Int. J. Nonlinear Anal. Appl. **1** (2010) No.1, 1–5
ISSN:2008-6822 (electronic)
http://www.ijnaa.com

UNIVALENT HOLOMORPHIC FUNCTIONS WITH FIXED FINITELY MANY COEFFICIENTS INVOLVING SALAGEAN OPERATOR

M. ACU[1] AND SH. NAJAFZADEH[2*]

ABSTRACT. By using generalized Salagean differential operator a new class of univalent holomorphic functions with fixed finitely many coefficients is defined. Coefficient estimates, extreme points, arithmetic mean, and weighted mean properties are investigated.

1. Introduction and preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=t+1}^{+\infty} a_k z^k \tag{1.1}$$

which are holomorphic in the unit disk $\Delta = \{z : \quad |z| < 1\}$.
We denote by N the subclass of A consisting of functions $f(z) \in A$ which are holomorphic univalent in Δ and are of the form

$$f(z) = z - \sum_{k=t+1}^{+\infty} a_k z^k, \qquad (a_k \geq 0). \tag{1.2}$$

The generalized Salagean operator is defined in [1] by

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z)$$

$$D_\lambda^n f(z) = D_\lambda^1(D_\lambda^{n-1} f(z)), \quad \lambda \geq 0$$

see also [2].
If $f(z)$ is given by (1.2), we see that

$$D_\lambda^n f(z) = z - \sum_{k=t+1}^{+\infty} [1+(k-1)\lambda]^n a_k z^k. \tag{1.3}$$

Date: Received: March 2009; Revised: November 2009.
2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.
Key words and phrases. Salagean operator; Univalent function; Coefficient estimate; Extreme point; Arithmetic and weighted mean.
* Corresponding author.

When $\lambda = 1$, we get the classic Salagean differential operator [3].
A function $f(z) \in N$ is said to be in $N_{n,\lambda}(\alpha,\beta,\gamma,\theta)$ if and only if

$$\left| \frac{[D_\lambda^{n+2} f(z)]' - \frac{1}{z} D_\lambda^{n+1} f(z)}{\frac{2\alpha}{z} D_\lambda^{n+1} f(z) - \beta(1+\theta)\alpha} \right| < \gamma, \tag{1.4}$$

where $\alpha,\beta,\gamma,\theta$ belong to [0,1].

Now we introduce the class $N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$, the subclass of $N_{n,\lambda}(\alpha,\beta,\gamma,\theta)$ consisting of functions with negative and fixed finitely many coefficient of the form

$$f(z) = z - \sum_{m=2}^{t} \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{[(1+(m-1)\lambda)^{n+1}(m^2\lambda + m(1-\lambda) - 1 + 2\alpha\lambda)]}. \tag{1.5}$$

Such type of work was recently carried out by Shams and Kulkani [4]. See also [5].
We need the following lemma for proving our main results.

Lemma 1.1. *A function $f(z)$ given by (1.2) is in the class $N_{n,\lambda}(\alpha,\beta,\gamma,\theta)$ if and only if*

$$\sum_{k=t+1}^{\infty} [(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)a_k] \leq \alpha\gamma(2-\beta(1+\theta)). \tag{1.6}$$

Proof. Let the inequality (1.6) holds true and suppose $|z| = 1$. Then we obtain

$$|(D_\lambda^{n+2} f(z))' - \tfrac{1}{z} D_\lambda^{n+1} f(z)| - \gamma|2\alpha - 2\alpha \textstyle\sum_{k=t+1}^{+\infty}(1+(k-1)\lambda)^{n+1} a_k z^{k-1} - \beta(1+\theta)\alpha|$$

$$= \textstyle\sum_{k=t+1}^{+\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)a_k - \alpha\gamma(2-\beta(1+\theta))] \leq 0.$$

Hence, by maximum modulus theorem, we conclude that $f(z) \in N_{n,\lambda}(\alpha,\beta,\gamma,\theta)$.

Conversely, let $f(z)$ defined by (1.2) be in the class $N_{n,\lambda}(\alpha,\beta,\gamma,\theta)$, so the condition (1.4) yields

$$\left| \frac{\sum_{k=t+1}^{+\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1]a_k z^{k-1}}{2\alpha - \sum_{k=t+1}^{\infty} 2\alpha(1+(k-1)\lambda)^{n+1} - \beta(1+\theta)\alpha} \right| < \gamma, \quad z \in \Delta.$$

Since for any z, $|Re(z)| < |z|$, then

$$Re\left\{ \frac{\sum_{k=t+1}^{+\infty}[(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1)]a_k z^{k-1}}{\alpha(2-\beta(1+\theta)) - \sum_{k=t+1}^{\infty} 2\alpha(1+(k-1)\lambda)^{n+1} a_k z^{k-1}} \right\} < \gamma.$$

By letting $z \to 1$ through real values, we get the required result. □

2. Main Results

We begin by proving a necessary and sufficient conditions for a function belonging to the class $N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$.

Theorem 2.1. *Let $f(z)$ defined by (1.2), then $f(z) \in N^{c_m}_{n,\lambda}(\alpha,\beta,\gamma,\theta)$ if and only if*

$$\sum_{k=t+1}^{+\infty} \frac{[(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\lambda)]a_k}{\alpha\gamma(2-\beta(1+\theta))} < 1-\sum_{m=2}^{t} c_m. \tag{2.1}$$

Proof. By letting

$$a_m = \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{(1+(m-1)\lambda)^{n+1}(m^2\lambda+m(1-\lambda)-1+2\alpha\gamma)}, \tag{2.2}$$

since $N^{c_m}_{n,\lambda}(\alpha,\beta,\gamma,\theta) \subset N_{n,\lambda}(\alpha,\beta,\gamma,\theta)$, so $f(z) \in N^{c_m}_{n,\lambda}(\alpha,\beta,\gamma,\theta)$ if and only if

$$\sum_{m=2}^{t} \frac{(1+(m-1)\lambda)^{n+1}(m^2\lambda+m(1-\lambda)-1+2\alpha\gamma)}{\alpha\gamma(2-\beta(1+\theta))} a_m$$
$$+\sum_{k=t+1}^{+\infty} \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}{\alpha\gamma(2-\beta(1+\theta))} a_k < 1$$

or

$$\sum_{k=t+1}^{+\infty} \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}{\alpha\gamma(2-\beta(1+\theta))} < 1-\sum_{m=2}^{t} c_m,$$

and this gives the result.

□

Corollary 2.2. *If $f(z)$ defined by (1.2) be in $N^{c_m}_{n,\lambda}(\alpha,\beta,\gamma,\theta)$ then for $k \geq t+1$ we have*

$$a_k \leq \frac{\alpha\gamma(2-\beta(1+\theta))(1-\sum_{m=2}^{t} c_m)}{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}, \tag{2.3}$$

and result is best possible for the function

$$g(z) = z - \sum_{m=2}^{t} \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{(1+(m-1)\lambda)^{n+1}(m^2\lambda+m(1-\lambda)-1+2\alpha\gamma)} z^m$$
$$-\frac{\alpha\gamma(2-\beta(1+\theta))(1-\sum_{m=2}^{t} c_m)}{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)} z^k \tag{2.4}$$

3. Extreme points and arithmetic mean structure

Now we find Extreme points and convolution structure for functions in $N^{c_m}_{n,\lambda}(\alpha,\beta,\gamma,\theta)$.

Theorem 3.1. *Let*

$$f_t(z) = z - \sum_{m=2}^{t} \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{(1+(m-1)\lambda)^{n+1}(m^2\lambda+m(1-\lambda)-1+2\alpha\gamma)} z^m$$

and for $k \geq t+1$

$$f_k(z) = z - \sum_{m=2}^{t} \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{(1+(m-1)\lambda)^{n+1}(m^2\lambda+m(1-\lambda)-1+2\alpha\gamma)} z^m$$
$$-\frac{\alpha\gamma(2-\beta(1+\theta))(1-\sum_{m=2}^{t} c_m)}{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)} z^k.$$

Then $F(z) \in N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$ *if and only if it can be expressed in the form*

$$F(z) = \sum_{k=t}^{+\infty} \sigma_k f_k(z)$$

where $\sigma_k \geq 0 \quad (k \geq t)$ *and* $\sum_{k=t}^{+\infty} \sigma_k = 1$.

Proof. Let $F(z) = \sum_{k=t}^{+\infty} \sigma_k f_k(z)$, then

$$F(z) = z - \sum_{m=2}^{t} \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{(1+(m-1)\lambda)^{n+1}(m^2\lambda + m(1-\lambda) - 1 + 2\alpha\gamma)} z^m$$

$$- \sum_{k=t+1}^{+\infty} \frac{(1-\sum_{m=2}^{+\infty} c_m)\alpha\gamma(2-\beta(1+\theta))\sigma_k}{(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)} z^k.$$

Finally we have

$$\sum_{k=t+1}^{\infty} \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)(1-\sum_{m=2}^{+\infty} c_m)\alpha\gamma(2-\beta(1+\theta))\sigma_k}{\alpha\gamma(2-\beta(1+\theta))(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)}$$

$$= (1 - \sum_{m=2}^{t} c_m) \sum_{k=t+1}^{+\infty} \sigma_k = (1 - \sum_{m=2}^{t} c_m)(1-\sigma_t) < 1 - \sum_{m=2}^{t} c_m.$$

Thus $F(z) \in N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$.
Conversely suppose $F(z) \in N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$, so

$$F(z) = z - \sum_{m=2}^{t} \frac{\alpha\gamma(2-\beta(1+\theta))c_m}{(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)} - \sum_{k=t+1}^{+\infty} a_k z^k.$$

By putting

$$\sigma_k = \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda + k(1-\lambda) - 1 + 2\alpha\gamma)}{\alpha\gamma(2-\beta(1+\theta))(1-\sum_{m=2}^{t} c_m)} a_k, \quad (k \geq t+1)$$

we have $\sigma_k \geq 0$ and if we put $\sigma_t = 1 - \sum_{k=t+1}^{+\infty} \sigma_k$, we conclude the required result. □

Theorem 3.2. *Let* $f_j(z) \quad (j = 1, 2, ..., l)$ *defined by*

$$f_j(z) = z - \sum_{m=2}^{l} \frac{\alpha\gamma(2-\beta(1+\theta))}{(1+(m-1)\lambda)^{n+1}(m^2\lambda + m(1-\lambda) - 1 + 2\alpha\gamma)} z^m - \sum_{\substack{k=t+1 \\ j=1,\dots,l}}^{+\infty} a_{k,j} z^k$$

be in $N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$*, then the function*

$$H(z) = z - \sum_{m=2}^{+\infty} \frac{\alpha\gamma(2-\beta(1+\theta))C_m}{(1+(m-1)\lambda)^{n+1}(m^2\lambda + m(1-\lambda) - 1 + 2\alpha\gamma)} z^m - \sum_{k=t+1}^{+\infty} d_k z^k, \quad (c_k \geq 0)$$

is also in $N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$*, where* $d_k = \frac{1}{l}\sum_{j=1}^{l} a_{k,j}$.

Proof. We have

$$\sum_{k=t+1}^{+\infty} \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}{\alpha\gamma(2-\beta(1+\theta))} d_k$$

$$= \sum_{k=t+1}^{+\infty} \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}{l\alpha\gamma(2-\beta(1+\theta))} (\sum_{j=1}^{l} a_{k,j})$$

$$= \frac{1}{l}\sum_{j=1}^{l}\left[\sum_{k=t+1}^{+\infty} \frac{(1+(k-1)\lambda)^{n+1}(k^2\lambda+k(1-\lambda)-1+2\alpha\gamma)}{\alpha\gamma(2-\beta(1+\theta))} a_{k,j}\right]$$

$$< \frac{1}{l}\sum_{j=1}^{l}\left(1-\sum_{m=2}^{t} c_m\right) = 1-\sum_{m=2}^{t} c_m$$

and the proof by Theorem 2.1 is complete. So $N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$ is closed under arithmetic mean. □

Remark 3.3. with the same calculation with theorem 3.2 we can prove that $N_{n,\lambda}^{c_m}(\alpha,\beta,\gamma,\theta)$ is closed under weighted mean.

Acknowledgements: The authors would like to thank the referees for their valuable comments.

References

1. F.M. Al-Oboudi, *On univalent functions by a generalized Salagean operator*, Int. J. Math. Math. Sci. 27 (2004), 1429-1436.
2. D. Raducanu, *On the Fekete Szego inequality for a class of analytic functions defined by using the generalized Salagean operator,* General Math. Vol. 10 (16) (2008), 19-27.
3. G. S. Salagean, *Subclasses of univalent functions,* Lect. Notes Math. 1013, (1983), 362-372
4. S, Shams and S. R. Kulkarni, *A class of univalent functions with negative and fixed finitely many coefficients,* Acta Ciencia Indica, XXIXM (3), (2003), 587-594.
5. Sh. Najafzadeh and S.R. Kulkarni, *Application of Ruscheweyh derivative on univalent functions with negative and fixed finitely many coefficients,* Rajasthan Acad. Phy. Sci. 5(1), (2006), 39-51.

[1] Department of Mathematics,Lucian Blaga University, 2400 Sibiu, Romania
E-mail address: Muger_Acu@yahoo.com

[2]Department of Mathematics, Faculty of Science,University of Maragheh, Maragheh-Iran
E-mail address: najafzadeh1234@yahoo.ie