A NEW RESTRUCTURED HARDY-LITTLEWOOD’S INEQUALITY

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ABSTRACT. In this paper, we reconstruct the Hardy-Littlewood’s inequality by using the method of the weight coefficient and the technic of real analysis including a best constant factor. An open problem is raised.

1. Introduction

In 1908, D. Hilbert published the following Hilbert’s inequality (cf. [1]): If \(0 < \sum_{n=1}^\infty a_n^2 < \infty\) and \(0 < \sum_{n=1}^\infty b_n^2 < \infty\), then
\[
\sum_{n=1}^\infty \sum_{m=1}^\infty a_m b_n \frac{1}{m+n} < \pi \left( \sum_{n=1}^\infty a_n^2 \right)^{1/2} \left( \sum_{n=1}^\infty b_n^2 \right)^{1/2},
\]
where the constant factor \(\pi\) is the best possible. The integral analogue of (1.1) known as Hilbert’s integral inequality is stated as follows:
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \left( \int_0^\infty f^2(t) \, dt \right)^{1/2} \left( \int_0^\infty g^2(t) \, dt \right)^{1/2},
\]
where the constant factor \(\pi\) is still the best possible.

In 1925, G. H. Hardy and M. Riesz [2] gave extensions of (1.1) and (1.2) by introducing one pair of conjugate exponents \((p, q)\) as:
\[
\sum_{n=1}^\infty \sum_{m=1}^\infty a_m b_n \frac{1}{m+n} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{n=1}^\infty a_n^p \right)^{1/p} \left( \sum_{n=1}^\infty b_n^q \right)^{1/q},
\]
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(t) \, dt \right)^{1/p} \left( \int_0^\infty g^q(t) \, dt \right)^{1/q},
\]
where the constant factor \(\frac{\pi}{\sin(\pi/p)}\) is the best possible. Inequalities (1.3) and (1.4) are respectively called Hardy-Hilbert’s inequality and Hardy-Hilbert’s integral inequality. Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [3], [4]).

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In 1998, by introducing an independent parameter $\lambda > 0$ and applying the way of weight functions, Yang gave an extension of (1.2) as (cf. [5], [6]):
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda}} dx dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\int_0^\infty t^{1-\lambda} f^2(t) dt \right)^{\frac{1}{2}} \left(\int_0^\infty t^{1-\lambda} g^2(t) dt\right)^{\frac{1}{2}},
\]
(1.5)
where the constant $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible, and $B(u, v)$ is the Beta function.

Since then several mathematicians studied this thesis, such as Jichang Kuang, Mingzhe Gao, W. T. Sulaiman and S. R. Salem et al.. In 2003, Yang and Rassias [7] studied the way of weight coefficient and the method of introducing some independent parameters to obtain a number of new improvements and best extensions of (1.1)-(1.5). In 2004, Yang [8] gave an extension of (1.4) by introducing a parameter $\lambda > 0$ and adding another pair of conjugate exponents $(r, s)(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ as:
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^r + y^s} dx dy < \frac{1}{\lambda} B\left(\frac{1}{r}, \frac{1}{s}\right) \times \left(\int_0^\infty t^{\lambda(1-\frac{1}{r})-1} f^p(t) dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^{\lambda(1-\frac{1}{s})-1} g^q(t) dt\right)^{\frac{1}{q}},
\]
(1.6)
where the constant factor $\frac{1}{\lambda} B\left(\frac{1}{r}, \frac{1}{s}\right)$ is the best possible, and for $\lambda = 1, r = q$, inequality (1.6) reduces to (1.4). For those Hilbert-type inequalities, which possess the general form of kernel or the particular homogeneous kernel of $-\lambda$-degree ($\lambda > 0$), Yang et al. [9], [10], [11], [12] used the Operator theory to study them and published many new interesting results.

The equivalent form of (1.3) with the best constant $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ is as follows:
\[
\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=1}^\infty a_n^p.
\]
(1.7)
Modifying the kernel of (1.7), Hardy’s inequality was given as (cf. [13]):
\[
\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{m=1}^n a_m\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p,
\]
(1.8)
where the constant factor $\left(\frac{p}{p-1}\right)^p$ is the best possible. The integral analogue of (1.8) is as follows (cf. [13]):
\[
\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x) dx.
\]
(1.9)
In the period 1927-1928, Hardy [14] provided an extension of (1.9) in the following form (cf. [2], Th. 330): If $p > 1, r \neq 1, 0 < \int_0^\infty x^{-r}(xf(x))^p dx < \infty$, setting $F(x)$ as: $F(x) = \int_0^x f(t)dt(r > 1), F(x) = \int_x^\infty f(t)dt(r < 1)$, then
\[
\int_0^\infty x^{-r} F^p(x) dx < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf(x))^p dx,
\]
(1.10)
where the constant $\left(\frac{p}{|r-1|}\right)^p$ is the best possible. Similarly to the type of (1.10), Hardy and Littlewood [15] proved the following inequality (cf. [2], Th. 346): Assuming
that $p > 1, r \neq 1, a_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{-r}(na_n)^p < \infty$, if (a) $r > 1, s_n = \sum_{k=1}^{n} a_k$, or
(b) $r < 1, s_n = \sum_{k=n}^{\infty} a_k$, then
\begin{equation}
\sum_{n=1}^{\infty} n^{-r} s_n^p \leq K^p \sum_{n=1}^{\infty} n^{-r} (na_n)^p,
\end{equation}
where the constant factor $K$ satisfies the following inequalities
\begin{equation}
\phi_n = \sum_{k=n}^{\infty} \frac{1}{k^r} \leq K n^{1-r} (r > 1); \tilde{\phi}_n = \sum_{k=1}^{n} \frac{1}{k^r} \leq K n^{1-r} (r < 1).
\end{equation}

Hardy et al. [2] did not obtained the expression of $K^p$ and proved that the constant factor is the best possible. But Hardy and Littlewood [16] pointed out some applications of (1.11) in the theory of functions, especially for $r = 2$.

The proof of (a) in (1.11) was described in Hardy et. al. [2] as follows:
For $r > 1, s_n = \sum_{k=1}^{n} a_k(s_0 = 0)$, by Abel’s transform and (1.12), one finds
\begin{equation}
\sum_{n=1}^{m} n^{-r} s_n^p = \sum_{n=1}^{m} (\phi_n - \phi_{n+1}) s_n^p
= \sum_{n=1}^{m} \phi_n (s_n^p - s_{n-1}^p) - \phi_{m+1} s_m^p \leq \sum_{n=1}^{m} \phi_n (s_n^p - s_{n-1}^p)
\leq K \sum_{n=1}^{m} n^{1-r} s_n^{p-1} a_n = K \sum_{n=1}^{m} n^{-r} (na_n)(s_n^{p-1}).
\end{equation}

Hence, by Hölder’s inequality with weight, it follows
\begin{equation}
\sum_{n=1}^{m} n^{-r} s_n^p \leq K \left\{ \sum_{n=1}^{m} n^{-r} (na_n)^p \right\}^{\frac{1}{r}} \left\{ \sum_{n=1}^{m} n^{-r} s_n^p \right\}^{\frac{1}{q}}.
\end{equation}

For more large enough $m \in \mathbb{N}$, we have $\sum_{n=1}^{m} n^{-r} s_n^p > 0$. Dividing by $\left\{ \sum_{n=1}^{m} n^{-r} s_n^p \right\}^{\frac{1}{r}}$ in both sides of (14), we obtain
\begin{equation}
\left\{ \sum_{n=1}^{m} n^{-r} s_n^p \right\}^{\frac{1}{r}} \leq K \left\{ \sum_{n=1}^{m} n^{-r} (na_n)^p \right\}^{\frac{1}{p}}.
\end{equation}

It follows that (a) in (1.11) is valid.

**Remark 1.1.** We find that the following inequality
\begin{equation}
s_n^p - s_{n-1}^p \leq s_{n-1}^{p-1} a_n
\end{equation}
is wrong. Hence we can’t reach the last inequality of (1.13). In fact, we can find that
\begin{equation}
\begin{aligned}
s_n^p - s_{n-1}^p &= s_n^p - s_{n-1}^p = s_n^{p-1} (s_n - a_n) - s_{n-1}^p \\
&= (s_n^{p-1} - s_{n-1}^{p-1}) s_{n-1} + s_{n-1}^{p-1} a_n \\
&= [(s_n^{p-1} - s_{n-1}^{p-1}) s_{n-1} + s_{n-1}^{p-1} a_n] s_{n-1}.
\end{aligned}
\end{equation}

Since $p > 1, s_n = \sum_{k=1}^{n} a_k$, in view of $\sum_{n=1}^{\infty} n^{-r} (na_n)^p > 0$, there exists $n \in \mathbb{N}$, such that $s_{n-1} > 0, a_n > 0$ and
\begin{equation}
[(s_n^{p-1} - s_{n-1}^{p-1}) s_{n-1} + s_{n-1}^{p-1} a_n] s_{n-1} = 0.
\end{equation}
Hence by (1.16), it follows
\[ s_n^p - s_{n-1}^p > s_{n-1}^{p-1}a_n, \] (1.17)
which contradicts (1.15). Therefore, inequality (1.15) is not valid by using the this way, and we can not prove (a) in (1.11).

If (b) \( r < 1, s_n = \sum_{k=n}^{\infty} a_k \), setting \( \phi_0 = 0 \), then following the front-way, we can meet the similar result of (1.17). In fact,
\[
\sum_{n=1}^{m} n^{-r} s_n^p = \sum_{n=1}^{m} (\tilde{\phi}_n - \tilde{\phi}_{n-1}) s_n^p = \sum_{n=1}^{m} \tilde{\phi}_n (s_n^p - s_{n+1}^p) - \tilde{\phi}_m s_{m+1}^p
\]
\[
= \sum_{n=1}^{m} \tilde{\phi}_n (s_n^{p-1} s_n - s_{n+1}^p) - \tilde{\phi}_m s_{m+1}^p
\]
\[
= \sum_{n=1}^{m} \tilde{\phi}_n [s_n^{p-1} (s_{n+1}^p + a_n) - s_{n+1}^p] - \tilde{\phi}_m s_{m+1}^p
\]
\[
= \sum_{n=1}^{m} \tilde{\phi}_n s_n^{p-1} a_n + \sum_{n=1}^{m} \tilde{\phi}_n s_{n+1}^p (s_n^{p-1} - s_{n+1}^{p-1}) - \tilde{\phi}_m s_{m+1}^p. \] (1.18)

Since \( s_n^{p-1} - s_{n+1}^{p-1} \geq 0 \), we can't prove the following inequality:
\[
\sum_{n=1}^{m} \tilde{\phi}_n s_{n+1}^p (s_n^{p-1} - s_{n+1}^{p-1}) - \tilde{\phi}_m s_{m+1}^p \leq 0,
\]
and then the inequality \( \sum_{n=1}^{m} n^{-r} s_n^p \leq \sum_{n=1}^{m} \tilde{\phi}_n s_n^{p-1} a_n \) is not valid by (1.18). So we cannot do more work for (b) in (1.11) following this way.

In this paper, by using (1.10), we reformulate (1.11) to obtain a new inequality with a best constant factor, by using the way of weight coefficient and the technic of real analysis. That is the following theorem:

**Theorem 1.2.** Assuming that \( r \neq 1, p > 1, a_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{-r} (na_n)^p < \infty \), if (a) \( r > 1, s_n = \sum_{m=1}^{n} a_m \), or (b) \( r < 1, s_n = \sum_{m=n}^{\infty} a_m \), then
\[
\sum_{n=1}^{\infty} \frac{n^{-r} s_n^p}{(1 + \frac{r-1}{pn})^{p-1}} < k_p^p \sum_{n=1}^{\infty} (1 + \frac{|r-1|}{pn}) n^{-r} (na_n)^p, \] (1.19)
where the constant factor \( k_p^p = \left( \frac{p}{|r-1|} \right)^p \) is the best possible and \( k_r := \frac{p}{|r-1|} \).

**Remark 1.3.** Inequality (1.19) is a new restructured Hardy-Littlewood's inequality with a best constant factor. For \( r = p, q = \frac{p}{p-1} \), we have
\[
\sum_{n=1}^{\infty} \left( \frac{qn}{1+qn} \right)^{p-1} \left( \frac{1}{n} \sum_{m=1}^{n} a_m \right)^p < q^p \sum_{n=1}^{\infty} \frac{1 + qn}{qn} a_n^p, \] (1.20)
which is weaker than (1.8) but with the same best constant factor as (1.8).
2. A Lemma and a Preliminary Theorem

Lemma 2.1. If $\alpha > 0, m, n \in \mathbb{N},$ then
\[
\sum_{n=m}^{\infty} \frac{1}{n^{1+\alpha}} < \frac{1}{\alpha m^\alpha} (1 + \frac{\alpha}{m});
\] (2.1)
\[
\frac{1}{\alpha} n^\alpha (1 - \frac{1}{n^\alpha}) < \sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} < \frac{1}{\alpha n^\alpha} (1 + \frac{\alpha}{n}).
\] (2.2)

Proof. For $\alpha > 0$, we obtain
\[
\sum_{n=m}^{\infty} \frac{1}{n^{1+\alpha}} = \frac{1}{m^{1+\alpha}} + \sum_{n=m+1}^{\infty} \frac{1}{n^{1+\alpha}} < \frac{1}{m^{1+\alpha}} + \int_{m}^{\infty} \frac{1}{x^{1+\alpha}} dx = \frac{1}{\alpha} (1 + \frac{\alpha}{m}) \frac{1}{m^\alpha}.
\]
Then inequality (2.1) is valid.

For $0 < \alpha \leq 1$, it follows
\[
\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} < \int_{0}^{n} \frac{1}{x^{1-\alpha}} dx = \frac{1}{\alpha} n^\alpha < \frac{1}{\alpha} n^\alpha (1 + \frac{\alpha}{n}),
\]
\[
\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} > \int_{1}^{n} \frac{1}{x^{1-\alpha}} dx = \frac{1}{\alpha} n^\alpha (1 - \frac{1}{n^\alpha});
\]
for $\alpha > 1$, we obtain
\[
\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} = \frac{1}{n^{1-\alpha}} + \sum_{m=1}^{n-1} m^{\alpha-1}
\]
\[
< \frac{1}{n^{1-\alpha}} + \int_{0}^{n} x^{\alpha-1} dx = \frac{1}{\alpha} n^\alpha (1 + \frac{\alpha}{n}),
\]
\[
\sum_{m=1}^{n} \frac{1}{m^{1-\alpha}} = \sum_{m=1}^{n} m^{\alpha-1} > \int_{1}^{n} x^{\alpha-1} dx = \frac{1}{\alpha} n^\alpha (1 - \frac{1}{n^\alpha}).
\]
Hence (2.2) is valid. The lemma is proved.

Theorem 2.2. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, a_n, b_n \geq 0, n \in \mathbb{N},$ such that
\[
0 < \sum_{n=1}^{\infty} n^{-r} (na_n)^p < \infty, 0 < \sum_{n=1}^{\infty} n^{-r} (n^r b_n)^q < \infty,
\] (2.3)
then the following inequality holds:
\[
I : = \sum_{n=1}^{\infty} \sum_{m=1}^{n} a_m b_n = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} a_m b_n < \frac{p}{r - 1}
\]
\[
\times \{ \sum_{n=1}^{\infty} (1 + \frac{r - 1}{pm}) n^{-r} (na_n)^p \}^{\frac{1}{p}} \{ \sum_{n=1}^{\infty} (1 + \frac{r - 1}{pm}) n^{-r} (n^r b_n)^q \}^{\frac{1}{q}},
\] (2.4)
where the constant factor $\frac{p}{r - 1}$ is the best possible.
Since \(1 < 1 + \frac{r - 1}{pm} \leq 1 + \frac{r - 1}{p}\), it is obvious that inequalities (2.3) are equivalent to the following:

\[
0 < \sum_{n=1}^{\infty} (1 + \frac{r - 1}{pn})n^{-r}(n\alpha_n)^p < \infty, \quad 0 < \sum_{n=1}^{\infty} (1 + \frac{r - 1}{pn})n^{-r}(n\beta_n)^q < \infty.
\]

By Hölder’s inequality (cf. [17]), we obtain

\[
I = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{m(1 - \frac{r - 1}{p})/q}{1 + \frac{r - 1}{p}} a_m \frac{m(1 - \frac{r - 1}{p})/q}{1 + \frac{r - 1}{p}} b_n
\]

\[
\leq \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{m(1 - \frac{r - 1}{p})(p-1)}{1 + \frac{r - 1}{p}} a_m \frac{m(1 - \frac{r - 1}{p})(q-1)}{1 + \frac{r - 1}{p}} b_n
\]

\[
= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{1}{1 + \frac{r - 1}{p}} m(1 - \frac{r - 1}{p})(p-1) a_m \frac{1}{1 + \frac{r - 1}{p}} m(1 - \frac{r - 1}{p})(q-1) b_n
\]

\[
= \frac{p}{r - 1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{1 + \frac{r - 1}{p}} m^{p-r} a_m \frac{1}{1 + \frac{r - 1}{p}} m^{q-r} b_n
\]

Then by (2.1) and (2.2), setting \(\alpha = \frac{r - 1}{p}(>0)\), we have

\[
I < \frac{p}{r - 1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1 + \frac{r - 1}{pn}}{m^{p-r} a_m} \frac{1 + \frac{r - 1}{pn}}{m^{q-r} b_n}
\]

\[
\leq \frac{p}{r - 1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1 + \frac{r - 1}{pn}}{m^{p-r} a_m} \frac{1 + \frac{r - 1}{pn}}{m^{q-r} b_n}
\]

\[
= \frac{p}{r - 1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1 + \frac{r - 1}{pn}}{m^{p-r} a_m} \frac{1 + \frac{r - 1}{pn}}{m^{q-r} b_n}
\]

Hence inequality (2.4) is valid.

For \(N \in \mathbb{N}\), setting \(\tilde{a}_n = n^{-\frac{1}{p}}\alpha_n\), \(\tilde{b}_n = n^{-\frac{1}{q}}\beta_n\), \(n \leq N\); \(\tilde{a}_n = \tilde{b}_n = 0\), \(n > N\), if there exists a positive number \(k \leq \frac{p}{r - 1}\), such that (2.4) is still valid as we replace \(\frac{p}{r - 1}\) by \(k\), then in particular, we have

\[
\tilde{I} : = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tilde{a}_m}{1 + \frac{r - 1}{pn}} \tilde{b}_n < k \left\{ \sum_{n=1}^{\infty} \left(1 + \frac{r - 1}{pn}\right)n^{-r}(n\tilde{a}_n)^p \right\}^{\frac{1}{q}}
\]

\[
\times \left\{ \sum_{n=1}^{\infty} \left(1 + \frac{r - 1}{pn}\right)n^{-r}(n\tilde{b}_n)^q \right\}^{\frac{1}{q}}
\]

\[
= k \sum_{n=1}^{N} \left(1 + \frac{r - 1}{pn}\right) \frac{1}{n} = k \left( \sum_{n=1}^{N} \frac{1}{n} + \frac{r - 1}{p} \sum_{n=1}^{N} \frac{1}{n^2} \right)
\]

\[
= k \left( \sum_{n=1}^{N} \frac{1}{n} \right) \left[ 1 + \frac{r - 1}{p} \left( \sum_{n=1}^{N} \frac{1}{n} \right)^{-1} \right] \left[ \sum_{n=1}^{N} \frac{1}{n^2} \right] \quad \text{(2.5)}
\]
On the other-hand, by (2.2), we obtain

\[ \tilde{I} = \sum_{n=1}^{N} \left( \sum_{m=1}^{n} \frac{r-1}{m^r} \right) n^{-\frac{1}{q}} \geq \frac{p}{r-1} \sum_{n=1}^{N} n^{-\frac{1}{r}} \left( 1 - \frac{1}{n^{\frac{1}{r}}} \right) n^{-\frac{1}{q}} \]

\[ = \frac{p}{r-1} \left( \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n^{\frac{1}{r}+1}} \right) \]

\[ = \frac{p}{r-1} \left( \sum_{n=1}^{N} \frac{1}{n} \right) \left[ 1 - \left( \sum_{n=1}^{N} \frac{1}{n} \right)^{-1} \sum_{n=1}^{N} \frac{1}{n^{\frac{1}{r}+1}} \right]. \quad (2.6) \]

Combining with (2.5) and (2.6) and dividing by \( \sum_{n=1}^{N} \frac{1}{n} \), we have

\[ \frac{p}{r-1} \left[ 1 - \left( \sum_{n=1}^{N} \frac{1}{n} \right)^{-1} \sum_{n=1}^{N} \frac{1}{n^{\frac{1}{r}+1}} \right] < k \left[ 1 + \frac{r-1}{p} \left( \sum_{n=1}^{N} \frac{1}{n} \right)^{-1} \sum_{n=1}^{N} \frac{1}{n^2} \right], \]

and then \( \frac{p}{r-1} < k \) (for \( N \to \infty \)). Hence \( k = \frac{p}{r-1} \) is the best value of (2.4) and the theorem is proved.

3. Main results

**Theorem 3.1.** If \( p > 1, \frac{1}{q} = 1, r > 1, a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{-r}(na_n)^p < \infty \) and \( 0 < \sum_{n=1}^{\infty} n^{-r}(n^rb_n)^q < \infty \), then

\[ J : = \sum_{n=1}^{\infty} \frac{n^{-r}}{(1 + \frac{r-1}{pn})^{p-1}} \left( \sum_{m=1}^{n} a_m \right)^p \]

\[ < \left( \frac{p}{r-1} \right)^p \sum_{n=1}^{\infty} (1 + \frac{r-1}{pn}) n^{-r}(na_n)^p; \quad (3.1) \]

\[ L : = \sum_{m=1}^{\infty} \frac{m^{r(q-1)-q}}{(1 + \frac{r-1}{pm})^{q-1}} \left( \sum_{n=m}^{\infty} b_n \right)^q \]

\[ < \left( \frac{p}{r-1} \right)^q \sum_{n=1}^{\infty} (1 + \frac{r-1}{pn}) n^{-r}(n^rb_n)^q, \quad (3.2) \]

where the constant factors \( \left( \frac{p}{r-1} \right)^p \) and \( \left( \frac{p}{r-1} \right)^q \) are the best possible. Inequalities (3.1), (3.2) and (2.4) are equivalent.

**Proof.** If \( J = 0 \), then (3.1) is naturally valid; if \( J > 0 \), then there exists \( n_0 \in \mathbb{N} \), such that for \( N \geq n_0, \sum_{n=1}^{N} n^{-r}(na_n)^p > 0 \) and \( J_N := \sum_{n=1}^{N} \frac{n^{-r}}{(1 + \frac{r-1}{pn})^{p-1}} \left( \sum_{m=1}^{n} a_m \right)^p > 0. \)
We set $b_n(N) := \frac{n^{-r}}{(1 + \frac{r-1}{pm})} \sum_{m=1}^{n} a_m)^{p-1}(n \leq N)$, and use (2.4) to obtain

$$0 < \sum_{n=1}^{N} (1 + \frac{r-1}{pm}) n^{-r} (n^rb_n(N))^q = J_N$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{n} a_m b_n(N) < \frac{p}{r-1} \{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(na_n)^p \}^{\frac{1}{p}}$$

$$\times \{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(n^rb_n(N))^q \}^{\frac{1}{q}}. \tag{3.3}$$

Dividing $\{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(n^rb_n(N))^q \}^{\frac{1}{q}}$ in both sides of (3.3), it follows

$$0 < \{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(n^rb_n(N))^q \}^{\frac{1}{q}} = J_N^{\frac{1}{p}}$$

$$< \frac{p}{r-1} \{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(na_n)^p \}^{\frac{1}{p}}$$

$$< \frac{p}{r-1} \{ \sum_{n=1}^{\infty} (1 + \frac{r-1}{pn}) n^{-r}(na_n)^p \}^{\frac{1}{p}} < \infty. \tag{3.4}$$

We conform that $0 < \sum_{n=1}^{\infty} n^{-r}(n^rb_n(\infty))^q < \infty$ and for $N \to \infty$, both (3.3) and (3.4) still preserve the strict sign-inequalities. Hence (3.1) follows.

By the same way, if $L = 0$, then (3.2) is naturally valid; if $L > 0$, then there exists $n_0$, such that for $N \geq n_0$, $\sum_{n=1}^{N} n^{-r}(n^rb_n)^q > 0$ and $L_N := \sum_{m=1}^{N} \frac{m^{q(r-1)-q}}{(1 + \frac{r-1}{pm})^{r}} \sum_{n=1}^{N} b_n^q > 0$. We set $a_m(N) := \frac{m^{q(r-1)-q}}{(1 + \frac{r-1}{pm})^{r}} \sum_{n=1}^{N} b_n^{-q}$. Then use (2.4) to obtain

$$0 < \sum_{m=1}^{N} (1 + \frac{r-1}{pm}) m^{-r}(ma_m(N))^p = L_N = \sum_{m=1}^{N} \sum_{n=m}^{\infty} a_m(N) b_n$$

$$< \frac{p}{r-1} \{ \sum_{m=1}^{N} (1 + \frac{r-1}{pm}) m^{-r}(ma_m(N))^p \}^{\frac{1}{p}}$$

$$\times \{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(n^rb_n)^q \}^{\frac{1}{q}}. \tag{3.5}$$

$$0 < \sum_{m=1}^{N} (1 + \frac{r-1}{pm}) m^{-r}(ma_m(N))^p$$

$$< \frac{p}{r-1} \{ \sum_{n=1}^{N} (1 + \frac{r-1}{pn}) n^{-r}(n^rb_n)^q \}^{\frac{1}{q}} < \infty. \tag{3.6}$$

We conform that $0 < \sum_{n=1}^{\infty} m^{-r}(ma_m(\infty))^p < \infty$, and for $N \to \infty$, both (3.5) and (3.6) still preserve the strict sign-inequalities. Hence we have (3.2).
By Hölder's inequality (cf. [17]), we have
\[ I = \sum_{n=1}^{\infty} \left( \left( \frac{n^{r-1}}{(1 + \frac{1 - r}{pn})^{\frac{1}{r}}} \right)^{\frac{1}{r}} \sum_{m=1}^{n} a_m \right) \left[ (1 + \frac{r - 1}{pn})^{\frac{1}{r}} n^{\frac{k}{r}} b_n \right]^{\frac{1}{r}} \leq J^{\frac{1}{r}} \left\{ \sum_{n=1}^{\infty} \left( \frac{n^{r-1}}{(1 + \frac{1 - r}{pn})^{\frac{1}{r}}} \right)^{\frac{1}{r}} (n^r b_n)^q \right\}^{\frac{1}{q}}; \quad (3.7) \]
\[ I = \sum_{m=1}^{\infty} \left[ (1 + \frac{r - 1}{pn})^{\frac{1}{r}} m^{\frac{1 - r}{r}} a_m \right] \sum_{n=m}^{\infty} b_n \]
\[ \leq \left\{ \sum_{n=1}^{\infty} \left( \frac{n^{r-1}}{(1 + \frac{1 - r}{pn})^{\frac{1}{r}}} \right)^{\frac{1}{r}} \right\}^{\frac{1}{r}} \left\{ \sum_{m=1}^{\infty} \left[ (1 + \frac{r - 1}{pn})^{\frac{1}{r}} m^{\frac{1 - r}{r}} a_m \right] \sum_{n=m}^{\infty} b_n \right\}^{\frac{1}{q}} \leq \left\{ \sum_{n=1}^{\infty} \left( \frac{n^{r-1}}{(1 + \frac{1 - r}{pn})^{\frac{1}{r}}} \right)^{\frac{1}{r}} \right\}^{\frac{1}{r}} \left\{ \sum_{m=1}^{\infty} \left[ (1 + \frac{r - 1}{pn})^{\frac{1}{r}} m^{\frac{1 - r}{r}} a_m \right] \sum_{n=m}^{\infty} b_n \right\}^{\frac{1}{q}}. \quad (3.8) \]

On the other hand, assuming that (3.1) (or (3.2)) is valid, by (3.7) (or (3.8)), we obtain (2.4). Hence (3.1), (3.2) and (2.4) are equivalent. We confirm that both constants \((\frac{r}{r-1})^p\) in (3.1) and \((\frac{r}{r-1})^q\) in (3.2) are the best possible, otherwise, we can obtain a contradiction by (3.7) or (3.8) that the constant factor in (2.4) is not the best possible. The theorem is proved. \(\square\)

**Proof of Theorem 1.** Exchange with \(m\) and \(n, a_m\) and \(b_n, p\) and \(q\) in (3.2), and putting \(R = r(> 1)\), we have
\[ \sum_{n=1}^{\infty} \frac{n^{R(p-1)-p}}{(1 + \frac{1}{R-1})^{p-1}} (\sum_{m=n}^{\infty} a_m)^p < \left( \frac{q}{R-1} \right)^p \sum_{m=1}^{\infty} (1 + \frac{R-1}{qm}) m^{-R} (m^R a_m)^p. \quad (3.9) \]
Setting \(r = p - R(p - 1)\) in (3.9), we obtain \(R(p - 1) = p - r, r < 1\) and
\[ \sum_{n=1}^{\infty} \frac{n^{-r}}{(1 + \frac{1-r}{pn})^{p-1}} (\sum_{m=n}^{\infty} a_m)^p < \left( \frac{p}{1-r} \right)^p \sum_{m=1}^{\infty} (1 + \frac{1-r}{pn}) m^{-r} (ma_m)^p. \quad (3.10) \]
Combining with (3.1) and (3.10), we have (1.19), and the constant factor is obviously the best possible. This proves the theorem.

**Open problem.** Since \(1 + \frac{|r-1|}{pm} \leq 1 + \frac{|r-1|}{p}\), if we set \(K_r = 1 + \frac{p}{|r-1|}\), then inequality (1.19) can be deduced to
\[ \sum_{n=1}^{\infty} n^{-r} s_n^p \leq K_r^p \sum_{n=1}^{\infty} n^{-r} (na_n)^p, \quad (3.11) \]
which is the same as (1.11), but obviously the constant factor \(K_r^p\) is not the best possible in (3.11) unless \(K_r = k_r = \frac{p}{|r-1|}\). If we replace \(K_r\) by \(\tilde{k}_r\), that makes (3.11) still valid, then by simple proof, we find \(\frac{p}{|r-1|} \leq \tilde{k}_r \leq 1 + \frac{p}{|r-1|}\) and in view of (1.8), it follows for \(r = p\), \(\inf \tilde{k}_p = k_p\). We conjecture that
\[ \inf \tilde{k}_r = k_r = \frac{p}{|r-1|}. \quad (3.12) \]
We leave behind it as an open problem.
References


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