Int. J. Nonlinear Anal. Appl. **1** (2010) No.2, 26–35
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

HYERS-ULAM STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION

E. ELQORACHI[1]*, Y. MANAR[1] AND TH. M. RASSIAS[2]

Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

Abstract. In the present paper a solution of the generalized quadratic functional equation

$$f(kx+y)+f(kx+\sigma(y))=2k^2f(x)+2f(y), \quad x,y\in E$$

is given where σ is an involution of the normed space E and k is a fixed positive integer. Furthermore we investigate the Hyers-Ulam-Rassias stability of the functional equation. The Hyers-Ulam stability on unbounded domains is also studied. Applications of the results for the asymptotic behavior of the generalized quadratic functional equation are provided.

1. Introduction

In 1940, S. M. Ulam [28] raised a question concerning the stability of group homomorphisms: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h: G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphisms $a: G_1\to G_2$ with $d(h(x),a(x))<\epsilon$ for all $x\in G_1$?*

The first partial solution to Ulam's problem was given by Hyers [8] under the assumption that G_1 and G_2 are Banach spaces.

Hyers proved that each solution of the functional inequality
$\|f(x+y)-f(x)-f(y)\|\le\epsilon$, for all x, $y\in G_1$, can be approximated by an additive function a, given by the formula $a(x)=\lim_{n\to+\infty}2^{-n}f(2^nx)$.

In 1978, Th. M. Rassias [16] provided a generalization of Hyers's stability theorem which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.1. [16] *Let $f\colon V\longrightarrow X$ be a mapping between Banach spaces and let $p<1$ be fixed. Suppose f satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\|\le\theta(\|x\|^p+\|y\|^p) \tag{1.1}$$

Date: Received: February 2010; Revised: May 2010.
2000 *Mathematics Subject Classification.* Primary 39B52; Secondary 39B82.
Key words and phrases. Hyers-Ulam-Rassias stability; quadratic functional equation.
*: Corresponding author.

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T\colon V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2^p - 2}\|x\|^p \tag{1.2}$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).
If, in addition, $f(tx)$ is continuous in t for each fixed $x \in V$, then T is linear.

A particular case of Rassias's theorem regarding the Hyers-Ulam stability of the additive mapping was proved by Aoki [1].
The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. Since then, a great deal of works has been published by a number of mathematicians for other functional equations (see for example [4], [5], [6], [7], [10], [14], [17], [25] and the references therein).
A Hyers-Ulam stability theorem for the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x, y \in E, \tag{1.3}$$

was proved by Skof [26] and later by S. M. Jung [13] on unbounded domains.
In [5], Czerwik proved the Hyers-Ulam-Rassias stability of equation (1.3).
Recently, the functional equation

$$f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y), \quad x, y \in E, \tag{1.4}$$

was solved by J.-R. Lee et al. [12]. Indeed ,they proved the Hyers-Ulam-Rassias stability theorem of equation (1.4).
Throughout this paper, let k denote a fixed positive integer. Let E and F be a vector space and a Banach space, respectively. Suppose $\sigma : E \to E$ is an automorphism of E such that $\sigma(\sigma(x)) = x$, for all $x \in E$.
The purpose of the present paper is to extend the results mentioned due to Jung Rye Lee et al [12] to the generalized quadratic functional equation

$$f(kx + y) + f(kx + \sigma(y)) = 2k^2 f(x) + 2f(y), \quad x, y \in E. \tag{1.5}$$

It's clear that equation (1.5) is a proper extension of equation (1.4) ($\sigma = -I$) and equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in E. \tag{1.6}$$

Equation (1.6) has been studied by Stetkær [27] and the Hyers-Ulam-Rassias Theorem has been obtained by Bouikhalene et al. [2, 3]. So in this paper we consider the case: $k \geq 2$.
Our results are organized as follows: In section 2, we determine the general solution of the functional equation (1.5). In section 3, we prove the Hyers-Ulam-Rassias stability of the equation (1.5) in Banach spaces. In section 4, we obtain the Hyers-Ulam stability of equation (1.5) on unbounded domains.

2. GENERAL SOLUTION OF THE GENERALIZED QUADRATIC FUNCTIONAL EQUATION (1.5)

In this section we solve the functional equation (1.5) by means of solutions of equation (1.6).

Theorem 2.1. *Let $k \in \mathbb{N}\backslash\{0,1\}$. Let E and F be two vector spaces. A mapping $f : E \to F$ satisfies the functional equation*

$$f(kx+y)+f(kx+\sigma(y)) = 2k^2f(x)+2f(y), \; x,y \in E \tag{2.1}$$

if and only if $f : E \to F$ satisfies

$$f(x+y)+f(x+\sigma(y)) = 2f(x)+2f(y) \; \text{ and } \; f(x+\sigma(x)) = 0 \text{ for all } x,y \in E. \tag{2.2}$$

Proof. Putting $x = 0$ and $y = 0$ in (2.1), we see that $f(0) = 0$.
Putting $y = 0$ in (2.1), we obtain

$$f(kx) = k^2f(x), \tag{2.3}$$

for all $x \in E$. Letting $x = 0$ in (2.1), we get $f(\sigma(y)) = f(y)$, for all $y \in E$.
We can now prove the first part of (2.2)

$$f(kx+y)+f(kx+\sigma(y)) = 2k^2f(x)+2f(y) = 2f(kx)+2f(y) \tag{2.4}$$

for all $x, y \in E$. So the mapping f satisfies (1.6).
Now, we will prove that f satisfies $f(x+\sigma(x)) = 0$ for all $x \in E$. By applying the inductive argument, we show that

$$f(n(x+\sigma(x))) = nf(x+\sigma(x)) \tag{2.5}$$

for all $x \in E$ and for all $n \in \mathbb{N}$. Replacing x and y by $x+\sigma(x)$ in (1.6), we find $f(2(x+\sigma(x))) = 2f(x+\sigma(x))$. Writting $n(x+\sigma(x))$ instead of x and $x+\sigma(x)$ instead of y in (1.6), we get

$$2f((n+1)(x+\sigma(x))) = 2f(n(x+\sigma(x)))+2f(x+\sigma(x)) = 2nf(x+\sigma(x))+2f(x+\sigma(x))$$
$$= 2(n+1)f(x+\sigma(x)).$$

This proves the validity of (2.5) for all $n \in \mathbb{N}$.
By using (2.3) and (2.5), we obtain $k^2f(x+\sigma(x)) = kf(x+\sigma(x))$, for all $x \in E$, since $k \geq 2$, then we get $f(x+\sigma(x)) = 0$, for all $x \in E$.
We shall now prove the converse. Let $f : E \to F$ be a solution of equation (2.2).
Replacing x by $(n-1)x$ and y by $x+\sigma(x)$ in (2.2), we obtain

$$f(nx+\sigma(x)) = f((n-1)x), \tag{2.6}$$

for all $x \in E$ and for all $n \in \mathbb{N}^*$.
We will prove by mathematical induction that

$$f(nx) = n^2f(x), \; n \in \mathbb{N}. \tag{2.7}$$

By letting $x = y$ in (2.2), we obtain (2.7) for $n = 2$. The inductive step must be demonstrated to hold true for the integer $n+1$.
By using (2.6) and (1.6) we find that

$$f(nx+x)+f(nx+\sigma(x)) = 2f(nx)+2f(x)$$
$$f((n+1)x)+f(nx+\sigma(x)) = 2n^2f(x)+2f(x)$$
$$f((n+1)x)+f((n-1)x) = 2n^2f(x)+2f(x)$$
$$f((n+1)x)+(n-1)^2f(x) = 2n^2f(x)+2f(x).$$

Finally, we get

$$f((n+1)x) = (n+1)^2f(x), \; \text{proving (2.7)}.$$

By using (1.6) and (2.7) we prove that f is a solution of equation (1.5).

$$f(kx+y)+f(kx+\sigma(y)) = 2f(kx)+2f(y) = 2k^2f(x)+2f(y), x,y\in E.$$

This completes the proof of Theorem 2.1. □

Corollary 2.2. [12] *Let $k\in\mathbb{N}\backslash\{0\}$. Let E and F be two vector spaces. A mapping $f: E\to F$ satisfies the functional equation*

$$f(kx+y)+f(kx-y) = 2k^2f(x)+2f(y),\ x,y\in E \tag{2.8}$$

if and only if $f: E\to F$ satisfies the quadratic functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y),\ x,y\in E. \tag{2.9}$$

Corollary 2.3. *Let $k\in\mathbb{N}\backslash\{0,1\}$. Let E and F be two vector spaces. A mapping $f: E\to F$ satisfies the functional equation*

$$f(kx+y) = k^2f(x)+f(y),\ x,y\in E \tag{2.10}$$

if and only if $f\equiv 0$.

3. Hyers-Ulam-Rassias stability of equation (1.5)

In this section we investigate the Hyers-Ulam-Rassias stability of the functional equation (1.5).

Theorem 3.1. *Let E be an abelian group, F a Banach space and $f: E\to F$ a mapping which satisfies the inequality*

$$\|f(kx+y)+f(kx+\sigma(y))-2k^2f(x)-2f(y)\|\le\delta, \tag{3.1}$$

for all $x,y\in E$ and for some $\delta\ge 0$. Then there exists a unique mapping $q: E\to F$ solution of (1.5) such that

$$\|f(x)-q(x)\|\le\frac{\delta}{2k^2}\frac{k^2+1}{k^2-1},\quad x\in E. \tag{3.2}$$

Proof. By letting $y=0$ (resp. $x=y=0$) in (3.1), we get respectively

$$\|f(x)-\frac{1}{k^2}\{f(kx)-f(0)\}\|\le\frac{\delta}{2k^2},\quad x\in E. \tag{3.3}$$

$$\|f(0)\|\le\frac{\delta}{2k^2},\quad x\in E. \tag{3.4}$$

By triangle inequality, we deduce that

$$\|f(x)-\frac{1}{k^2}\{f(kx)\}\|\le\frac{\delta}{2k^2}+\frac{\delta}{2k^4},\quad x\in E. \tag{3.5}$$

By applying the inductive assumption we prove

$$\|f(x)-\frac{1}{k^{2n}}\{f(k^nx)\}\|\le\frac{\delta}{2k^2}(1+\frac{1}{k^2})[1+\frac{1}{k^2}+\ldots+\frac{1}{k^{2(n-1)}}] \tag{3.6}$$

for all $n\in\mathbb{N}$. From (3.5) it follows that (3.6) is true for $n=1$. Assume now that (3.6) holds for $n\in\mathbb{N}$. The inductive step must be demonstrated to hold for $n+1$, that is

$$\|f(x)-\frac{1}{k^{2(n+1)}}\{f(k^{n+1}x)\}\|$$

$$\leq \|f(x)-\frac{1}{k^{2n}}\{f(k^nx)\}\|+\frac{1}{k^{2n}}\|f(k^nx)-\frac{1}{k^2}\{f(k^{n+1}x)\}\|$$

$$\leq \frac{\delta}{2k^2}(1+\frac{1}{k^2})[1+\frac{1}{k^2}+\ldots+\frac{1}{k^{2(n-1)}}]+\frac{1}{k^{2n}}\frac{\delta}{2k^2}(1+\frac{1}{k^2})$$

$$=\frac{\delta}{2k^2}(1+\frac{1}{k^2})[1+\frac{1}{k^2}+\ldots+\frac{1}{k^{2n}}].$$

This proves the validity of the inequality (3.6).
Let us define the sequence of functions

$$f_n(x)=\frac{1}{k^{2n}}\{f(k^nx)\},\ x\in E,\ n\in\mathbb{N}.$$

We will show that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x\in E$. In fact, by using (3.5) one has

$$\|f_{n+1}(x)-f_n(x)\|=\|\frac{1}{k^{2(n+1)}}\{f(k^{n+1}x)\}-\frac{1}{k^{2n}}\{f(k^nx)\}\|$$

$$=\frac{1}{k^{2n}}\|f(k^nx)-\frac{1}{k^2}\{f(k^{n+1}x)\}\|\leq\frac{\delta}{2k^2}(1+\frac{1}{k^2})\frac{1}{k^{2n}}.$$

Since $\frac{1}{k}<1$, it follows that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x\in E$. However, F is a complete normed space, thus the limit function $q(x)=\lim_{n\to+\infty}f_n(x)$ exists for every $x\in E$.
We will now prove that q is a solution of equation (1.5). Let x,y be two elements of E. From (3.1) and the definition of f_n it follows that

$$\|f_n(kx+y)+f_n(kx+\sigma(y))-2k^2f_n(x)-2f_n(y)\|$$

$$=\frac{1}{k^{2n}}\|f(kk^nx+k^ny)+f(kk^nx+\sigma(k^ny))-2k^2f(k^nx)-2f(k^ny)\|\leq\frac{\delta}{k^{2n}}.$$

By letting $n\to+\infty$, we get the equality

$$q(kx+y)+q(kx+\sigma(y))=2k^2q(x)+2q(y),\quad x,y\in E.$$

Assume now that there exist two mapping $q_i:E\to F$ $(i=1,2)$ satisfying (1.5) and (3.2).
By mathematical induction, we can easily verify that

$$q_i(k^nx)=k^{2n}q_i(x),\ (i=1,2).\tag{3.7}$$

For all $x\in E$ and all $n\in\mathbb{N}$, we have

$$\|q_1(x)-q_2(x)\|=\frac{1}{k^{2n}}\|q_1(k^nx)-q_2(k^nx)\|$$

$$\leq\frac{1}{k^{2n}}\|q_1(k^nx)-f(k^nx)\|+\frac{1}{k^{2n}}\|q_2(k^nx)-f(k^nx)\|$$

$$\leq\frac{\delta}{k^{2(n+1)}}\frac{k^2+1}{k^2-1}.$$

If we let $n\longrightarrow+\infty$, we get $q_1(x)=q_2(x)$ for all $x\in E$. This ends the proof of the theorem.
By using Theorem 3.1 and Corollary 2.2, we get

Corollary 3.2. [12] *Let E be a vector space, F a Banach space and $f : E \to F$ a mapping which satisfies the inequality*

$$\|f(kx+y)+f(kx-y)-2k^2f(x)-2f(y)\| \le \delta, \tag{3.8}$$

for all $x, y \in E$ and for some $\delta \ge 0$. Then there exists a unique mapping $q : E \to F$ solution of (1.3) such that

$$\|f(x)-q(x)\| \le \frac{\delta}{2k^2}\frac{k^2+1}{k^2-1}, \quad x \in E. \tag{3.9}$$

□

Theorem 3.3. *Let E be a normed space and F a Banach space. Suppose a mapping $f : E \to F$ satisfies the inequality*

$$\|f(kx+y)+f(kx+\sigma(y))-2k^2f(x)-2f(y)\| \le \theta(\|x\|^p+\|y\|^p), \tag{3.10}$$

for some $\theta \ge 0$, $p \in]0,2[$ and for all $x, y \in E$. Then there exists a unique mapping $q : E \to F$, that is a solution of the functional equation (1.5) such that

$$\|f(x)-q(x)\| \le \frac{\theta}{2}\frac{\|x\|^p}{k^2-k^p}, \tag{3.11}$$

for all $x \in E$.

Proof. Suppose that f satisfies the inequality (3.10). Letting $x = y = 0$ in (3.10), we get $f(0) = 0$. Putting $y = 0$ in (3.10), we get

$$\|2f(kx)-2k^2f(x)\| \le \theta\|x\|^p, \tag{3.12}$$

for all $x \in E$. So

$$\|f(x)-\frac{1}{k^2}f(kx)\| \le \frac{\theta}{2k^2}\|x\|^p, \tag{3.13}$$

for all $x \in E$.
By mathematical induction we verify that

$$\|f(x)-\frac{1}{k^{2n}}f(k^nx)\| \le \frac{\theta}{2k^2}[1+\frac{1}{k^{2-p}}+\ldots+\frac{1}{k^{(n-1)(2-p)}}]\|x\|^p, \tag{3.14}$$

holds for all $n \in \mathbb{N}$. Next, we will show that the sequence of functions $f_n(x) = \frac{1}{k^{2n}}f(k^nx)$ is a Cauchy sequence for every $x \in E$. By using the inequality (3.13), we get

$$\begin{aligned}\|f_{n+1}(x)-f_n(x)\| &= \|\frac{1}{k^{2(n+1)}}f(k^{n+1}x)-\frac{1}{k^{2n}}f(k^nx)\| \\ &= \frac{1}{k^{2n}}\|f(k^nx)-\frac{1}{k^2}f(k^{n+1}x)\| \le \frac{1}{k^{n(2-p)}}\frac{\theta}{2k^2}\|x\|^p.\end{aligned}$$

Consequently, $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for all $x \in E$. Since F is a complete normed space, the limit function $q(x) = \lim_{n\to+\infty} f_n(x)$ exists for every $x \in E$. Let us now show that q is a solution of equation (1.5). Indeed,

$$\begin{aligned}&\|f_n(kx+y)+f_n(kx+\sigma(y))-2k^2f_n(x)-2f_n(y)\| \\ &= \frac{1}{k^{2n}}\|f(kk^nx+k^ny)+f(kk^nx+\sigma(k^ny))-2k^2f(k^nx)-2f(k^ny)\| \\ &\le \frac{\theta}{k^{n(2-p)}}(\|x\|^p+\|y\|^p).\end{aligned}$$

By letting $n \to +\infty$, we get the desired result.
The uniqueness of the mapping q can be proved by using a similar argument as in the proof of the previous theorem. This completes the proof of the theorem. □

Corollary 3.4. [12] *Let E be a normed space and F a Banach space. Assume a function $f : E \to F$ satisfies the inequality*

$$\|f(kx + y) + f(kx - y) - 2k^2 f(x) - 2f(y)\| \le \theta(\|x\|^p + \|y\|^p), \tag{3.15}$$

for some $\theta \ge 0$, $p \in]0, 2[$ and for all $x, y \in E$. Then there exists a unique mapping $q : E \to F$, solution of the quadratic functional equation (1.3) and satisfying the inequality

$$\|f(x) - q(x)\| \le \frac{\theta}{2} \frac{\|x\|^p}{k^2 - k^p} \tag{3.16}$$

for all $x \in E$.

Theorem 3.5. *Let E be a normed vector space, F a Banach space and $f : E \to F$ a mapping which satisfies the inequality (3.10) for some $\theta \ge 0$, $p > 2$ and for all $x, y \in E$. Then there exists a unique mapping $q : E \to F$, that is a solution of the functional equation (1.5) and satisfying*

$$\|f(x) - q(x)\| \le \frac{\theta}{2} \frac{\|x\|^p}{k^p - k^2} \tag{3.17}$$

for all $x \in E$.

Corollary 3.6. [12] *Let E be a normed vector space, F a Banach space. Suppose a function $f : E \to F$ satisfies the inequality*

$$\|f(kx + y) + f(kx - y) - 2k^2 f(x) - 2f(y)\| \le \theta(\|x\|^p + \|y\|^p), \tag{3.18}$$

for some $\theta \ge 0$, $p > 2$ and for all $x, y \in E$. Then there exists a unique mapping $q : E \to F$, that is a solution of the functional equation (1.3), such that

$$\|f(x) - q(x)\| \le \frac{\theta}{2} \frac{\|x\|^p}{k^p - k^2} \tag{3.19}$$

for all $x \in E$.

4. Hyers-Ulam stability of equation (1.5) on unbounded domains

In this section, we will investigate the Hyers-Ulam stability of equation (1.5) on unbounded domains: $\{(x, y) \in E^2 : \ \|x\| + \|y\| \ge d\}$.

Theorem 4.1. *Let $d > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(kx + y) + f(kx + \sigma(y)) - 2k^2 f(x) - 2f(y)\| \le \delta, \tag{4.1}$$

for all $x, y \in E$ with $\|x\| + \|y\| \ge d$. Then, there exists a unique mapping $Q : E \to F$ solution of equation (1.5) such that

$$\|f(x) - Q(x)\| \le \frac{2\delta}{k^2} \frac{k^2 + 1}{k^2 - 1}, \quad x \in E. \tag{4.2}$$

Proof. Let $x, y \in E$ such that $0 < \|x\| + \|y\| < d$. We choose $z = 2^n x$ if $x \neq 0$ or $z = 2^n y$ if $y \neq 0$ with n large enough.
Clearly, we see

$$\|\frac{z}{k}\| + \|kx + y\| \geq d, \|\frac{z}{k}\| + \|kx + \sigma(y)\| \geq d, \|x\| + \|z + \sigma(y)\| \geq d,$$

$$\|x\| + \|y + z\| \geq d, \|\frac{z}{k}\| + \|y\| \geq d, \|kx + y + \sigma(z)\| \geq d, \|kx + \sigma(y) + \sigma(z)\| \geq d.$$

From inequality (4.1), the triangle inequality and the following equation

$$\begin{aligned}
&2[f(kx + y) + f(kx + \sigma(y)) - 2k^2 f(x) - 2f(y)] \\
&= -[f(z + kx + y) + f(z + \sigma(kx) + \sigma(y)) - 2k^2 f(\frac{z}{k}) - 2f(kx + y)] \\
&-[f(z + kx + \sigma(y)) + f(z + \sigma(kx) + y) - 2k^2 f(\frac{z}{k}) - 2f(kx + \sigma(y))] \\
&+[f(kx + z + \sigma(y)) + f(kx + \sigma(z) + y) - 2k^2 f(x) - 2f(z + \sigma(y))] \\
&+[f(kx + y + z) + f(kx + \sigma(y) + \sigma(z)) - 2k^2 f(x) - 2f(y + z)] \\
&+2[f(z + y) + f(z + \sigma(y)) - 2k^2 f(\frac{z}{k}) - 2f(y)] \\
&+[f(z + \sigma(kx) + \sigma(y)) - f(kx + y + \sigma(z))] - 2k^2 f(0) \\
&+[f(\sigma(kx) + z + y) - f(kx + \sigma(y) + \sigma(z))] + 2k^2 f(0)
\end{aligned}$$

we get

$$\|f(kx + y) + f(kx + \sigma(y)) - 2k^2 f(x) - 2f(y)\| \leq 4\delta,$$

for $x, y \in E$ with $x \neq 0$ and $y \neq 0$.
Now, if $x = y = 0$, we use the following relation with an arbitrary $z \in E$ such that $\|z\| = kd$

$$\begin{aligned}
&2[f(0) + f(0) - 2k^2 f(0) - 2f(0)] \\
&= [f(z) + f(\sigma(z)) - 2k^2 f(0) - 2f(z)] + [f(z) - f(\sigma(z)) - 2k^2 f(0)]
\end{aligned}$$

to obtain

$$\|2k^2 f(0)\| \leq \delta.$$

Consequently, the inequality

$$\|f(kx + y) + f(kx + \sigma(y)) - 2k^2 f(x) - 2f(y)\| \leq 4\delta, \tag{4.3}$$

holds for all $x, y \in E$. Therefore, by using Theorem 3.1, we get the rest of the proof. □

Corollary 4.2. *Let $d > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(kx + y) + f(kx - y) - 2k^2 f(x) - 2f(y)\| \leq \delta, \tag{4.4}$$

for all $x, y \in E$ with $\|x\| + \|y\| \geq d$. Then, there exists a unique mapping $Q : E \to F$ solution of equation (1.3) such that

$$\|f(x) - Q(x)\| \leq \frac{2\delta}{k^2} \frac{k^2 + 1}{k^2 - 1}, \quad x \in E. \tag{4.5}$$

Corollary 4.3. *A mapping $f : E \to F$ is a solution of equation (1.5) if and only if*

$$\|f(kx + y) + f(kx + \sigma(y)) - 2k^2 f(x) - 2f(y)\| \to 0 \quad as \quad \|x\| + \|y\| \to +\infty. \tag{4.6}$$

Corollary 4.4. *A mapping $f : E \to F$ is a solution of equation (1.4) if and only if*

$$\|f(kx+y)+f(kx-y)-2k^2f(x)-2f(y)\| \to 0 \quad as \quad \|x\|+\|y\| \to +\infty. \tag{4.7}$$

Corollary 4.5. *A mapping $f : E \to F$ is a solution of equation (1.3) if and only if*

$$\|f(kx+y)+f(kx-y)-2k^2f(x)-2f(y)\| \to 0 \quad as \quad \|x\|+\|y\| \to +\infty. \tag{4.8}$$

Corollary 4.6. [13]*A mapping $f : E \to F$ is a solution of equation (1.3) if and only if*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \to 0 \quad as \quad \|x\|+\|y\| \to +\infty. \tag{4.9}$$

References

1. T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, **2** (1950), 64-66.
2. B. Bouikhalene, E. Elqorachi and Th. M. Rassias, On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution, *Nonlinear Funct. Anal. Appl.,* **12**. no 2 (2007), 247-262.
3. B. Bouikhalene, E. Elqorachi and Th. M. Rassias, On the Hyers-Ulam stability of approximately Pexider mappings. *Math. Inequal. Appl.* **11** (2008), 805-818.
4. P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76-86.
5. S. Czerwik, On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg.* **62** (1992), 59-64.
6. P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184** (1994), 431-436.
7. Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.* **14** (1991), 431-434.
8. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941), 222-224.
9. D. H. Hyers, G. Isac and Th. M. Rassias, On the asymptoticity aspect of Hyers-Ulam stability of mappings, *Proc. Amer. Math. Soc.* **126** (1998),425-430.
10. D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.,* **44** (1992), 125-153.
11. D. H. Hyers G. I. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, *Birkhäuser, Basel,* 1998.
12. J.-R. Lee, J.-S. An and C.-Park, On the stability of quadratic functional equations, *Abs. Appl. Anal.* doi: 10.1155/2008/628178.
13. S.-M. Jung, On the Hyers-Ulam stability of the functional equation that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998), 126-137.
14. S.-M. Jung, Stability of the quadratic equation of Pexider type, *Abh. Math. Sem. Univ. Hamburg,* **70** (2000), 175-190.
15. S.-M. Jung and P. K. Sahoo, Hyers-Ulam-Rassias stability of the quadratic equation of Pexider type, *J. Korean Math. Soc.,* **38** (2001), No. 3, 645-656.
16. Th. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300.
17. Th. M. Rassias, On a modified Hyers-Ulam sequence, *J. Math. Anal. Appl.* **158** (1991), 106-113.
18. Th. M. Rassias, Functional Equations and Inequalities, *Kluwer Academic Publishers*, Dordrecht, Boston, London, 2001.

19. Th. M. Rassias, Functional Equations, Inequalities and Applications, *Kluwer Academic Publishers*, Dordercht, Boston, London, 2003.
20. Th. M. Rassias, The problem of S. M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.* **246** (2000), 352-378.
21. Th. M. Rassias, On the stability of minimum points, *Mathematica,* **45** (68)(1)(2003), 93-104.
22. Th. M. Rassias, On the stability of the functional equations and a problem of Ulam, *Acta Applicandae Mathematicae.* **62** (2000), 23-130.
23. Th. M. Rassias, and P. Šemrl, On the behavior of mappings which do not satisfy Hyers-Ulam stability, *Proc. Amer. Math. Soc.* **114** (1992), 989-993.
24. Th. M. Rassias and P. Šemrl, On the Hyers-Ulam stability of linear mappings, *J. Math. Anal. Appl.* **173** (1993), 325-338.
25. Th. M. Rassias and J. Tabor, Stability of Mappings of Hyers-Ulam Type, *Hardronic Press, Inc., Palm Harbor, Florida*, 1994.
26. F. Skof, Approssimazione di funzioni δ-quadratic su dominio restretto, *Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **118** (1984), 58-70.
27. H. Stetkær, Functional equations on abelian groups with involution. *Aequationes Math.* **54** (1997), 144-172.
28. S. M. Ulam, A Collection of Mathematical Problems, *Interscience Publ. New York,* 1961. Problems in Modern Mathematics, *Wiley, New York* 1964.

[1] Department of Mathematics, Faculty of Sciences, University Ibn Zohr, Agadir, Morocco,

E-mail address: elqorachi@hotmail.com; manaryoussefr32@hotmail.fr,,

[2] Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780, Athens Greece,

E-mail address: trassias@math.ntua.gr