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SOME PROPERTIES OF CONTINUOUS LINEAR OPERATORS IN TOPOLOGICAL VECTOR PN - SPACES

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Abstract. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance. Probabilistic Metric space was introduced by Karl Menger. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [1]. In this note we study the PN spaces which are topological vector spaces and the open mapping and closed graph Theorems in this spaces are proved.

1. Introduction and Preliminaries

Probabilistic normed spaces were first defined by Šerstnev in 1962 (see [16]). Their definition was generalized in [1]. We recall the definition of probabilistic normed space briefly as given in [1], together with the notation that will be needed (see [12]). We shall consider the space of all distance probability distribution functions (briefly, d.f.'s), namely the set of all left-continuous and non-decreasing functions from $\overline{\mathbb{R}}$ into $[0,1]$ such that $F(0) = 0$ and $F(+\infty) = 1$. Here as usual, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. The spaces of these functions will be denoted by Δ^+, while the subset $D^+ \subseteq \Delta^+$ will denote the set of all proper distance d.f.'s, namely those for which $\ell^- F(+\infty) = 1$. Here $\ell^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions i.e., $F \le G$ if and only if $F(x) \le G(x)$ for all x in $\mathbb{R}$. For any $a \ge 0$, ε_a is the d.f. given by

$$\varepsilon_a = \begin{cases} 0, & \text{if } x \le a, \\ 1, & \text{if } x > a. \end{cases}$$

The space Δ^+ can be metrized in several ways [12], but we shall here adopt the Sibley metric d_S. If F, G are d.f.'s and h is in $]0,1[$, let $(F, G; h)$ denote the condition:

$$G(x) \le F(x+h) + h \text{ for all} x \in \left]0, \frac{1}{h}\right[.$$

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Then the Sibley metric d_S is defined by

$$d_S(F,G) := \inf\{h \in]0,1[: \text{ both } (F,G;h) \text{ and } (G,F;h) \text{ hold}\}.$$

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+. A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, nondecreasing in each place and has ε_0 as identity, that is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$,
(TF2) $\tau(F,G) = \tau(G,F)$,
(TF3) $F \leq G \Longrightarrow \tau(F,H) \leq \tau(G,H)$,
(TF4) $\tau(F,\varepsilon_0) = \tau(\varepsilon_0,F) = F$.

Moreover, a triangle function is *continuous* if it is continuous in the metric space (Δ^+, d_S).

Typical continuous triangle functions are

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)),$$

and

$$\tau_{T^*}(F,G) = \inf_{s+t=x} T^*(F(s),G(t)).$$

Here T is a continuous t-norm, i.e. a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x,1-y)$. Let us recall among the triangular function one has the function defined via $T(x,y) = \min(x,y) = M(x,y)$ and $T^*(x,y) = \max(x,y)$ or $T(x,y) = \pi(x,y) = xy$ and $T^*(x,y) = \pi^*(x,y) = x + y - xy$.

Definition 1.1. A *Probabilistic Normed space* (briefly, PN space) is a (V,ν,τ,τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν is a mapping (the *probabilistic norm*) from V into Δ^+, such that for every choice of p and q in V the following hold:

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A PN space is called a Šerstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{\mid \alpha \mid}\right),$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ and $x > 0$. There is a natural topology in a PN-space (V,ν,τ,τ^*), called *strong topology*; it is defined for $p \in V$ and $t > 0$, by the neighborhoods

$$N_p(t) := \{q \in V; \nu_{q-p}(t) > 1 - t\} = \{q \in V; d_s(\nu_{q-p}, \varepsilon_0) < t\}.$$

A set A in the PN space (V, ν, τ, τ^*) is said to be D–*bounded* if its *probabilistic radius*, R_A belongs to D^+. The probabilistic radius of A is defined by

$$R_A(x) := \begin{cases} \ell^- \inf\{\nu_p(x) : p \in A\}, & x \in [0, +\infty) \\ 1, & x = +\infty. \end{cases}$$

Of course, if V is a normed space under the norm $\|\cdot\|$, then the set A may be bounded when regarded as a subset of the normed space $(V, \|\cdot\|)$. The two notions need not coincide (see [8]).

Definition 1.2. A subset A of a PN space which is a TV space is said to be *bounded* if for every $m \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that

$$A \subset k\mathcal{N}_\theta(1/m).$$

The papers [4, 17, 6, 5] on the relationship between the two types of boundedness ought also to be kept in mind.

The concept of paranorm is a generalization of that of absolute value. The paranorm of x may be thought of as the distance from x to 0.

Definition 1.3. [18] A paranorm is a real function $p : V \to \mathbb{R}$ where V is a vector space, and satisfying conditions (i) through (v) for all vectors a, b in V.

(i) $p(\theta) = 0$
(ii) $p(a) \geq 0$
(iii) $p(-a) = p(a)$
(iv) $p(a + b) \leq p(a) + p(b)$
(v) If t_n is a sequence of scalars with $t_n \to t$ and u_n is a sequence of vectors with $u_n \to u$, then $p(t_n u_n - tu) \to 0$ (continuity of multiplication).

A paranorm p for which $p(a) = 0$ implies $a = \theta$ will be called *total*.

One owes the following result to Alsina, Schweizer & Sklar ([2]).

Theorem 1.4. *Every* PN *space* (V, ν, τ, τ^*), *when it is endowed with the strong topology induced by the probabilistic norm* ν, *is a topological vector space if, and only if, for every* $p \in V$ *the map from* $\mathbb{R}$ *into* V *defined by*

$$\lambda \mapsto \lambda p \tag{1.1}$$

is continuous.

It was proved in [2, Theorem 4], that, if the triangle function τ^* is Archimedean, i.e., if τ^* admits no idempotents other than ϵ_0 and ϵ_∞ ([12]), then the mapping (1.1) is continuous and, as a consequence, the PN space (V, ν, τ, τ^*) is a TV space.

2. Some properties of continuous linear operator in topological vector PN spaces

Definition 2.1. A Probabilistic Total Paranormed space (briefly P tp-N space) is a triple (V, ν, τ), where V is real vector space, τ is a continuous triangle function and ν is a mapping (the probabilistic total paranorm) from V in to $\triangle^+$, such that for every choise of p and q in V the following hold:

(p1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(p2) $\nu_{-p} = \nu_p$;
(p3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(p4) If u_n and α_n be two sequence of vectors and scalars respectively and $u_n \to u$, $\alpha_n \to \alpha$; Then $\nu_{\alpha_n u_n - \alpha u} \to \varepsilon_0$.

Example 2.2. Suppose p is a total paranorm on the vector space V and let τ be a triangle function such that $\tau(\epsilon_a, \epsilon_b) \leq \epsilon_{a+b}$ and $\nu_a = \epsilon_{p(a)}$. Then (V, ν, τ) is a P tp-N space.

Proof. (p1) $\nu_a = \epsilon_0 \Leftrightarrow \epsilon_{p(a)} = \epsilon_0 \Leftrightarrow a = 0$.
(p2) $\nu_a = \epsilon_{p(a)} = \epsilon_{p(-a)} = \nu_{-a}$.
(p3) $\nu_{a+b} = \epsilon_{p(a+b)} \geq \epsilon_{p(a)+p(b)} \geq \tau(\epsilon_{p(a)}, \epsilon_{p(b)})$.
(p4) Suppose u_n and α_n be two sequences of vectors and scalars respectively and $u_n \to u$, $\alpha_n \to \alpha$.Then $\nu_{\alpha_n u_n - \alpha u} = \epsilon_{p(\alpha_n u_n - \alpha u)} \geq \epsilon_{\alpha_n(u_n - u) + p(\alpha_n - \alpha)} \to \epsilon_0$. □

Theorem 2.3. [3] *A PN-space is a topological vector space if, and only if, it is a P tp-N space.*

Definition 2.4. A continuous linear map on V into W, where V and W are topological vector space over $\mathbb{R}$, is called topological homomorphism,if each open subset $G \subset V$, the image $u(G)$ is an open subset of $u(V)$.

Definition 2.5. If X is a topological space, a set $E \subset X$ is meager set if E is a countable union of nowhere dense sets.

Theorem 2.6. *Let (V, ν_1, τ_1) and (W, ν_2, τ_2) be two P tp-N spaces. $T : V \to W$ be a continuous map of V with range dense in W. Then either $T(V)$ is meager in W or else $T(V) = W$. Also T is topological homomorphism.*

Proof. Let (V, ν_1, τ_1) be a P tp-N space. By theorem 2.3, (V, ν_1, τ_1) is topological vector space and the set $N_\theta(\frac{1}{n}), n \in N$ is a θ-neighborhood base. Let $N_\theta(\frac{1}{n_i}), i \in N$ be a base of circled θ-neighborhoods satisfying

$$N_\theta(\frac{1}{n_{i+1}}) + N_\theta(\frac{1}{n_{i+1}}) \subset N_\theta(\frac{1}{n_i}) \quad n \in N \tag{2.1}$$

For each nonempty finite subset K of N, define the circled θ-neighborhood V_K by $V_K = \sum_{n \in K} N_\theta(\frac{1}{n})$ and the real number P_K by $P_K = \sum_{n \in K} 2^{-n}$. We define the real valued function $x \to \mid x \mid$ on V by $\mid x \mid = 1$ if x is not contained in any V_K and otherwise by

$$\mid x \mid = \inf_K \{P_K, x \in V_K\}.$$

Then $\mid \cdot \mid$ is a pseudo-norm on V, similarly we can find a pseudo-norm $\mid \cdot \mid$ on W. The family $\{S_r, r > 0\}$ is a θ-neighborhood in V. For fixed r, let $U_1 = S_r$ and $U_2 = S_{\frac{r}{2}}$. Then $U_2 + U_2 \subset U_1$ and $T(V) = \cup_1^\infty nT(U_2)$.
since by assumption $T(V)$ is a complete metric space there exist $n \in N$ such that $\overline{[nT(U_2)]}$ has an interior point; Hence $\overline{[T(U_2)]}$ has an interior point. Now

$$\overline{[T(U_2)]} + \overline{[T(U_2)]} \subset \overline{[T(U_2) + T(U_2)]}.$$

Hence there exist $\zeta > 0$ such that $T(V) \cap S_\zeta \subset T(S_{r+\varepsilon})$ for every $\varepsilon > 0$. Thus T is a topological homomorphism and $T(V) = W$. □

Corollary 2.7. *A continuous linear map T of a P tp-N space (V, ν_1, τ_1) in to another such space (W, ν_2, τ_2) is a topological homomorphism iff $T(V)$ is closed in W.*

Proof. $T(V)$ is isomorphic with $\frac{V}{T^{-1}(0)}$ and hence in W. conversely if $T(V)$ is closed in W, then it is P tp-N space and hence can replace W in preceding theorem. □

Theorem 2.8. *If (V, ν_1, τ_1) and (W, ν_2, τ_2) are two P tp-N spaces. $T : V \to W$ is a continuous surjection and G is open in V then $T(G)$ is open in W.*

Proof. We first show for every $r > 0$;

$$o \in int\overline{[T(N_\theta(r))]}. \tag{2.2}$$

Not that because T is surjective we have,

$$W = \cup_{k=1}^{\infty}\overline{[T(N_\theta(k\frac{r}{2}))]} = \cup_{k=1}^{\infty}k\overline{[T(N_\theta(\frac{r}{2}))]}.$$

By the Baire Category theorem, there is a $k \geq 1$ such that $k\overline{[T(N_\theta(\frac{r}{2}))]}$ has nonempty interior thus $M = int\overline{[T(N_\theta(\frac{r}{2}))]} \neq \emptyset$. If $q_0 \in M$, let $s > 0$ such that $N_{q_0}(S) \subseteq \overline{[T(N_\theta(\frac{r}{2}))]}$. Let $q \in W$, $q \in N'_\theta(S)$ where θ' is the null vector in W. Since $q_0 \in \overline{[T(N_\theta(\frac{r}{2}))]}$, there exist a sequence $\{p_n\}$ in $N_\theta(\frac{r}{2})$ such that $T(p_n) \to q_0$. There is also a sequence $\{p'_n\}$ in $N_\theta(\frac{r}{2})$ such that $T(p'_n) \to q_0 + q$. Thus $T(p'_n - p_n) \to q$ and $\{p'_n - p_n\} \subseteq N_\theta(r)$; that is $N_\theta(r) \subseteq \overline{[T(N_\theta(r))]}$. This establish claim (2.2). Now it will be shown that

$$\overline{[T(N_\theta(\frac{r}{2}))]} \subseteq \overline{[T(N_\theta(r))]}. \tag{2.3}$$

Note that if (2.3) is proved, then claim (2.2) implies that $o \in int[T(N_\theta(r))]$, for any $r > 0$ and the theorem is proved. Indeed, if G is an open subset of V, then for every $p \in G$ let $r_p > 0$ such that $N_p(r_p) \subseteq G$. But $o \in int[T(N_\theta(r_p))]$ and so $T(p) \in intT(N_p(r_p))$.
Thus there is an $s_p > 0$ such that $U_p \equiv N_{T(p)}(s_p) \subseteq T(N_p(r_p))$. Therefore $\cup_{p\in G}\{U_p\} \subseteq T(G)$. But $T(p) \in U_p$ so $\cup_{p\in G}\{U_p\} = T(G)$ and hence $T(G)$ is open.
To prove (2.3) fix $q_1 \in \overline{[T(N_\theta(\frac{r}{2}))]}$, by (2.2) $0 \in int\overline{[T(N_\theta(\frac{r}{4}))]}$. Hence

$$[q_1 - \overline{[T(N_\theta(\frac{r}{2}))]}] \cap T(N_\theta(\frac{r}{2})) \neq \emptyset.$$

Let $p_1 \in N_\theta(\frac{r}{2})$ such that $T(p_1) \in [q_1 - \overline{[T(N_\theta(\frac{r}{4}))]}]$; Now $T(p_1) = q_1 - q_2$ where $q_2 \in \overline{[T(N_\theta(\frac{r}{4}))]}$. Using induction, we obtain a sequence $\{p_n\}$ in V and a sequence $\{q_n\}$ in W such that,

(i) $p_n \in N_\theta(\frac{r}{2^n})$.
(ii) $q_n \in \overline{[T(N_\theta(\frac{r}{2^n}))]}$.
(iii) $q_{n+1} = q_n - T(q_n)$.

But $d_s(\nu_{p_n}, \varepsilon_0) < \frac{r}{2^n}$, so $\sum_1^\infty d_s(\nu_{p_n}, \varepsilon_0) < \infty$. Hence $p = \sum_{n=1}^\infty p_n$ exist in V and $d_s(\nu_p, \varepsilon_0) < r$. Also

$$\sum_{k=1}^{n} T(p_n) = \sum_{k=1}^{n}(q_n - q_{n+1}) = q_1 - q_{n+1}.$$

The relation (iii) implies that $q_n \to 0$, therefore

$$q_1 = \sum_{1}^{\infty} T(p_k) = T(p) \in T(N_\theta(r)).$$

This proving (2.3) and completing the proof of the theorem. □

Theorem 2.9. (Prochaska)[7]. *Šerstnev space (V, ν, τ) with $\nu(V) \subseteq D^+$ and $\tau = \tau_M$ is a locally convex space.*

Corollary 2.10. *If (V, ν_1, τ_M) and (W, ν_2, τ_M) be two Šerstnev spaces with $\nu_1(V) \subseteq D^+$ and $\nu_2(W) \subseteq D^+$. Let $T : V \to W$ is a continuous surjection, and G is an open subset of V, then $T(G)$ is open.*

Proof. Theorems 2.7 and 2.8. □

Definition 2.11. [8]. A linear map $T : (V, \nu_1, \tau_1) \to (W, \nu_2, \tau_2)$ between two PN spaces is said to be strongly bounded if there exist a constant $k > 0$ such that for every $p \in V$ and foe every $x > 0$,

$$\nu_{2_{T(p)}}(x) \geq \nu_{1_p}(\frac{x}{k}).$$

Lemma 2.12. [8]. *Every strongly bounded linear operator $T : (V, \nu_1, \tau_1) \to (W, \nu_2, \tau_2)$ between two PN spaces is continuous with respect to the strong topology in (V, ν_1, τ_1) and (W, ν_2, τ_2) respectively.*

Theorem 2.13. *Let (V, ν_1, τ_1) and (W, ν_2, τ_2) be two P tp-N spaces and $T : V \to W$ is strongly abounded linear operator that is bijective, then T^{-1} is strongly bounded.*

Proof. The operator T is continuous by Lemma 2. 11. Now the result follows from theorem 2.7. □

Theorem 2.14. *Let (V, ν_1, τ) and (W, ν_2, τ) be two P tp-N spaces under the same triangular function τ, and $T : V \to W$ is a linear operator such that the graph of T is closed, where*

$$graph\ T \equiv \{(p, T(p)), p \in V\}.$$

Then T is continuous.

Proof. Let $G = graphT$, since $(V \times W, \mu^\tau, \tau)$ where $\mu^\tau : V \times W \longrightarrow \triangle^+$ is defined by $\mu^\tau(p, q) = \tau(\nu_p, \nu'_q)$, is a P tp-N space and G is closed, G is complete. Define $\rho : G \to V$ by $\rho(p, T(p)) = p$. It is easy to check that ρ is bounded and bijective. By the preceding theorem, $\rho^{-1} : V \longrightarrow G$ is continuous. $T : V \to W$ is the composition of the continuous map $\rho^{-1} : V \longrightarrow G$ and the continuous map of $G \to W$ defined by $(p, T(p)) \longrightarrow T(p)$. Therefore T is continuous. □

Corollary 2.15. *Let (V, ν_1, τ_M) and (W, ν_2, τ_M) be two Šerstnev spaces with $\nu_1(V) \subseteq D^+$ and $\nu_2(W) \subseteq D^+$ and $T : V \to W$ be a linear operator such that graph of T is closed. Then T is continuous.*

Proof. It is obviously obtained from Theorems 2.8 and Theorem 2.13. □

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