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HYERS–ULAM STABILITY OF VOLTERRA INTEGRAL EQUATION

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

ABSTRACT. We will apply the successive approximation method for proving the Hyers–Ulam stability of a linear integral equation of the second kind.

1. Introduction

We say a functional equation is stable if for every approximate solution, there exists an exact solution near it. In 1940 Ulam [13] posed the following problem concerning the stability of functional equations: We are given a group G and a metric group G' with metric $\rho(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$\rho(f(xy), f(x)f(y)) < \delta,$$

for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$? The problem for the case of the approximately additive mappings was solved by Hyers [4], when G and G' are Banach space. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [15, 7, 8, 12, 10]). The interested reader can also find further details in the book of Kuczma ([9], chapter XVII). In this paper, we study the Hyers–Ulam stability for the linear Volterra integral equation of second kind. Jung was the author who investigated the Hyers–Ulam stability of Volterra integral equation on any compact interval. In 2007, he proved in [7] the following:

Given $a \in \mathbb{R}$ and $r > 0$, let $I(a; r)$ denoted a closed interval $\{x \in \mathbb{R} \mid a - r \leq x \leq a + r\}$ and let $f : I(a; r) \times \mathbb{C} \to \mathbb{C}$ be a continuous function which satisfies a Lipschitz condition $|f(x, y) - f(x, z)| \leq L|y - z|$ for all $x \in I(a; r)$ and $y, z \in \mathbb{C}$, where L is a constant with $0 < Lr < 1$. If a continuous function $y : I(a; r) \to \mathbb{C}$ satisfies

$$|y(x) - b - \int_a^x f(x, t, u(t))dt| \leq \epsilon,$$

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for all $x \in I(a;r)$ and for some $\epsilon \geq 0$, where b is a complex number, then there exists a unique continuous function $u : I(a;r) \to \mathbb{C}$ such that

$$y(x) = b + \int_a^x f(x,t,u(t))dt, \quad |u(x) - y(x)| \leq \frac{\epsilon}{1 - Lr},$$

for all $x \in I(a;r)$. Recently, Y. Li and L. Hua [10] proved the stability of Banach's fixed point theorem.

The purpose of the this work is to discuss the Hyers–Ulam stability of the non homogeneous linear Volterra integral equation (2.1), where $x \in I = [a,b]$, $-\infty \leq a < b \leq \infty$. We will use the successive approximation method, to prove that equation (2.1) has the Hyers–Ulam stability under some appropriate conditions. The method of this work is distinctive. It is simpler and clearer than the previous work.

2. Basic Concepts

Consider the following Volterra integral equation of the second kind,

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \equiv T(u). \tag{2.1}$$

We assume that f is a continuous function on the interval $[a,b]$, and also k is continuous on the triangular $D = \{(x,t) : \; x \in [a,b] \; , \; t \in [a,x]\}$. We work with the complete metric space $X = C[a,b]$ of continuous functions that its metric $d(x,y)$ is

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|, \qquad f,g \in C[a,b].$$

Definition 2.1. (cf. [15, 7]). We say that equation (2.1) has the Hyers–Ulam stability if there exists a constant $K \geq 0$ with the following property: for every $\epsilon \geq 0$, $y \in C[a,b]$, if

$$|y(x) - f(x) - \lambda \int_a^x k(x,t)u(t)dt| \leq \epsilon,$$

then there exists some $u \in C[a,b]$ satisfying $u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt$ such that

$$|u(x) - y(x)| \leq K\epsilon.$$

We call such K a Hyers–Ulam stability constant for equation (2.1).

3. A Contractive Mapping for the Volterra Equation

We will show here that T^n of (2.1) is contractive when n is large.

Theorem 3.1. *The mapping T^n is contractive when n is sufficiently large.*

Proof: We write

$$\begin{aligned} T(u) &= f(x) + \lambda \int_a^x k(x,\zeta)u(\zeta)d\zeta, \\ T^2(u) &= f(x) + \lambda \int_a^x k(x,\zeta)[f(\zeta) + \lambda \int_a^\zeta k(\zeta,t)u(t)dt]d\zeta \\ &= f(x) + \lambda \int_a^x k(x,\zeta)f(\zeta)d\zeta + \lambda^2 \int_a^x \int_a^\zeta k(x,\zeta)k(\zeta,t)u(t)dtd\zeta \\ &= f(x) + \lambda \int_a^x k(x,\zeta)f(\zeta)d\zeta + \lambda^2 \int_a^x k_2(x,\zeta)u(\zeta)d\zeta, \end{aligned}$$

where $k_2(x,\zeta) = \int_\zeta^x k(x,t)k(t,\zeta)dt$.

If we repeat this successive process to $T^n(u)$, we have

$$\begin{aligned} T^n(u) = f(x) &+ \lambda \int_a^x k_1(x,\zeta)f(\zeta)d\zeta + \lambda^2 \int_a^x k_2(x,\zeta)f(\zeta)d\zeta \\ &+ \cdots + \lambda^{n-1} \int_a^x k_{n-1}(x,\zeta)f(\zeta)d\zeta + \lambda^n \int_a^x k_n(x,\zeta)u(\zeta)d\zeta, \end{aligned}$$

where $k_{n+1}(x,\zeta) = \int_\zeta^x k(x,t)k_n(t,\zeta)dt$, $k_1(x,\zeta) = k(x,\zeta)$.

$$\begin{aligned} T^n(u) = f(x) &+ \lambda \int_a^x k_1(x,\zeta)f(\zeta)d\zeta + \lambda^2 \int_a^x k_2(x,\zeta)f(\zeta)d\zeta \\ &+ \cdots + \lambda^{n-1} \int_a^x k_{n-1}(x,\zeta)f(\zeta)d\zeta + \lambda^n \int_a^x k_n(x,\zeta)u(\zeta)d\zeta, \end{aligned}$$

where $k_{n+1}(x,\zeta) = \int_\zeta^x k(x,t)k_n(t,\zeta)dt$, $k_1(x,\zeta) = k(x,\zeta)$.

Clearly, we have

$$|T^n(u) - T^n(v)| \leq |\lambda|^n \int_a^x |k_n(x,\zeta)||u(\zeta) - v((\zeta)|d\zeta, \tag{3.1}$$

since $k_1(x,\zeta) = k(x,\zeta)$ is assumed continuous on domain D, we can conclude that $k_1(x,\zeta)$ is bounded by some positive number M, $|k_1(x,\zeta)| \leq M$. In the other hand, we can show the following bound for the iterated kernel $k_n(x,\zeta)$:

$$|k_n(x,\zeta)| \leq \frac{M^n}{(n-1)!}(x-\zeta)^{n-1}, \qquad a \leq \zeta \leq x. \tag{3.2}$$

With this result (3.2) and the result (3.1), we write

$$\begin{aligned} d(T^n(u),T^n(v)) &= \max_x |T^n(u)-T^n(v)| \\ &= \max_x |\lambda^n|| \int_a^x k_n(x,\zeta)[u(\zeta)-v(\zeta)]d\zeta| \\ &\le |\lambda|^n \max_x \int_a^x |k_n(x,\zeta)||u(\zeta)-v(\zeta)|d\zeta \\ &\le |\lambda|^n \max_x \int_a^x \frac{M^n}{(n-1)!}(x-\zeta)^{n-1}|u(\zeta)-v(\zeta)|d\zeta \\ &\le |\lambda|^n M^n \max_x \{|u(\zeta)-v(\zeta)| \int_a^x \frac{(x-\zeta)^{n-1}}{(n-1)!}d\zeta\} \\ &\le |\lambda|^n M^n \frac{(b-a)^n}{n!} d(u,v). \\ & \quad d(T^n(u),T^n(v)) \le \alpha d(u,v), \end{aligned}$$

where $\alpha = |\lambda|^n M^n \frac{(b-a)^n}{n!}$. Clearly, for sufficiently large n, $\alpha < 1$. Hence T^n is a contractive operator. □

4. Main Results

Theorem 4.1. *The mapping $T : X \to X$ defined in (2.1), has a unique fixed point, u, and $\{T^n(x)\}_1^\infty$ converges to u for each $x \in X$.*

Proof: By theorem (3.1) for enough large n, T^n is a contractive mapping. Let $T^n \equiv S$. Hence the equation $Sx = x$ has a unique fixed point u. This means that with the initial estimation of ξ, we have the sequence $u_{k+1} = S(u_k) = S^k(\xi)$ converging to u, that is,

$$u = \lim_{k\to\infty} u_{k+1} = \lim_{k\to\infty} S^k(\xi) = \lim_{k\to\infty} (T^n)^k(\xi) = \lim_{k\to\infty} T^{nk}(\xi). \tag{4.1}$$

In (4.1), ξ is arbitrary, so we may choose it to be $\xi = T(u)$,

$$u = \lim_{k\to\infty} T^{nk}(\xi) = \lim_{k\to\infty} T^{nk}(T(u)) = \lim_{k\to\infty} T[T^{nk}(u)] = T[\lim_{k\to\infty} T^{nk}(u)] = T(u). \tag{4.2}$$

Hence (4.2) concludes the existence of the solution u to $T(u) = u$. To prove that u is unique, let γ, β, be two different solution to equation $T(x) = x$ [*i.e.*, $\gamma = T(\gamma), \beta = T(\beta)$]. But since $\gamma = T(\gamma)$, then

$$T^n(\gamma) = T^{n-1}(T(\gamma)) = T^{n-1}(\gamma) = \cdots = T(\gamma) = \gamma.$$

The same can be shown for β,

$$T^n(\beta) = \beta.$$

But since T^n is known to be contractive, it must have a unique solution which forces $\gamma = \beta$. Hence the equation $T(x) = x$ has a unique solution. □

Theorem 4.2. *The equation $(T - I)x = 0$, defined by (2.1), has the Hyers–Ulam stability, that is for $\epsilon \ge 0$, if*

$$d(T\xi, \xi) \le \epsilon,$$

then there exists an unique $u \in X$ *satisfying*

$$Tu - u = 0,$$

with

$$d(\xi, u) \leq K\epsilon,$$

for some $K \geq 0$.

Proof: In first, we consider the iterative integral equation

$$u_{n+1}(x) = f(x) + \lambda \int_a^x k(x,t)u_n(t)dt \equiv T(u_n), \qquad n = 1, 2, \cdots .$$

Hence

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &= |\lambda \int_a^x k(x,t)(u_n(t) - u_{n-1}(t))dt| \\ &\leq |\lambda| \int_a^x |k(x,t)||u_n(t) - u_{n-1}(t)|dt \\ &\leq |\lambda| M \int_a^x |u_n(t) - u_{n-1}(t)|dt \end{aligned}$$

For $n = 2$, we have

$$\begin{aligned} |u_3(x) - u_2(x)| &\leq |\lambda| M \int_a^x |u_2(t) - u_1(t)|dt \\ &\leq |\lambda| M \; d(Tu, u) \int_a^x dt \\ &\leq |\lambda| M(x-a) \; d(Tu, u). \end{aligned}$$

$$d(T^2u, Tu) = d(u_3, u_2) \leq |\lambda| M(b-a) \; d(Tu, u).$$

If we repeat this process, we have

$$\begin{aligned} d(T^n u, T^{n-1}u) &\leq \frac{(|\lambda| M(b-a))^{n-1}}{(n-1)!} \; d(Tu, u) \\ &= \frac{(L)^{n-1}}{(n-1)!} \; d(Tu, u), \end{aligned}$$

where $L = |\lambda| M(b-a)$. Now by using theorem (4.1), T has a unique fixed point $u \in X$, and $\{T^n(x)\}_1^\infty$ converges to u for each $x \in X$. Hence the equation $Tx = x$ has a unique solution on X. If $\epsilon \geq 0$ is given and $d(T\xi, \xi) \leq \epsilon$, then there is a integer number n such that $d(T^n\xi, u) \leq \epsilon$. Thus

$$\begin{aligned} d(\xi, u) &\leq d(\xi, T^n\xi) + d(T^n\xi, u) \\ &\leq d(\xi, T\xi) + d(T\xi, T^2\xi) + d(T^2\xi, T^3\xi) + ... + d(T^{n-1}\xi, T^n\xi) + d(T^n\xi, u) \\ &\leq d(\xi, T\xi) + \frac{L}{1!}d(\xi, T\xi) + \frac{L^2}{2!}d(\xi, T\xi) + \cdots + \frac{L^{n-1}}{(n-1)!}d(\xi, T\xi) + d(T^n\xi, u) \\ &\leq d(\xi, T\xi)(1 + \frac{L}{1!} + \frac{L^2}{2!} + \cdots + \frac{L^{n-1}}{(n-1)!}) + \epsilon \\ &\leq \epsilon(e^L) + \epsilon = (1 + e^L)\epsilon = K\epsilon. \end{aligned}$$

which completes the proof. □

Corollary 4.3. *For infinite interval, the theorem (4.2), is not true necessarily. For example, the exact solution of the integral Equation.* $u(x) = 1 + \int_a^x u(t)dt \equiv T(u)$, $x \in [0, \infty)$, *is* $u(x) = e^x$. *By choosing* $\epsilon = 1$ *and* $\xi(x) = 0$, $T(\xi) = 1$ *is obtained, so* $d(T(\xi), \xi) \leq \epsilon = 1$, $d(\xi, u) = \infty$. *Hence, there exists no Hyers–Ulam stability constant* $K \geq 0$ *such that the relation* $d(\xi, u) \leq K\epsilon$ *be true.*

Corollary 4.4. *The theorem (4.2) holds for every finite interval* $[a, b]$, $[a, b)$, $(a, b]$ *and* (a, b), *when* $-\infty < a < b < \infty$.

Corollary 4.5. *If we apply the successive approximation method for solving (2.1) and* $u_i(x) = u_{i+1}(x)$ *for some* $i = 1, 2, \cdots$, *then* $u(x) = u_i(x)$, *such that* $u(x)$ *is the exact solution of (2.1).*

5. CONCLUSION

Let $I = [a, b]$ be a finite interval, $X = C[a, b]$ and $y = Ty$ be an integral equation which $T : X \to X$ is a linear integral map. In this paper, we showed that T has the Hyers–Ulam stability by means that, if y° be an approximate solution of the integral equation and $d(y^\circ, Ty^\circ) \leq \epsilon$ for all $t \in I$ and $\epsilon \geq 0$, then $d(y^*, y^\circ) \leq K\epsilon$, which y^* is the exact solution and K is positive constant.

6. IDEAS

We extend (2.1) into

$$u(x) = f(x) + \varphi(\int_a^x F(x, t, u(t))dt), \tag{6.1}$$

where $\varphi : X = C[a, b] \to X = C[a, b]$ is a map. It is an open problem that " What we can say about the Hyers–Ulam stability of the general nonlinear Volterra integral equation (6.1)?"

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