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# THE STRUCTURE OF MODULE CONTRACTIBLE BANACH ALGEBRAS

ABASALT BODAGHI

Abstract. In this paper we study the module contractibility of Banach algebras and characterize them in terms the concepts splitting and admissibility of short exact sequences. Also we investigate module contractibility of Banach algebras with the concept of the module diagonal.

## 1. Introduction and preliminaries

A Banach algebra $\mathcal{A}$ is *contractible (super-amenable)* if every bounded derivation from $\mathcal{A}$ into any Banach $A$-module is inner, equivalently if $H^1(\mathcal{A}, X) = \{0\}$ for every Banach $A$-module $X$, where $H^1(\mathcal{A}, X)$ is the *first Hochschild cohomology group* of $A$ with coefficients in $X$. This concept was introduced by Barry Johnson in [11].

The author and Amini in [4] introduced the concept of module contractibility and showed that for an inverse semigroup $S$ with set of idempotents $E_S$, the semigroup algebra $\ell^1(S)$ is module contractible if and only if the group homomorphic image $S/\approx$ is finite, where $s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set

$$\{\delta_{set} - \delta_{st} : s, t \in S, e \in E_S\}.$$

In this paper we study the module contractibility of Banach algebras and characterize them in terms the concepts splitting and admissibility of short exact sequences. Finally we investigate module contractibility of Banach algebras with the concept of the module diagonal. In fact we prove that, if the Banach algebra $\mathcal{B}$ is the cluster of the image a module contractible Banach algebra $\mathcal{A}$ under a continuous module morphism, then $\mathcal{B}$ is module contractible.

## 2. Module Contractibility

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \ (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

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Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,\ (\alpha\cdot x)\cdot a=\alpha\cdot(x\cdot a)\quad (a\in\mathcal{A},\alpha\in\mathfrak{A},x\in\mathcal{X})$$

and the same for the right or two-sided actions. Then we say that $\mathcal{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha\cdot x=x\cdot\alpha\quad(\alpha\in\mathfrak{A},x\in\mathcal{X})$$

then $\mathcal{X}$ is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If $\mathcal{X}$ is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is $\mathcal{X}^*$, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on $\mathcal{X}^*$ are defined by

$$\langle\alpha\cdot f,x\rangle=\langle f,x\cdot\alpha\rangle,\ \langle a\cdot f,x\rangle=\langle f,x\cdot a\rangle\quad(a\in\mathcal{A},\alpha\in\mathfrak{A},x\in\mathcal{X},f\in\mathcal{X}^*)$$

and the same for the right actions. Let $\mathcal{Y}$ be another $\mathcal{A}$-$\mathfrak{A}$-module, then a $\mathcal{A}$-$\mathfrak{A}$-module morphism from $\mathcal{X}$ to $\mathcal{Y}$ is a norm-continuous map $\varphi:\mathcal{X}\longrightarrow\mathcal{Y}$ with $\varphi(x\pm y)=\varphi(x)\pm\varphi(y)$ and

$$\varphi(\alpha\cdot x)=\alpha\cdot\varphi(x),\ \varphi(x\cdot\alpha)=\varphi(x)\cdot\alpha,\ \varphi(a\cdot x)=a\cdot\varphi(x),\varphi(x\cdot a)=\varphi(x)\cdot a,$$

for $x,y\in\mathcal{X},a\in\mathcal{A}$, and $\alpha\in\mathfrak{A}$.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a\cdot(\alpha\cdot b)=(a\cdot\alpha)\cdot b\quad(\alpha\in\mathfrak{A},a,b\in\mathcal{A}).$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

If $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then so is the dual space $\mathcal{A}^*$. If moreover $\mathcal{A}$ is a commutative $\mathfrak{A}$-module, then $\mathcal{A}^*$ is commutative $\mathcal{A}$-$\mathfrak{A}$-module.

Consider the projective tensor product $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$. It is well known that $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ is a Banach algebra with respect to the canonical multiplication map defined by

$$(a\otimes b)(c\otimes d)=(ac\otimes bd)$$

and extended by bi-linearity and continuity [6]. Then $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with canonical actions. Let $I$ be the closed ideal of the projective tensor product $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ generated by elements of the form $\alpha\cdot a\otimes b-a\otimes b\cdot\alpha$ for $\alpha\in\mathfrak{A},a,b\in\mathcal{A}$. Consider the map $\omega\in\mathcal{L}(\mathcal{A}\widehat{\bigotimes}\mathcal{A},\mathcal{A})$ defined by $\omega(a\otimes b)=ab$ and extended by linearity and continuity. Let $J$ be the closed ideal of $\mathcal{A}$ generated by $\omega(I)$. Then the module projective tensor product $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}\cong(\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I$ and the quotient Banach algebra $\mathcal{A}/J$ are Banach $\mathfrak{A}$-modules with compatible actions. We have $(\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A})^*=\mathcal{L}_{\mathfrak{A}}(\mathcal{A},\mathcal{A}^*)$ where the right hand side is the space of all $\mathfrak{A}$-module morphism from $A$ to $A^*$[12]. Also the map $\widetilde{\omega}\in\mathcal{L}(\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A},\mathcal{A}/J)$ defined by $\widetilde{\omega}(a\otimes b+I)=ab+J$ extends to an $\mathfrak{A}$-module morphism. If $\mathcal{A}/J$ and $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}$ are commutative $\mathfrak{A}$-module, then $\mathcal{A}/J$ and $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}$ are $\mathcal{A}/J$-$\mathfrak{A}$-module and $\widetilde{\omega}$ is $\mathcal{A}/J$-$\mathfrak{A}$-module morphism.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as in the above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Let $I$ and $J$ be the corresponding closed ideals of $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ and $\mathcal{A}$, respectively. A bounded map $D:\mathcal{A}\longrightarrow\mathcal{X}$ is called a *module derivation* if

$$D(a\pm b)=D(a)\pm D(b),\ \ D(ab)=D(a)\cdot b+a\cdot D(b)\quad(a,b\in\mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Although $D$ is not necessary linear, but still its boundedness implies its norm continuity (since it preserves subtraction). When $\mathcal{X}$ is commutative, each $x \in \mathcal{X}$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called *inner* module derivations. The Banach algebra $\mathcal{A}$ is called *contractible* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $\mathcal{X}$, each module derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is inner [4].

## 3. Characterization of Module Contractible Banach algebras

Let $\widetilde{\omega}$ be as in the above section. Let $I$ and $J$ be the corresponding closed ideals of $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ and $\mathcal{A}$, respectively. By the definition of $I$, if $x = \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A}\widehat{\bigotimes}\mathcal{A}$, then $\alpha \cdot x - x \cdot \alpha \in I$ for all $\alpha \in \mathfrak{A}$. Therefore $\alpha \cdot x + I = x \cdot \alpha + I$, that is $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}$ is always a commutative $\mathfrak{A}$-module.

We say that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\alpha_j\}$ in $\mathfrak{A}$ such that $\|\alpha_j.a - a\| \rightarrow 0$ and $\|a.\alpha_j - a\| \rightarrow 0$, for each $a \in \mathcal{A}$.

Let $X$, $Y$ and $Z$ be Banach $\mathcal{A}/J$-$\mathfrak{A}$-modules. Then the short, exact sequence

$$\{0\} \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow \{0\} \tag{2.1}$$

is *admissible* if $\psi$ has a bounded right inverse which is $\mathfrak{A}$-module homomorphism, and *splits* if $\psi$ has a bounded right inverse which is a $\mathcal{A}/J$-$\mathfrak{A}$-module morphism. Obviously, the short, exact sequence (2.1) is admissible if and only if $\varphi$ has a bounded left inverse which is $\mathcal{A}/J$-$\mathfrak{A}$-module morphism.

We set $K = \ker(\widetilde{\omega})$. If $\mathcal{A}/J$ has a bounded approximate identity, then the following sequences are exact.

$$\{0\} \longrightarrow K \xrightarrow{i} (\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I \xrightarrow{\widetilde{\omega}} \mathcal{A}/J \longrightarrow \{0\} \tag{2.2}$$

Recall that an element $\mathcal{M} \in \mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}$ is called a *module diagonal* if $\widetilde{\omega}(\mathcal{M})$ is an identity of $\mathcal{A}/J$ and $a \cdot \mathcal{M} = \mathcal{M} \cdot a$, for all $a \in \mathcal{A}$. It is proved in [4, Theorem 2.6] that if $\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module, then $\mathcal{A}$ is module contractible if and only if $\mathcal{A}$ has a module diagonal. Also it is shown in [4, Proposition 2.3] that if $\mathcal{A}$ be a commutative Banach module super-amenable as an $\mathfrak{A}$-module, then it is unital.

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$, where $f$ is a continuous linear functional on $\mathfrak{A}$. The following lemma is proved in [3, Lemma 3.1].

**Lemma 3.1.** *Let $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from left (right) and $J_0$ be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has a left (right) identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $a \cdot \alpha - f(\alpha)a \in J_0$ $(\alpha \cdot a - f(\alpha)a \in J_0, respectively)$.*

**Lemma 3.2.** *Let $\mathcal{A}/J$ be commutative Banach $\mathfrak{A}$-modules. If $\mathcal{A}/J$ has an identity, the exact sequences (2.2) is admissible.*

*Proof.* Suppose that $\mathcal{A}/J$ has an identity $e + J$, the $\widetilde{\rho} : \mathcal{A}/J \longrightarrow (\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I$; $\widetilde{\rho}(a + J) = a \otimes e + I$ is an $\mathfrak{A}$-module morphism and right inverse for $\widetilde{\omega}$. □

**Theorem 3.3.** *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with trivial left action and let $\mathcal{A}/J$ be a commutative Banach $\mathfrak{A}$-module. Then $\mathcal{A}$ is module contractible if and only if*
*(i) $\mathcal{A}/J$ has an identity, and*
*(ii) The exact sequence (2.2) splits.*

*Proof.* Assume that $\mathcal{A}$ is module contractible. Then $\mathcal{A}/J$ is contractible [4, Lemma 2.7], so $\mathcal{A}/J$ has identity [6, Theorem 2.8.48]. Also $\mathcal{A}$ has a module diagonal $\mathcal{M}$ [4, Theorem 2.6]. Define $\widetilde{\rho}: \mathcal{A}/J \longrightarrow (\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I$ by $\widetilde{\rho}(a+J) := a \cdot \mathcal{M}$, for all $a \in \mathcal{A}$. Since $a \cdot \mathcal{M} = \mathcal{M} \cdot a$, for all $a, b, c, d \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned} c \otimes d \cdot (\alpha \cdot (ab) - (ab) \cdot \alpha) &= c \otimes d \cdot (f(\alpha)ab) - c \otimes d \cdot (ab) \cdot \alpha \\ &= f(\alpha)c \otimes dab - c \otimes (dab) \cdot \alpha \\ &= \alpha \cdot c \otimes dab - c \otimes (dab) \cdot \alpha \in I, \end{aligned}$$

so $\widetilde{\rho}$ is well-defined. Also for all $a, b \in \mathcal{A}$ we have

$$\begin{aligned} \widetilde{\rho}((a+J) \cdot (b+J)) &= ab \cdot \mathcal{M} \\ &= (a+J) \cdot (b \cdot \mathcal{M}) \\ &= (a+J) \cdot \rho(b+J), \end{aligned}$$

and

$$\begin{aligned} \widetilde{\rho}((b+J) \cdot (a+J)) &= ba \cdot \mathcal{M} = \mathcal{M} \cdot ba \\ &= (\mathcal{M} \cdot b).(a+J) \\ &= (b \cdot \mathcal{M}) \cdot (a+J) \\ &= \widetilde{\rho}(b+J) \cdot (a+J). \end{aligned}$$

It follows from Lemma 3.1 that $\widetilde{\rho}$ is a $\mathcal{A}/J$- $\mathfrak{A}$-module morphism. Also

$$(\widetilde{\omega} \circ \widetilde{\rho})(a+J) = \widetilde{\omega}(a \cdot \mathcal{M}) = (a+J) \cdot \widetilde{\omega}(\mathcal{M}) = a+J.$$

So $\widetilde{\rho}$ is a bounded right inverse for $\widetilde{\omega}$. Therefore the exact sequence (2.2) splits. Conversely, if $e+J$ is an identity for $\mathcal{A}/J$ and the exact sequence (2.2) splits, then $\widetilde{\rho}$ is a bounded right inverse for $\widetilde{\omega}$ which is $\mathcal{A}/J$-$\mathfrak{A}$-module morphism, it is easy that show $\widetilde{\rho}(e+J)$ is a module diagonal for $\mathcal{A}$. Now the module contractibility of $\mathcal{A}$ follows from [4, Theorem 2.6]. □

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\widetilde{\omega}_{\mathcal{A}}$ and $\widetilde{\omega}_{\mathcal{B}}$ be the corresponding $\widetilde{\omega}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively. Also we consider $I_{\mathcal{A}}$ and $J_{\mathcal{A}}$ the corresponding closed ideals of $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ and $\mathcal{A}$, respectively. Similarly, for $\mathcal{B}$.

In the next result we give an alternative proof of [3, Proposition 3.6] with using from the concept of the module diagonal.

**Proposition 3.4.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and Banach $\mathfrak{A}$-modules with compatible actions. If $\mathcal{A}$ is $\mathfrak{A}$-module contractible and $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous Banach $\mathfrak{A}$-module morphism with dense range, then $\mathcal{B}$ is also $\mathfrak{A}$-module contractible.*

*Proof.* Assume that $M = \sum_{i=1}^{n} x_i \otimes y_i + I_{\mathcal{A}}$ is a module diagonal for $\mathcal{A}$. We show that $\mathcal{B}$ has the module diagonal $N = \sum_{i=1}^{n} \varphi(x_i) \otimes \varphi(y_i) + I_{\mathcal{B}}$. Consider the map $\widetilde{\varphi}: \mathcal{A}/J_{\mathcal{A}} \longrightarrow \mathcal{B}/J_{\mathcal{B}}$ ; $(a + J_{\mathcal{A}} \longrightarrow \varphi(a) + J_{\mathcal{B}})$. For each $a_1, a_2 \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we have

$$\varphi(\alpha \cdot a_1a_2 - a_1a_2 \cdot \alpha) = \alpha \cdot \varphi(a_1)\varphi(a_2) - \varphi(a_1)\varphi(a_2) \cdot \alpha \in J_{\mathcal{B}}.$$

Thus $\widetilde{\varphi}$ is well defined and it is easy that show $\widetilde{\varphi}$ is $\mathfrak{A}$-module continuous morphism. So for each $a \in \mathcal{A}$ we have

$$\begin{aligned}(\varphi(a) + J_{\mathcal{B}}) \cdot \widetilde{\omega}_{\mathcal{B}}(N) &= (\varphi(a) + J_{\mathcal{B}}) \cdot (\sum_{i=1}^{n} \varphi(x_i)\varphi(y_i) + J_{\mathcal{B}}) \\ &= \sum_{i=1}^{n} \varphi(ax_iy_i) + J_{\mathcal{B}} \\ &= \widetilde{\varphi}((a + J_{\mathcal{A}}) \cdot (\sum_{i=1}^{n} x_iy_i + J_{\mathcal{A}})) \\ &= \widetilde{\varphi}((a + J_{\mathcal{A}}) \cdot \widetilde{\omega}_{\mathcal{A}}(M)) \\ &= \widetilde{\varphi}(a + J_{\mathcal{A}}) = \varphi(a) + J_{\mathcal{B}}.\end{aligned}$$

Given $b \in \mathcal{B}$, take the bounded net $(a_j)$ in $\mathcal{A}$ with $\varphi(a_j) \longrightarrow b$, then

$$(b + J_{\mathcal{B}}) \cdot \widetilde{\omega}_{\mathcal{B}}(N) = \lim_j (\varphi(a_j) + J_{\mathcal{B}}) \cdot \widetilde{\omega}_{\mathcal{B}}(N) = \lim_j \varphi(a_j) + J_{\mathcal{B}} = b + J_{\mathcal{B}}.$$

Define $\widehat{\varphi} : (\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I_{\mathcal{A}} \longrightarrow (\mathcal{B}\widehat{\bigotimes}\mathcal{B})/I_{\mathcal{B}}$ via

$$\widehat{\varphi}(a \otimes b + I_{\mathcal{A}}) := \varphi(a) \otimes \varphi(b) + I_{\mathcal{B}}, \quad (a, b \in \mathcal{A}).$$

We can show that $\widehat{\varphi}$ is well defined and $\mathfrak{A}$-module morphism. Also for each $a \in \mathcal{A}$ we have

$$\begin{aligned}\varphi(a) \cdot N &= \varphi(a) \cdot (\sum_{i=1}^{n} \varphi(x_i) \otimes \varphi(y_i) + I_{\mathcal{B}}) \\ &= \sum_{i=1}^{n} \varphi(ax_i) \otimes \varphi(y_i) + I_{\mathcal{B}} \\ &= \widehat{\varphi}(\sum_{i=1}^{n} ax_i \otimes y_i + I_{\mathcal{A}}) \\ &= \widehat{\varphi}(a \cdot \sum_{i=1}^{n} x_i \otimes y_i + I_{\mathcal{A}}) \\ &= \widehat{\varphi}(a \cdot M) = \widehat{\varphi}(M \cdot a) \\ &= \sum_{i=1}^{n} \varphi(x_i) \otimes \varphi(y_i)\varphi(a) + I_{\mathcal{B}} \\ &= N \cdot \varphi(a).\end{aligned}$$

By density of the range of $\varphi$ and continuity it we have $b \cdot N = N \cdot b$, for all $b \in \mathcal{B}$. Therefore $\mathcal{B}$ is module contractible. □

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Department of Mathematics, Islamic Azad University, Garmsar Branch, Garmsar, Iran.

*E-mail address*: abasalt.bodaghi@gmail.com