q_1, \)
\( Q_1(P_5 \cdots P_{2n-1}) = (P_5 \cdots P_{2n-1})Q_1, \)
\[
\vdots
\]
\( Q_1 P_{2n-1} = P_{2n-1} Q_1; \)

(3) \( P_2 \cdots P_{2n} \) or \( Q_0 \) is continuous;

(4) The pair \( (Q_0, P_2 \cdots P_{2n}) \) is compatible and the pair \( (Q_1, P_1 \cdots P_{2n-1}) \) is weakly compatible;

(5) There exists \( \varphi \in \Phi \) such that
\[
d(Q_0u, Q_1v) \leq M(u, v) - \varphi(M(u, v)), \forall u, v \in X,
\]
where
\[
M(u, v) = \max\{d(P_2 P_4 \cdots P_{2n} u, Q_0 u), d(P_1 P_3 \cdots P_{2n-1} v, Q_1 v),
\]
\[
\frac{1}{2} \left[ d(P_1 P_3 \cdots P_{2n-1} v, Q_0 u) + d(P_2 P_4 \cdots P_{2n} u, Q_1 v) \right]
\]
for all \( u, v \in X \). Then \( P_1, P_2, \ldots, P_{2n}, Q_0 \) and \( Q_1 \) have a unique common fixed point in \( X \).

Proof. Let \( x_0 \in X \), from condition (1) there exist \( x_1, x_2 \in X \) such that \( Q_0 x_0 = P_1 P_3 \cdots P_{2n-1} x_1 = y_0 \) and \( Q_1 x_1 = P_2 P_4 \cdots P_{2n} x_2 = y_1 \). Inductively we can construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \):
\[
Q_0 x_{2k} = P_1 P_3 \cdots P_{2n-1} x_{2k+1} = y_{2k}
\]
and
\[
Q_1 x_{2k+1} = P_2 P_4 \cdots P_{2n} x_{2k+2} = y_{2k+1},
\]
for \( k \in \mathbb{N} \).

Putting \( u = x_p = x_{2k}, v = x_{q+1} = x_{2m+1}, G_1 = P_1 P_3 \cdots P_{2n} \) and \( G_2 = P_2 P_4 \cdots P_{2n-1} \) in condition (5), we have
\[
d(Q_0 x_{2k}, Q_1 x_{2m+1}) \leq M(x_{2k}, x_{2m+1}) - \varphi(M(x_{2k}, x_{2m+1})) \leq \max\{d(G_1 x_{2k}, Q_0 x_{2k}), d(G_2 x_{2m+1}, Q_1 x_{2m+1}),
\]
\[
d(G_1 x_{2k}, G_2 x_{2m+1}),
\]
\[
\frac{1}{2} \left[ d(G_2 x_{2m+1}, Q_0 x_{2k}) + d(G_1 x_{2k}, Q_1 x_{2m+1}) \right]\]
 Thus
\[ d(y_{2k}, y_{2m+1}) \leq \max \{ d(y_{2k-1}, y_{2k}), d(y_{2m}, y_{2m+1}), d(y_{2k-1}, y_{2m}), \frac{1}{2} [d(y_{2m}, y_{2k}) + d(y_{2k-1}, y_{2m+1})] \}. \]

Thus
\[ d(y_p, y_{q+1}) \leq \max \{ d(y_{p-1}, y_p), d(y_q, y_{q+1}), d(y_{p-1}, y_q), \frac{1}{2} [d(y_q, y_p) + d(y_{p-1}, y_{q+1})] \}. \]

If \( q = p \), then
\[ \frac{1}{2} [d(y_p, y_p) + d(y_{p-1}, y_{p+1})] \leq \frac{1}{2} [d(y_{p-1}, y_p) + d(y_p, y_{p+1})] \leq \max \{ d(y_{p-1}, y_p), d(y_p, y_{p+1}) \}. \]

Thus \( (y_p, y_{p+1}) \leq d(y_{p-1}, y_p) \) as the inequality \( d(y_p, y_{p+1}) > d(y_{p-1}, y_p) \) implies \( M(p, p+1) = d(y_p, y_{p+1}) \) and furthermore,
\[ d(y_p, y_{p+1}) \leq d(y_p, y_{p+1}) - \varphi(d(y_p, y_{p+1})). \]

So \( \varphi(d(y_p, y_{p+1})) = 0 \). This is a contradiction. Hence
\[ d(y_{2k}, y_{2k+1}) \leq M(x_{2k}, x_{2k+1}) \leq d(y_{2k}, y_{2k-1}). \]

Similarly,
\[ d(y_{2k+1}, y_{2k+2}) \leq M(x_{2k+1}, x_{2k+2}) \leq d(y_{2k}, y_{2k+1}). \]

Therefore, for all \( n \in \mathbb{N} \), even or odd,
\[ d(y_n, y_{n+1}) \leq M(x_n, x_{n+1}) \leq d(y_{n-1}, y_n). \]

Thus \( \{d(y_n, y_{n+1})\} \) is a decreasing and bounded below sequence. So, there exists \( r \geq 0 \) such that
\[ \lim_{n \to \infty} d(y_n, y_{n+1}) = \lim_{n \to \infty} M(x_n, x_{n+1}) = r. \]

Then (by semi-continuity of \( \varphi \))
\[ \varphi(r) \leq \liminf_{n \to \infty} \varphi(M(x_n, x_{n+1})). \]

We claim that \( r = 0 \). We know
\[ d(y_n, y_{n+1}) \leq M(x_n, x_{n+1}) - \varphi(M(x_n, x_{n+1})). \]

So
\[ r \leq r - \liminf_{n \to \infty} \varphi(M(x_n, x_{n+1})) \leq r - \varphi(r), \]
i.e., \( \varphi(r) \leq 0 \). Thus \( \varphi(r) = 0 \) by the property of the function \( \varphi \) and furthermore,
\[ \lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \]

Next, we show that \( \{y_n\} \) is a cauchy sequence. Let
\[ C_n = \sup \{d(y_j, y_k) : k, j \geq n\}. \]

Then \( \{C_n\} \) is decreasing. If \( \lim_{n \to \infty} C_n = 0 \), then we are done. Assume that \( \lim_{n \to \infty} C_n = C > 0 \). Choose \( \varepsilon < \frac{C}{8} \) small enough and select \( N \) such that for all \( n \geq N \),
\[ d(y_n, y_{n+1}) < \varepsilon \quad \text{and} \quad C_n < C + \varepsilon. \]
By the definition of $C_{N+1}$, there exist $m, n \geq N + 1$ such that $d(y_m, y_n) > C_n - \varepsilon \geq C - \varepsilon$. Replace $y_m$ by $y_{m+1}$ if necessary. We may assume that $m$ is even, $n$ is odd and $d(y_m, y_n) > C - 2\varepsilon$. Then $d(y_{m-1}, y_{n-1}) > C - 4\varepsilon$ and

$$d(y_m, y_n) \leq M(x_m, x_n) - \varphi(M(x_m, x_n))$$

$$\leq \max\{d(y_{m-1}, y_m), d(y_{n-1}, y_n), d(y_{m-1}, y_{n-1}),$$

$$\frac{1}{2}[d(y_{n-1}, y_m) + d(y_{m-1}, y_n)]\} - \varphi\left(\frac{C}{2}\right).$$

i.e.,

$$C - 2\varepsilon < d(y_m, y_n) \leq \max\{\varepsilon, \varepsilon, d(y_{m-1}, y_{n-1}), C_N\} - \varphi\left(\frac{C}{2}\right).$$

So

$$C - 2\varepsilon < C_N - \varphi\left(\frac{C}{2}\right) \leq C - \varphi\left(\frac{C}{2}\right).$$

This is impossible if $\varepsilon$ be small enough. Thus, we must have $c = 0$. Therefore, the sequence $\{y_n\}$ is a cauchy sequence. Since $X$ is complete, there exists some $z \in X$ such that $y_n \rightarrow z$. Also, for it’s subsequence we have

$$Q_0x_{2k} \rightarrow z, P_2P_4 \cdots P_{2n}x_{2k} \rightarrow z$$

and

$$Q_1x_{2k+1} \rightarrow z, P_1P_3 \cdots P_{2n-1}x_{2k+1} \rightarrow z.$$

Case 1. $P_2P_4 \cdots P_{2n}$ is continuous.

Define $G_1 = P_2P_4 \cdots P_{2n}$. Since $G_1$ is continuous, $G_1^2x_{2k} \rightarrow G_1z$ and $G_1Q_0x_{2k} \rightarrow G_1z$. Also, as $(Q_0, G_1)$ is compatible, this implies that $Q_0G_1x_{2k} \rightarrow G_1z$.

(a) Putting $u = P_2P_4 \cdots P_{2n}x_{2k}, v = x_{2k+1}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

$$d(Q_0G_1x_{2k}, Q_1x_{2k+1}) \leq M(G_1x_{2k}, x_{2k+1}) - \varphi(M(G_1x_{2k}, x_{2k+1}))$$

$$= \max\{d(G_1^2x_{2k}, Q_0G_1x_{2k}), d(G_2x_{2k+1}, Q_1x_{2k+1}),$$

$$d(G_1^2x_{2k}, Q_0G_1x_{2k}), d(G_2x_{2k+1}, Q_1x_{2k+1}),$$

$$\frac{1}{2}[d(G_2x_{2k+1}, Q_0G_1x_{2k}) + d(G_1^2x_{2k}, Q_1x_{2k+1})]\} - \varphi(M(G_1x_{2k}, x_{2k+1})).$$

Letting $k \rightarrow \infty$ (taking lower limit), we get

$$d(G_1z, z) \leq \max\{d(Gz, Gz), d(z, z), d(z, G_1z), \frac{1}{2}[d(G_1z, z) + d(G_1z, z)]\}$$

$$- \lim_{n \rightarrow \infty} \varphi(M(G_1x_{2k}, x_{2k+1}))$$

$$\leq d(G_1z, z) - \varphi(d(G_1z, z)).$$

So $G_1z = z$. Thus $P_2P_4 \cdots P_{2n}z = z$.

(b) Putting $u = z, v = x_{2k+1}, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

$$d(Q_0z, Q_1x_{2k+1}) \leq M(z, x_{2k+1}) - \varphi(M(z, x_{2k+1}))$$

$$= \max\{d(G_1z, Q_0z), d(G_2x_{2k+1}, Q_1x_{2k+1}), d(G_1z, G_2x_{2k+1}),$$

$$\frac{1}{2}[d(G_2x_{2k+1}, Q_0z) + d(G_1z, Q_1x_{2k+1})]\} - \varphi(M(z, x_{2k+1})).$$

Letting $k \rightarrow \infty$ (taking lower limit), we get

$$d(Q_0z, z) \leq \max\{d(z, Q_0z), d(z, z), d(z, z), \frac{1}{2}d(z, Q_0z)\}$$

$$- \varphi(M(z, Q_0z)).$$

So $d(Q_0z, z) \leq d(z, Q_0z) - \varphi(M(z, Q_0z))$. Hence $Q_0z = z$. Therefore $Q_0z = P_2P_4 \cdots P_{2n}z = z$. 
(c) Putting $u = P_4 \cdots P_{2n} z, v = x_{2k+1}, G_1 = P_2 P_4 \cdots P_{2n}$ and $G_2 = P_1 P_3 \cdots P_{2n-1}$ in condition (5) and using the condition $P_2( P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n}) P_2$ and $Q_0(P_4 \cdots P_{2n}) = (P_4 \cdots P_{2n})Q_0$ in condition (2), we get

$$d(Q_0 P_4 \cdots P_{2n} z, Q_1 x_{2k+1}) \leq M(P_4 \cdots P_{2n} z, x_{2k+1}) - \varphi(M(P_4 \cdots P_{2n} z, x_{2k+1}))$$

$$= \max\{d(G_1 P_4 \cdots P_{2n} z, G_2 x_{2k+1}), d(G_2 x_{2k+1}, Q_1 x_{2k+1}), d(G_1 P_4 \cdots P_{2n} z, Q_0 P_4 \cdots P_{2n}), \frac{1}{2}[d(G_2 x_{2k+1}, Q_0 P_4 \cdots P_{2n} z) + d(G_1 P_4 \cdots P_{2n} z, Q_1 x_{2k+1})] - \varphi(M(P_4 \cdots P_{2n} z, x_{2k+1}))\}.$$ 

Letting $k \to \infty$, (taking lower limit) we get

$$d(P_4 \cdots P_{2n} z, z) \leq \max\{d(P_4 \cdots P_{2n} z, P_4 \cdots P_{2n} z), d(z, z), d(P_4 \cdots P_{2n} z, z), \frac{1}{2}[d(z, P_4 \cdots P_{2n} z) + d(P_4 \cdots P_{2n} z, z)]\}$$

$$- \varphi(M(P_4 \cdots P_{2n} z, z)).$$

Hence, it follows that $P_4 \cdots P_{2n} z = z$. Then $P_2( P_4 \cdots P_{2n}) z = P_2 z = z$. Continuing this procedure, we obtain $Q_0 z = P_2 z = P_4 z = \cdots = P_{2n} z = z$.

(d) As $Q_0(X) \subseteq P_1 P_3 \cdots P_{2n-1}(X)$, there exists $v \in X$ such that $P_1 P_3 \cdots P_{2n-1} v = Q_0 z = z$. Putting $u = x_{2k}, G_1 = P_2 P_4 \cdots P_{2n}$ and $G_2 = P_1 P_3 \cdots P_{2n-1}$ in condition (5), we have

$$d(Q_0 x_{2k}, Q_1 v) \leq M(x_{2k}, v) - \varphi(M(x_{2k}, v))$$

$$= \max\{d(G_1 x_{2k}, Q_0 x_{2k}), d(G_2 v, Q_1 v), d(G_1 x_{2k}, G_2 v), \frac{1}{2}[d(G_2 v, Q_0 x_{2k}) + d(G_1 x_{2k}, Q_1 v)] - \varphi(M(x_{2k}, v)).$$

Letting $k \to \infty$, (taking lower limit) we get

$$d(z, Q_1 v) \leq \max\{d(z, z), d(z, Q_1 v), d(z, z), \frac{1}{2}[d(z, z) + d(z, Q_1 v)]\}$$

$$- \varphi(d(z, Q_1 v)).$$

So $Q_1 v = z$. Hence $P_1 P_3 \cdots P_{2n-1} v = Q_1 v = z$. As $(Q_1, P_1 P_3 \cdots P_{2n-1})$ is weakly compatible, we have

$$P_1 P_3 \cdots P_{2n-1} Q_1 v = Q_1 P_1 P_3 \cdots P_{2n-1} v.$$ 

Thus $P_1 P_3 \cdots P_{2n-1} z = Q_1 z$.

(e) Putting $u = x_{2k}, v = z, G_1 = P_2 P_4 \cdots P_{2n}$ and $G_2 = P_1 P_3 \cdots P_{2n-1}$ in condition (5), we have

$$d(Q_0 x_{2k}, Q_1 z) \leq M(x_{2k}, z) - \varphi(M(x_{2k}, z))$$

$$= \max\{d(G_1 x_{2k}, Q_0 x_{2k}), d(G_2 z, Q_1 z), d(G_1 x_{2k}, G_2 z), \frac{1}{2}[d(G_2 z, Q_0 x_{2k}) + d(G_1 x_{2k}, Q_1 z)] - \varphi(M(x_{2k}, z)).$$

Letting $k \to \infty$, (taking lower limit) we get

$$d(z, Q_1 z) \leq \max\{d(z, z), d(Q_1 z, Q_1 z), d(z, Q_1 z), \frac{1}{2}[d(Q_1 z, z) + d(z, Q_1 z)]\}$$

$$- \varphi(d(Q_1 z, z)).$$

Therefore $Q_1 z = z$. Hence $P_1 P_3 \cdots P_{2n-1} z = Q_1 z = z$.

(f) Putting $u = x_{2k}, v = P_3 \cdots P_{2n-1} z, G_1 = P_2 P_4 \cdots P_{2n}$ and $G_2 = P_1 P_3 \cdots P_{2n-1}$ in condition (5) and using the conditions $P_1(P_3 \cdots P_{2n-1}) = (P_3 \cdots P_{2n-1}) P_1$ and
Letting $k \to \infty$, (taking lower limit) we get

\[ d(z, P_3 \cdots P_{2n-1}z) \leq \max\{d(P_3 \cdots P_{2n-1}z, P_3 \cdots P_{2n-1}z), d(z, P_3 \cdots P_{2n-1}z), d(z, z), \frac{1}{2}d(P_3 \cdots P_{2n-1}z, z)\} \]

\[ -\varphi(d(z, P_3 \cdots P_{2n-1}z)). \]

So $P_3 \cdots P_{2n-1}z = z$. Therefore $P_1(P_3 \cdots P_{2n-1}z) = P_1z = z$. Continuing this procedure, we have

\[ Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z = z. \]

Thus, we have proved

\[ Q_0z = Q_1z = P_1z = P_2z = \cdots = P_{2n-1}z = P_{2n}z = z. \]

Case 2. $Q_0$ is continuous.

Since $Q_0$ is continuous, $Q_0^2x_{2k} \to Q_0z$. As $(Q_0, P_0P_1 \cdots P_{2n})$ is compatible, we have

\[ P_2P_4 \cdots P_{2n}Q_0x_{2k} \to Q_0z. \]

(g) Putting $u = Q_0x_{2k}, v = x_{2k+1}, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

\[ d(Q_0^2x_{2k}, Q_1x_{2k+1}) \leq \max\{d(G_1Q_0x_{2k}, Q_0^2x_{2k}), d(G_2x_{2k+1}, Q_1x_{2k+1}), d(G_1Q_0x_{2k}G_2x_{2k+1}), \frac{1}{2}[d(G_2x_{2k+1}, Q_0x_{2k}) + d(G_1Q_0x_{2k}Q_1x_{2k+1})]\} \]

\[ -\varphi(M(Q_0x_{2k}, x_{2k+1})). \]

Letting $k \to \infty$, (taking lower limit) we get

\[ d(Q_0z, z) \leq \max\{d(Q_0z, Q_0z), d(z, z), d(Q_0z, z), \frac{1}{2}[d(z, Q_0z) + d(Q_0z, z)]\} \]

\[ -\varphi(d(Q_0z, z)). \]

Therefore $Q_0z = z$. Now using step (d), (e), (f) and continuing step (f) gives us

\[ Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z = z \]

(h) As $Q_1(X) \subseteq P_2P_4 \cdots P_{2n}(X)$, there exists $w \in X$ such that $P_2P_4 \cdots P_{2n}w = Q_1z = z$. Putting $u = w, v = x_{2k+1}, G_1 = P_2P_4 \cdots P_{2n}$ and $G_2 = P_1P_3 \cdots P_{2n-1}$ in condition (5), we have

\[ d(Q_0w, Q_1x_{2k+1}) \leq \max\{d(G_1w, Q_0w), d(G_2x_{2k+1}, Q_1x_{2k+1}), d(G_1w, G_2x_{2k+1}), \frac{1}{2}[d(G_2x_{2k+1}, Q_0w) + d(G_1w, Q_1x_{2k+1})]\} \]

\[ -\varphi(M(w, x_{2k+1})). \]
Letting \( k \to \infty \), (taking lower limit) we get

\[
d(Q_0w, z) \leq \max\{d(z, Q_0w), d(z, z), \frac{1}{2}[d(z, Q_0w) + d(z, z)]\} - \varphi(M(z, Q_0w)).
\]

So \( Q_0w = z \). Hence \( Q_0w = P_2P_4 \cdots P_{2n}w = z \). As \((Q_0, P_2P_4 \cdots P_{2n})\) is weakly compatible, we have

\[
Q_0P_2P_4 \cdots P_{2n}w = P_2P_4 \cdots P_{2n}Q_0w.
\]

Hence \( Q_0z = P_2P_4 \cdots P_{2n}z = z \). Similarly to in step (c) it can be shown that \( Q_0z = P_2z = \cdots = P_{2n}z = z \). Thus, we have proved that

\[
Q_0z = Q_1z = P_1z = P_2z = \cdots = P_{2n-1}z = P_{2n}z = z.
\]

To prove the uniqueness property of \( z \), let \( z' \) be another common fixed point of the aforementioned maps; then

\[
Q_0z' = Q_1z' = P_1z' = P_2z' = \cdots = P_{2n-1}z' = P_{2n}z' = z'.
\]

Putting \( u = z, v = z' \), \( G_1 = P_2P_4 \cdots P_{2n} \) and \( G_2 = P_1P_3 \cdots P_{2n-1} \) in condition (5), we have

\[
d(Q_0z, Q_1z') \leq M(z, z') - \varphi(M(z, z'))
= \max\{d(G_1z, Q_0z), d(G_2z', Q_1z'), d(G_1z, G_2z'),
\frac{1}{2}[d(G_2z', Q_0z) + d(G_1z, Q_1z')]\} - \varphi(M(z, z')).
\]

Then \( d(z, z') \leq d(z, z') - \varphi(d(z, z')). \) So \( z = z' \) and this shows that \( z \) is a unique common fixed point of the maps. \( \square \)

**Remark 2.6.** Theorem 1.2 is a special case of Theorem 2.5 with \( Q_0 = S, Q_1 = T \)
and \( P_i = I \) (identity map) for all \( 1 \leq i \leq 2n \). Also, Theorem 2.5 is a generalization
of Theorem 2.4 with \( \varphi(t) = (1 - k)t. \)

**Theorem 2.7.** Let \((X, d)\) be a complete metric space and let \( \{T_\alpha\}_{\alpha \in J} \) and \( \{P_i\}_{i=1}^{2n} \)
be two families of self-mappings on \( X \). Suppose, there exists a fixed \( \beta \in J \) such that

1. \( T_\alpha(X) \subseteq P_2P_4, \cdots P_{2n}(X) \) for each \( \alpha \in J \) and \( T_\beta(X) \subseteq P_1P_3, \cdots P_{2n-1}(X); \)

2. \( \frac{1}{2}[d(G_2z', Q_0z) + d(G_1z, Q_1z')]\} - \varphi(M(z, z')).
\]

Then \( d(z, z') \leq d(z, z') - \varphi(d(z, z')). \) So \( z = z' \) and this shows that \( z \) is a unique common fixed point of the maps. \( \square \)

**Remark 2.6.** Theorem 1.2 is a special case of Theorem 2.5 with \( Q_0 = S, Q_1 = T \)
and \( P_i = I \) (identity map) for all \( 1 \leq i \leq 2n \). Also, Theorem 2.5 is a generalization
of Theorem 2.4 with \( \varphi(t) = (1 - k)t. \)
The pair proved. □

There exists $\varphi \in \Phi$ compatible; $P_\beta$ is continuous; the pairs $(T_\alpha, P_1 \cdots P_{2n-1})$ are weakly compatible;

There exists $\varphi \in \Phi$ such that
\[
d(T_\beta u, T_\alpha v) \leq M(u, v) - \varphi(M(u, v)), \quad \text{for all } u, v \in X \text{ and for all } \alpha \in J,\]
where
\[
M(u, v) = \max\{d(P_2 P_4 \cdots P_{2n} u, T_\beta u), d(P_1 P_3 \cdots P_{2n-1} v, T_\alpha v),
\frac{1}{2}[d(P_1 P_3 \cdots P_{2n-1} v, T_\beta u) + d(P_2 \cdots P_{2n} u, T_\alpha v)]\}
\]

Then, all $P_i$ and $T_\alpha$ have a unique common fixed point in $X$.

Proof. Let $T_{\alpha_0}$ be a fixed element of $\{T_\alpha\}_{\alpha \in J}$. By Theorem 2.5 with $Q_0 = T_\beta$ and $Q_1 = T_{\alpha_0}$ it follows that there exists some $z \in X$ such that $T_\beta z = T_{\alpha_0} z = P_1 P_3 \cdots P_{2n-1} z = P_2 P_4 \cdots P_{2n} z = z$. Let $\alpha \in J$ be arbitrary. Then from condition (5),
\[
d(T_\beta z, T_\alpha z) \leq \max\{d(P_2 P_4 \cdots P_{2n} z, T_\beta z), d(P_1 P_3 \cdots P_{2n-1} z, T_\alpha z),
\frac{1}{2}[d(P_1 P_3 \cdots P_{2n-1} z, T_\beta z) + d(P_2 \cdots P_{2n} z, T_\alpha z)]\} - \varphi(M(z, z)).
\]
So $d(z, T_\alpha z) \leq d(z, T_\alpha z) - \varphi(d(z, T_\alpha z))$. Thus $T_\alpha z = z$ for each $\alpha \in J$. Since condition (5) implies the uniqueness of the common fixed point, Theorem 2.7 is proved.

Remark 2.8. Theorem 2.1 is a special case of Theorem 2.7 with $P_i = I$ (identity map), for all $1 \leq i \leq 2n$ and $\varphi(t) = (1 - \lambda)t$.

Now, we prove a common fixed point for any number of mappings.

Corollary 2.9. Let $P_0, P_1, P_2, \cdots, P_n$ be self-maps on a complete metric space $(X, d)$ satisfying conditions:
(1) $P_0(X) \subseteq P_1 P_2 \cdots P_n(X)$;
\( (2) \)
\[
\begin{align*}
P_1(P_2 \cdots P_n) &= (P_2 \cdots P_n)P_1, \\
P_1P_2(P_3 \cdots P_n) &= (P_3 \cdots P_n)P_1P_2, \\
&\vdots \\
P_1 \cdots P_{n-1}(P_n) &= (P_n)P_1 \cdots P_{n-1};
\end{align*}
\]

\( (3) \) There exists \( \varphi \in \Phi \) such that
\[
d(P_0u, v) \leq M(u, v) - \varphi(M(u, v)), \quad \text{for all } u, v \in X \text{ where}
\]
\[
M(u, v) = \max\{d(u, P_0u), d(P_1P_2 \cdots P_nv, v), \\
d(u, P_1P_2 \cdots P_nv), \frac{1}{2}[d(P_1P_2 \cdots P_nv, P_0u) + d(u, v)]\}. 
\]

Then, \( P_0, P_1, P_2, \ldots, P_n \) have a unique common fixed point in \( X \).

\textbf{References}


1 Department of Mathematics, Faculty of Science, I. Kh. International University, P.O. Box: 34149-16818, Qazvin, Iran.
E-mail address: razani@ikiu.ac.ir

2 Department of Mathematics, Faculty of Science, I. Kh. International University, P.O. Box: 34149-16818, Qazvin, Iran.
E-mail address: msh_yazdi@ikiu.ac.ir