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# Solution and stability of Tribonacci functional equation in non-Archimedean Banach spaces

M. Eshaghi Gordji[a,*], M. Naderi Parizi[b,*], Th. M. Rassias[c]

[a]*Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran.*
[b]*Payame Noor University, Rafsanjan, Iran.*
[c]*Department of Mathematics, National Technical University of Athens, Greece.*

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**Abstract**

In this paper, we prove Hyers–Ulam stability of Tribonacci functional equation

$$f(x) = f(x-1) + f(x-2) + f(x-3)$$

in the class of functions $f : \mathbb{R} \longrightarrow X$ where $X$ is a real non-archimedean Banach space.

*Keywords:* Hyers–Ulam Stability, Real Non-Archimedean Banach Space, Tribonacci Functional Equation.
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## 1. Introduction

The stability of functional equations originated from a question of Ulam [17] in 1940. In the next year, Hyers [9] provided the solution of Ulam's problem for the special case of the Cauchy functional equation.

Stability problems related to Ulam's problem for functional equations have been extensively investigated worldwide by several mathematicians (cf. [5]–[16]).

*Corresponding author

*Email addresses:* madjid.eshaghi@gmail.com & meshaghi@semnan.ac.ir (M. Eshaghi Gordji ), mrym_ndr@yahoo.com (M. Naderi Parizi ), trassias@math.ntua.gr (Th. M. Rassias)

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Recently, S.-M.Jung investigated the Hyers–Ulam stability of Fibonacci functional equation

$$f(x) = f(x-1) + f(x-2).$$

More recently, M. Bidkham and M. Hosseini [1] and M. Bidkham, M. Hosseini, C. Park and M. Eshaghi Gordji [2] succeeded to provide a proof of the Hyers-Ulam stability of $k$-Fibonacci and $k, s-$Fibonacci functional equations.

Throughout this paper, we denote by $F'_n$ the $n$th Tribonacci number, for $n \in \mathbb{N}$. In particular, we define $F'_0 = 0$, $F'_1 = 0$ and $F'_2 = 1$, and putting $F'_n = F'_{n-1} + F'_{n+2} + F'_{n-3}$ for $n \geq 3$. From this famous formula, we may derive a functional equation

$$f(x) = f(x-1) + f(x-2) + f(x-3) \tag{1.1}$$

which may be called the Tribonacci functional equation. Let $X$ be a vector space. A function $f : \mathbb{R} \longrightarrow X$ will be called a Tribonacci function if it satisfies (1.1), for all $x \in \mathbb{R}$.

By $\alpha, \beta$ and $\gamma$ we denote the roots of the equation $x^3 - x^2 - x - 1 = 0$. $\alpha$ is greater than one and $\beta, \gamma \in \mathbb{C}$ and $|\beta| = |\gamma|$.

We have $\alpha + \beta + \gamma = 1$, $\beta\gamma + \alpha\beta + \alpha\gamma = -1$ and $\alpha\beta\gamma = 1$. For each $x \in \mathbb{R}$, $[x]$ stands for the largest integer that does not exceed $x$. Here, we will solve the Tribonacci functional equation (1.1) and we prove the Hyers–Ulam stability of functional equation (1.1).

## 2. Preliminaries

Since $\alpha + \beta + \gamma = 1$, $\beta\gamma + \alpha\beta + \alpha\gamma = -1$ and $\alpha\beta\gamma = 1$, it follows from (1.1) that

$$f(x) - \alpha[f(x-1) - \gamma f(x-2)] - \gamma f(x-1) =$$

$$\beta[f(x-1) - (\gamma + \alpha) f(x-2) + \alpha\gamma f(x-3)] \tag{2.1}$$

for all $x \geq 0$. By mathematical induction, we verify that for all $x \geq 0$ and all $n$ belonging to the set $\{0, 1, 2, ...........\}$, we have

$$\begin{aligned} f(x) - \alpha[f(x-1) - \gamma f(x-2)] - \gamma f(x-1) = \beta^n[&f(x-n) - \gamma f(x-n-1) \\ &+ \alpha\gamma f(x-n-2) - \alpha f(x-n-1)] \end{aligned}$$

$$f(x) - \gamma[f(x-1) - \beta f(x-2)] - \beta f(x-1) = \alpha^n[f(x-n) - \gamma f(x-n-1)$$

$$+\beta\gamma f(x-n-2)-\beta f(x-n-1)]$$

$$f(x)-\beta[f(x-1)-\alpha f(x-2)]-\alpha f(x-1)=\gamma^n[f(x-n)-\alpha f(x-n-1)$$

$$+\beta\alpha f(x-n-2)-\beta f(x-n-1)] \quad (2.2)$$

for all $x \geq 0$ and all $n \in \{0,1,2,\ldots\}$.

**Definition 2.1.** *Let $K$ be a field.A non-Archimedean absolute value on $K$ is a function $|.|: K \to [0,+\infty]$ such that for any $a,b \in K$,*

*(i)$|a| \geq 0$ and equality holds if and only if $|a|=0$;*

*(ii) $|ab|=|a||b|$;*

*(iii) $|a+b| \leq \max\{|a|,|b|\}$*

## 3. Stability

As already stated, $\alpha$ denotes the positive root of the equation $x^3-x^2-x-1=0$ and $\beta,\gamma$ are its complex conjugate roots. In the following we provide a proof of the Hyers-Ulam stability of the Tribonacci functional equation (1.1).

**Theorem 3.1.** *Let $(X,\|\cdot\|)$ be a non-Archimedean Banach space. If a function $f:\mathbb{R} \longrightarrow X$ satisfies the inequality*

$$\|f(x)-f(x-1)-f(x-2)-f(x-3)\| \leq \epsilon \quad (3.1)$$

*for all $x \in \mathbb{R}$ and for some $\epsilon > 0$, then there exists a Tribonacci function $G:\mathbb{R} \longrightarrow X$ such that*

$$\|f(x)-G(x)\| \leq \frac{2(1+|\beta|)+|\beta|^2}{\|\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\gamma)} \times \frac{\epsilon}{1-|\beta|^2} \quad (3.2)$$

*for all $x \in \mathbb{R}$.*

**Proof .** Analogous to the first equation of (2.3), it follows from (3.1) that

$$\|f(x)-(\alpha+\beta+\gamma)f(x-1)+(\alpha\beta+\beta\gamma+\alpha\gamma)f(x-2)-\alpha\beta\gamma f(x-3)\| \leq \epsilon$$

for all $x \in \mathbb{R}$. If we replace $x$ by $x-k$ in the last inequality, then we have

$$\|f(x-k)-\alpha(f(x-k-1)-\gamma f(x-k-2))-\gamma f(x-k-1)$$

$$-\beta[f(x-k-1)-(\gamma+\alpha)f(x-k-2)+\alpha\gamma f(x-k-3)]\| \le \epsilon$$

for all $x \in \mathbb{R}$, and furthermore,

$$\|\beta^k(f(x-k)-\alpha(f(x-k-1)-\gamma f(x-k-2))-\gamma f(x-k-1))$$

$$-\beta^{k+1}(f(x-k-1)-(\gamma+\alpha)f(x-k-2)+\alpha\gamma f(x-k-3))\| \le |\beta^k|\epsilon \tag{3.3}$$

for all $x \in \mathbb{R}$. We obviously have

$$\|f(x)-\alpha(f(x-1)-\gamma f(x-2))-\gamma f(x-1)-\beta^n[f(x-n)$$

$$-(\gamma+\alpha)f(x-n-1)+\alpha\gamma f(x-n-2)]\|$$

$$\le \max_{0\le k\le n-1}\{\|\beta^k(f(x-k)-\alpha f(x-k-1))-\beta^{k+1}(f(x-k-1)-\alpha f(x-k-2))\|\}$$

$$\le \max_{0\le k\le n-1}\{|\beta|^k\epsilon\} = \epsilon, \tag{3.4}$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Let $x \in \mathbb{R}$ be fixed. Then (3.3) implies that

$$\{\beta^n(f(x-n)-\alpha(f(x-n-1)-\gamma f(x-n-2))-\gamma f(x-n-1)\}$$

is a Cauchy sequence (note that $|\beta| < 1$). Therefore, by completeness of $X$, we can define a function $G_1 : \mathbb{R} \longrightarrow X$ by

$$G_1(x) := \lim_{n\to\infty} \beta^n[f(x-n)-\alpha(f(x-n-1)-\gamma f(x-n-2))-\gamma f(x-n-1)]$$

for all $x \in \mathbb{R}$. In view of the above definition of $G_1$, we obtain that

$$G_1(x-1)+G_1(x-2)+G_1(x-3)$$

$$=\beta^{-1}\lim_{n\to\infty}\beta^{n+1}[f(x-(n+1)-\alpha(f(x-(n+1)-1)-\gamma f(x-(n+1)-2))-\gamma f(x-(n+1)-1))]$$

$$+\beta^{-2}\lim_{n\to\infty}\beta^{n+2}[f(x-(n+2)-\alpha(f(x-(n+2)-1)-\gamma f(x-(n+2)-2))-\gamma f(x-(n+2)-1))]$$

$$=\beta^{-3}\lim_{n\to\infty}\beta^{n+3}[f(x-(n+3)-\alpha(f(x-(n+3)-1)-\gamma f(x-(n+3)-2))-\gamma f(x-(n+3)-1))]$$

$$=\beta^{-1}G_1(x)+\beta^{-2}G_1(x)+\beta^{-3}G_1(x)=G_1(x)$$

for all $x \in \mathbb{R}$. Hence, $G_1$ is a Tribonacci function. If $n$ tends to infinity, then (3.4) yields

$$\|f(x) - \alpha(f(x-1) - \gamma f(x-2)) - \gamma f(x-1) - G_1\| \leq \frac{1}{1-|\beta|}\epsilon \tag{3.5}$$

for all $x \in \mathbb{R}$. On the other hand, it follows from (3.1) that

$$\|f(x) - \beta(f(x-1) - \alpha f(x-2)) - \alpha f(x-1) - \gamma[f(x-1) - \alpha f(x-2)$$
$$+ \alpha\beta f(x-3) - \beta f(x-2)]\| \leq \epsilon$$

for all $x \in \mathbb{R}$. Analogous to (3.3), replacing $x$ by $x-k$ in the above inequality, we obtain

$$\|f(x-k) - \beta[f(x-k-1) - \alpha f(x-k-2)] - \alpha f(x-k-1) - \gamma$$
$$[f(x-k-1) - \alpha f(x-k-2) + \alpha\beta f(x-k-3) - \beta f(x-k-2)]\| \leq \epsilon$$

and

$$\|\gamma^k[f(x-k) - \beta(f(x-k-1) - \alpha f(x-k-2)) - \alpha f(x-k-1))]$$
$$- \gamma^{k+1}(f(x-k-1) - \alpha f(x-k-2) + \alpha\beta f(x-k-3) - \beta f(x-k-2))\| \leq |\gamma|^k\epsilon \tag{3.6}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By using (3.6), we further obtain that

$$\|f(x) - \beta[f(x-1) - \alpha f(x-2)] - \alpha f(x-1) - \gamma^n[f(x-n)$$
$$-\beta(f(x-n-1) - \alpha f(x-n-2)) - \alpha f(x-n-1)]\| \leq \max_{1\leq k\leq n}\{\|\gamma^k(f(x-k) - \beta(f(x-k-1)$$
$$-\alpha f(x-k-2)) - \alpha f(x-k-1)) - \gamma^{k+1}(f(x-(k+1)) - \beta(f(x-(k+1)-1)$$
$$- \alpha f(x-(k+1)-2)) - \alpha f(x-(k+1)-1))\|\}- \leq \max_{0\leq k\leq n-1}\{|\gamma|^k\epsilon\} = \epsilon \tag{3.7}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. We obviously have

$$\{\gamma^n(f(x-n) - \beta(f(x-n-1) - \alpha f(x-n-2)) - \alpha f(x-n-2)) - \alpha f(x-n-1)\}$$

is a Cauchy sequence, for all $x \in \mathbb{R}$. Hence, we can define a function $G_2 : \mathbb{R} \longrightarrow X$ by

$$G_2(x) := \lim_{n\to\infty} \gamma^n[f(x-n) - \beta(f(x-n-1) - \alpha f(x-n-2)) - \alpha f(x-n-1)]$$

for all $x \in \mathbb{R}$. Using the above definition of $G_2$, we get

$$G_2(x-1) + G_2(x-2) + G_2(x-3)$$

$$= \gamma^{-1} \lim_{n\to\infty} \gamma^{n+1}[f(x-(n+1)) - \beta(f(x-(n+1)-1) - \alpha f(x-(n+1)-2))$$

$$-\alpha f(x-(n+1)-1))] + \gamma^{-2} \lim_{n\to\infty} \gamma^{n+2}[f(x-(n+2)) - \beta(f(x-(n+2)-1)$$

$$-\alpha f(x-(n+2)-2)) - \alpha f(x-(n+2)-1))] = \gamma^{-3} \lim_{n\to\infty} \gamma^{n+3}[f(x-(n+3))$$

$$-\beta(f(x-(n+3)-1) - \alpha f(x-(n+3)-2)) - \alpha f(x-(n+3)-1))]$$

$$= \gamma^{-1}G_2(x) + \gamma^{-2}G_2(x) + \gamma^{-3}G_2(x) = G_2(x),$$

for all $x \in \mathbb{R}$. So, $G_2$ is also a Tribonacci function. If $n$ tends to infinity, then it follows from (3.7) that

$$\|f(x) - \beta[f(x-1) - \alpha f(x-2)] - \alpha f(x-1) - G_2(x)\| \le \|f11 - |\gamma|\|\epsilon = \|f11 - |\beta|\|\epsilon, \quad (3.8)$$

for all $x \in \mathbb{R}$. Finally, it follows from (3.1) that

$$\|f(x) - \gamma(f(x-1) - \beta f(x-2)) - \beta f(x-1) - \alpha[f(x-1) - \gamma f(x-2)$$

$$+ \beta\gamma f(x-3) - \beta f(x-2)]\| \le \epsilon$$

for all $x \in \mathbb{R}$ (see the second equation in (2.3) for $n = 1$). If we replace $x$ by $x + k$ in the above inequality, then we have

$$\|f(x+k) - \gamma(f(x+k-1) - \beta f(x+k-2) - \beta f(x+k-1) - \alpha[f(x+k-1)\gamma f(x+k-2)$$

$$+\beta\gamma f(x+k-2) - \beta f(x+k-2))]\| \le \epsilon$$

and

$$\|\alpha^{-k}[f(x+k) - \gamma(f(x+k-1) - \beta f(x+k-2)) - \beta f(x+k-1) - \alpha^{-k+1}]$$

$$[f(x+k-1) - \gamma f(x+k-2) + \beta\gamma f(x+k-3) - \beta f(x+k-2)]\| \le |\alpha^{-1}|^k\epsilon. \quad (3.9)$$

for all $x \in \mathbb{R}$ and all $k \in \mathbb{Z}$. By using (3.9), we further obtain that

$$\|\alpha^{-n}[f(x+n)-\gamma(f(x+n-1)-\beta f(x+n-2))-\beta f(x+n-1)]-[f(x)-\gamma(f(x-1)$$

$$+\beta\gamma f(x-2))-\beta f(x-1)]\| \leq \max_{1\leq k\leq n}\{\|\alpha^{-k}[f(x+k)-\gamma(f(x+k-1)-\beta f(x+k-2))$$

$$-\beta f(x+k-1)]-\alpha^{-k+1}[f(x+k-1)-\gamma(f(x+k-2)+\beta\gamma f(x+k-3))-\beta f(x+k-2)]\|\}$$

$$\leq \max_{1\leq k\leq n}\{|\alpha^{-1}|^k\epsilon\} = \alpha^{-1}\epsilon \tag{3.10}$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$By applying ( 3.10) we see that

$$\{\alpha^{-n}[f(x+n)-\gamma(f(x+n-1)-\beta f(x+n-2))-\beta f(x+n-1)]\}$$

is a Cauchy sequence, for a fixed $x \in \mathbb{R}$. Hence, we can define a function $G_3 : \mathbb{R} \longrightarrow X$ by

$$G_3(x) := \lim_{n\to\infty} \alpha^{-n}[f(x+n)-\gamma(f(x+n-1)-\beta f(x+n-2))-\beta f(x+n-1)]$$

for all $x \in \mathbb{R}$. In view of the above definition of $G_3$, we obtain

$$G_3(x-1)+G_3(x-2)+G_3(x-3)$$

$$=\alpha^{-1}\lim_{n\to\infty}\alpha^{-(n-1)}[f(x+n-1)-\gamma(f(x+(n-1)-1)-\beta f(x+(n-1)-2))$$

$$-\beta f(x+(n-1)-1)]+\alpha^{-2}\lim_{n\to\infty}\alpha^{-(n-2)}[f(x+n-2)-\gamma(f(x+(n-2)-1)$$

$$-\beta f(x+(n-2)-2))-\beta f(x+(n-2)-1)]+\alpha^{-3}\lim_{n\to\infty}\alpha^{-(n-3)}[f(x+n-3)$$

$$-\gamma(f(x+(n-2)-1)-\beta f(x+(n-3)-2))-\beta f(x+(n-3)-1)]$$

$$=\alpha^{-1}G_3(x)+\alpha^{-2}G_3(x)+\alpha^{-3}G_3(x)=G_3(x),$$

for all $x \in \mathbb{R}$. This means that $G_3$ is also a Tribonacci function. If we let $n$ tends to infinity in (3.10), then we have

$$\|G_3(x)-f(x)+\gamma(f(x-1)-\beta f(x-2))+\beta f(x-1)\| \leq \alpha^{-1}\epsilon \tag{3.11}$$

for all $x \in \mathbb{R}$. By (3.5), (3.8) and (3.11), we obtain that

$$\|f(x)-\left[\frac{\beta^2(\gamma-\alpha)G_1+\gamma^2(\alpha-\beta)G_2-\alpha^2(\beta-\gamma)G_3}{\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\gamma)}\right]\|$$

$$= \frac{1}{|\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\gamma)|}$$

$$\|(\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\gamma))f(x)-\beta^2(\gamma-\alpha)G_1-\gamma^2(\alpha-\beta)G_2+\alpha^2(\beta-\gamma)G_3\|$$

$$\leq \frac{1}{|\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\gamma)|}$$

$$[\|\beta^2(\gamma-\alpha)f(x)-\beta^2(\gamma^2-\alpha^2)f(x-1)+\beta^2(\gamma-\alpha)\alpha\gamma f(x-2)-\beta^2(\gamma-\alpha)G_1\|$$

$$+\|\gamma^2(\alpha-\beta)f(x)-\gamma^2(\alpha^2-\beta^2)f(x-1)+\gamma^2(\alpha-\beta)\beta\alpha f(x-2)-\gamma^2(\alpha-\beta)G_2\|$$

$$+\|\alpha^2(\beta-\gamma)f(x)-\alpha^2(\beta^2-\gamma^2)f(x-1)+\alpha^2(\beta-\gamma)\beta\gamma f(x-2)-\alpha^2(\beta-\gamma)G_3\|]$$

$$\leq \{\frac{1}{(\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\alpha))|}\epsilon \max\{1,\alpha^{-1}\}$$

for all $x \in \mathbb{R}$. Putting

$$G(x) := \frac{\beta^2(\gamma-\alpha)G_1+\gamma^2(\alpha-\beta)G_2-\alpha^2(\beta-\gamma)G_3}{\beta^2(\gamma-\alpha)+\gamma^2(\alpha-\beta)+\alpha^2(\beta-\alpha)}$$

for all $x \in \mathbb{R}$. It is easy to show that $G$ is a Tribonacci function satisfying (3.2). □

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