LIE *-DOUBLE DERIVATIONS ON LIE C*-ALGEBRAS

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Dedicated to the 70th Anniversary of S.M.Ulam’s Problem for Approximate Homomorphisms

ABSTRACT. A unital C* – algebra \(A\), endowed with the Lie product \([x, y] = xy - yx\) on \(A\), is called a Lie C* – algebra. Let \(A\) be a Lie C* – algebra and \(g, h : A \rightarrow A\) be \(\mathbb{C}\) – linear mappings. A \(\mathbb{C}\) – linear mapping \(f : A \rightarrow A\) is called a Lie \((g, h)\) – double derivation if
\[
 f(a, b) = [f(a), b] + [a, f(b)] + [g(a), h(b)] + [h(a), g(b)]
\]
for all \(a, b \in A\).

In this paper, our main purpose is to prove the generalized Hyers - Ulam - Rassias stability of Lie * - double derivations on Lie C* - algebras associated with the following additive mapping:
\[
\sum_{k=2}^{n} \left( \sum_{i_1=2}^{n} \sum_{i_2=1+1}^{i_1-k+1} \ldots \sum_{i_{n-k+1}=1+1}^{i_{n-k+1}} \right) f(\sum_{i=1}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}) + f(\sum_{i=1}^{n} x_i) = 2^{n-1} f(x_1)
\]
for a fixed positive integer \(n\) with \(n \geq 2\).

1. INTRODUCTION AND PRELIMINARIES

It seems that the stability problem was first studied by D.H. Hyers [11], which was raised by S.M. Ulam [31] For what metric groups \(G\) is it true that an \(\epsilon\)–automorphism of \(G\) is necessarily near to a strict automorphism? An answer has been given in the following way. Let \(E_1, E_2\) be two real Banach spaces and \(f : E_1 \rightarrow E_2\) be a mapping. In 1941, Hyers [11] gave an answer to the problem above as follows: if there exists an \(\epsilon \geq 0\) such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]
for all \(x, y \in E_1\), then there exists a unique additive mapping \(T : E_1 \rightarrow E_2\) such that \(\|f(x) - T(x)\| \leq \epsilon\) for every \(x \in E_1\). This result is called the Hyers – Ulam stability of the additive Cauchy equation \(g(x + y) = g(x) + g(y)\). In 1978, Th.M. Rassias [26] introduced a new functional inequality that we call Cauchy – Rassias inequality and succeeded to extend the result of Hyers by weakening the condition for the Cauchy difference to be unbounded: if there exist an \(\epsilon \geq 0\) and \(0 \leq p < 1\) such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in E_1 \), then there exists a unique additive mapping \( T : E_1 \rightarrow E_2 \) such that

\[
\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2 - 2^p|} \|x\|^p
\]

for every \( x \in E_1 \) (see [12, 13, 27, 28, 29]). This stability phenomenon of this kind is called the Hyers – Ulam – Rassias stability. In 1991, Z. Gajda [9] solved the problem for \( 1 < p \), which was raised by Rassias. In fact, the result of Rassias is valid for \( 1 < p \); moreover, Gajda gave an example that a similar stability result does not hold for \( p = 1 \). Another example was given by Th.M. Rassias and P. Šemrl [30]. J.M. Rassias [23] followed the innovative approach of Rassias’ theorem [26] in which he replaced the factor \( \|x\|^p + \|y\|^q \) by \( \|x\|^p \cdot \|y\|^q \) for \( p, q \in \mathbb{R} \) with \( p + q \neq 1 \).

In 1994, a generalization of the Rassias’ theorem was obtained by Găvruta as follows [10]. Suppose \((G,+)\) is an abelian group, \( E \) is a Banach space, and that the so-called admissible control function \( \varphi : G \times G \rightarrow \mathbb{R} \) satisfies

\[
\tilde{\varphi}(x, y) := 2^{-1} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^ny) < \infty
\]

for all \( x, y \in G \). If \( f : G \rightarrow E \) is a mapping with

\[
\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)
\]

for all \( x, y \in G \), then there exists a unique mapping \( T : G \rightarrow E \) such that \( T(x+y) = T(x) + T(y) \) and \( \|f(x) - T(x)\| \leq \tilde{\varphi}(x, x) \) for all \( x, y \in G \).

Let \( \mathcal{A} \) be a subalgebra of an algebra \( \mathcal{B}, \mathcal{X} \) and be a \( \mathcal{B} \) – module \( \sigma : \mathcal{A} \rightarrow \mathcal{B} \) be a linear mapping. A linear mapping \( f : \mathcal{A} \rightarrow \mathcal{B} \) is called \( \sigma \) – derivation (see [17, 18]) if

\[
f(ab) = f(a)\sigma(b) + \sigma(a)f(b)
\]

(1.1)

for all \( a, b \in \mathcal{A} \).

Clearly, if \( \sigma = id \), the identity mapping on \( \mathcal{A} \), then a \( \sigma \) – derivation an ordinary derivation. On the other hand, each homomorphism \( f \) is a \( \frac{1}{2} \) – derivation. Thus, the theory of \( \sigma \) – derivations combines the theory of derivations and homomorphisms.

If \( g : \mathcal{A} \rightarrow \mathcal{A} \) is an ordinary derivation and \( \sigma : \mathcal{A} \rightarrow \mathcal{A} \) is a homomorphism, then \( f = g\sigma \) is a \( \sigma \) – derivation. Although, a \( \sigma \) – derivation is not necessarily of the form \( g\sigma \), but it seems that the generalized Leibniz rule, \( f(ab) = f(a)\sigma(b) + \sigma(a)f(b) \), comes from this observation.

M. Mirzavaziri and E. Omidvar Tehran [16] took ideas from above fact, and considered two derivations \( g, h \) to find a similar rule, for \( f = gh \). In this case, they saw that \( f \) satisfies

\[
f(ab) = f(a)b + af(b) + g(a)h(b) + h(a)g(b)
\]

(1.2)

for all \( a, b \in \mathcal{A} \). They said that a linear mapping \( f : \mathcal{A} \rightarrow \mathcal{A} \) is a \((g, h)\) – double derivation if satisfies (1.2). Moreover, by a \( f \) – double derivation they called a \((f, f)\) – derivation and proved that if \( \mathcal{A} \) is a \( C^* \) – algebra, \( f : \mathcal{A} \rightarrow \mathcal{A} \) is a \(*\) – linear mapping and \( g : \mathcal{A} \rightarrow \mathcal{A} \) is a continuous \( f \) – double derivation then \( f \) is continuous.

A unital \( C^* \) – algebra \( \mathcal{A} \), endowed with the Lie product \( [x, y] = xy - yx \) on \( \mathcal{A} \), is called a Lie \( C^* \) – algebra. Let \( \mathcal{A} \) be a Lie \( C^* \) – algebra and \( g, h : \mathcal{A} \rightarrow \mathcal{A} \) be \( \mathbb{C} \) – linear mappings. A \( \mathbb{C} \) – linear mapping \( f : \mathcal{A} \rightarrow \mathcal{A} \) is called a Lie \((g, h)\) – double derivation if \( f([a, b]) = [f(a), b] + [a, f(b)] + [g(a), h(b)] + [h(a), g(b)] \) for all \( a, b \in \mathcal{A} \).
M. Eshaghi Gordji, H. Khodaei, R. Saadati and Gh. Sadeghi [8] found the general $n$–dimensional additive functional equation as follows:

$$
\sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} \right) f(\sum_{i=1}^{n} x_i)
- \sum_{r=1}^{n-k+1} x_{i_r} + f(\sum_{i=1}^{n} x_i) = 2^{n-1} f(x_1)
$$

(1.3)

for a fixed positive integer $n$ with $n \geq 2$, and investigated stability of functional equation (1.3) in random normed spaces via fixed point method.

In this paper, our main purpose is to prove the generalized Hyers–Ulam–Rassias stability of Lie $\ast$–double derivations on Lie $C^\ast$–algebras associated with the functional equation (1.3).

Throughout this paper, assume that $\mathcal{A}$ is a Lie $C^\ast$–algebra and $U(\mathcal{A}) = \{ u \in \mathcal{A} | uu^* = u^*u = e \}$.

### 2. MAIN RESULTS

For given mappings $f, g, h : \mathcal{A} \to \mathcal{A}$, we define the difference operators $D_\mu f : \mathcal{A}^n \to \mathcal{A}$ and $C_{f,g,h} : \mathcal{A}^2 \to \mathcal{A}$ by

$$
D_\mu f(x_1, \ldots, x_n) := \sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} \right) f(\sum_{i=1}^{n} \mu x_i)
- \sum_{r=1}^{n-k+1} \mu x_{i_r} + f(\sum_{i=1}^{n} \mu x_i) = 2^{n-1} f(\mu x_1)
$$

and

$$
C_{f,g,h}(a, b) := f([a, b]) - [f(a), b] - [a, f(b)] - [g(a), h(b)] - [h(a), g(b)]
$$

for all $\mu \in \mathbb{T}^1 := \{ \lambda : |\lambda| = 1 \}$ and all $a, b, x_i \in \mathcal{A}$ ($i = 1, 2, \ldots, n$).

Throughout this section, assume that $f(0) = g(0) = h(0) = 0$.

We are going to investigate the generalized Hyers–Ulam–Rassias stability of Lie $\ast$–double derivations on Lie $C^\ast$–algebras for functional equation (1.3).

**Definition 2.1.** Let $\mathcal{A}$ be a Lie $C^\ast$–algebra and $g, h : \mathcal{A} \to \mathcal{A}$ be $\mathbb{C}$–linear mappings. A $\mathbb{C}$–linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a Lie $(g, h)$–double derivation if $f([a, b]) = [f(a), b] + [a, f(b)] + [g(a), h(b)] + [h(a), g(b)]$ for all $a, b \in \mathcal{A}$.

We will use the following lemma in this paper.

**Lemma 2.2.** [8] A function $f : \mathcal{A} \to \mathcal{A}$ with $f(0) = 0$ satisfies the functional equation (1.3) if and only if $f : \mathcal{A} \to \mathcal{A}$ is additive.

**Theorem 2.3.** If $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings for which there exists function $\varphi : \mathcal{A}^{n+2} \to [0, \infty)$ such that

$$
\tilde{\varphi}(x) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j x, \ldots, 0, 0, 0) < \infty,
$$

(2.1)
\[
\lim_{j \to \infty} \frac{1}{2^j} \varphi(2^j x_1, 2^j x_2, \ldots, 2^j x_n, 2^j a, 2^j b) = 0, \tag{2.2}
\]

\[
\max \{ \| D_\mu f(x_1, x_2, \ldots, x_n) - C_{f,g,h}(u, b), D_\mu g(x_1, x_2, \ldots, x_n) \\
- C_{f,g,h}(u, b), D_\mu h(x_1, x_2, \ldots, x_n) - C_{f,g,h}(u, b) \} \leq \varphi(x_1, x_2, \ldots, x_n, u, b), \tag{2.3}
\]

\[
\max \{ f(2^m u^*) - f(2^m u)^*, g(2^m u^*) - g(2^m u)^*, h(2^m u^*) - h(2^m u)^* \}
\leq \varphi(2^m u, 2^m u, 2^m u, 2^m u) \tag{2.4}
\]

for all \( \mu \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C}; |\lambda| = 1 \} \), all \( u \in U(A) \), \( m = 0, 1, \ldots \), and all \( a, b, x_i \in A \) (\( i = 1, 2, \ldots, n \)). Then there exist unique \( \mathbb{C} \) – linear \(*\) – mappings \( d, \delta, \epsilon : A \to A \) such that

\[
\max \{ \| f(x) - d(x) \|, \| g(x) - \delta(x) \|, \| h(x) - \epsilon(x) \| \} \leq \frac{1}{2^{n-1}} \tilde{\varphi}(x) \tag{2.5}
\]

for all \( x \in A \). Moreover, \( d : A \to A \) is a Lie \(*\) – \((\delta, \epsilon)\) – double derivation on \( A \).

**Proof.** It follows from the inequality (2.3) that

\[
\| D_\mu f(x_1, x_2, \ldots, x_n) - C_{f,g,h}(u, b) \| \leq \varphi(x_1, x_2, \ldots, x_n, u, b), \tag{2.6}
\]

\[
\| D_\mu g(x_1, x_2, \ldots, x_n) - C_{f,g,h}(u, b) \| \leq \varphi(x_1, x_2, \ldots, x_n, u, b), \tag{2.7}
\]

\[
\| D_\mu h(x_1, x_2, \ldots, x_n) - C_{f,g,h}(u, b) \| \leq \varphi(x_1, x_2, \ldots, x_n, u, b) \tag{2.8}
\]

for all \( a, x_i \in A \) (\( i = 1, 2, \ldots, n \)), all \( u \in U(A) \) and all \( \mu \in \mathbb{T}^1 \). Let \( \mu = 1 \). We use the relation

\[
1 + \sum_{k=1}^{n-k} \binom{n-k}{k} = \sum_{k=0}^{n-k} \binom{n-k}{k} = 2^{n-k} \tag{2.9}
\]

for all \( n > k \) and put \( x_1 = x_2 = x \) and \( b = u = x_i = 0 \) (\( i = 3, \ldots, n \)) in (2.6). Then we obtain

\[
\| 2^{n-2} f(2x) - 2^{n-1} f(x) \| \leq \varphi(x, x, \ldots, 0, 0, 0) \tag{2.10}
\]

for all \( x \in A \). So

\[
\| f(2x) \| - \varphi(x, x, \ldots, 0, 0, 0) \leq \frac{1}{2^{n-1}} \varphi(x, x, \ldots, 0, 0, 0) \tag{2.11}
\]

for all \( x \in A \). By induction on \( m \), we shall show that

\[
\| \frac{f(2^m x)}{2^m} - f(x) \| \leq \frac{1}{2^{m-1}} \sum_{j=0}^{m-1} \frac{1}{2^j} \varphi(2^j x, 2^j x, 0, \ldots, 0, 0, 0) \tag{2.12}
\]

for all \( x \in A \). It follows from (2.1) and (2.12) that the sequence \( \{ \frac{f(2^m x)}{2^m} \} \) is a Cauchy sequence for all \( x \in A \). Since \( A \) is complete, the sequence \( \{ \frac{f(2^m x)}{2^m} \} \) converges. Therefore, one can define the function \( d : A \to A \) by

\[
d(x) := \lim_{m \to \infty} \frac{f(2^m x)}{2^m}
\]
for all $x \in \mathcal{A}$. In the inequality (2.6), assume that $b = u = 0$ and $\mu = 1$. Then By

\[ \|D_1d(x_1, \ldots, x_n)\| = \lim_{m \to \infty} \frac{1}{2^m} \|D_1f(2^m x_1, \ldots, 2^m x_n)\| \]

\[ \leq \lim_{m \to \infty} \frac{1}{2^m} \varphi(2^m x_1, \ldots, 2^m x_n, 0, 0) = 0 \]

for all $x_1, \ldots, x_n \in \mathcal{A}$. So $D_1d(x_1, \ldots, x_n) = 0$. By Lemma 2.2, the function $d: \mathcal{A} \to \mathcal{A}$ is additive. Moreover, passing the limit $m \to \infty$ in (2.12), we get the inequality (2.5). Now, let $d': \mathcal{A} \to \mathcal{A}$ be another additive function satisfying (1.3) and (2.5). So

\[ \|d(x) - d'(x)\| = \frac{1}{2^m} \|d(2^m x) - d'(2^m x)\| \]

\[ \leq \frac{1}{2^m} (\|d(2^m x) - f(2^m x)\| + \|d'(2^m x) - f(2^m x)\|) \]

\[ \leq \frac{2}{2^m 2^{n-1}} \varphi(2^m x) \]

which tends to zero as $m \to \infty$ for all $x \in \mathcal{A}$. So we can conclude that $d(x) = d'(x)$ for all $x \in \mathcal{A}$. This proves the uniqueness of $d$.

A similar argument shows that there exist unique additive mappings $\delta, \epsilon: \mathcal{A} \to \mathcal{A}$ satisfying (2.5). The additive mappings $\delta, \epsilon: \mathcal{A} \to \mathcal{A}$ are by

\[ \delta(x) := \lim_{m \to \infty} \frac{g(2^m x)}{2^m} \]  

(2.13)

and

\[ \epsilon(x) := \lim_{m \to \infty} \frac{h(2^m x)}{2^m} \]  

(2.14)

for all $x \in \mathcal{A}$.

Let $\mu \in \mathbb{T}^1$. Set $x_1 = x$ and $u = b = x_i = 0$ ($i = 2, \ldots, n$) in (2.6). Then by the relation (2.9), we get

\[ \|2^{n-1} f(\mu x) - 2^{n-1} \mu f(x)\| \leq \varphi(x, 0, \ldots, 0, 0, 0) \]  

(2.15)

for all $x \in \mathcal{A}$. So that

\[ \|2^{-m}(f(2^m \mu x) - \mu f(2^m x))\| \leq \frac{2^{-m}}{2^{m-1}} \varphi(2^m x, 0, \ldots, 0, 0, 0) \]

for all $x \in \mathcal{A}$. Since the right hand side tends to zero as $m \to \infty$, we have

\[ d(\mu x) = \lim_{m \to \infty} \frac{f(2^m \mu x)}{2^m} = \lim_{m \to \infty} \frac{\mu f(2^m x)}{2^m} = \mu d(x) \]

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Obviously, $d(0x) = 0 = 0d(x)$.

Now, let $\gamma \in \mathbb{C}$ ($\gamma \neq 0$) and $L$ an integer greater than $4|\gamma|$. Then $|\frac{2}{L}| < \frac{1}{4} < \frac{1}{3}$.

By Theorem 1 of [14], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\mu_1 = \gamma \mu_2 = \gamma \mu_3$.
Thus
\[ d(\gamma x) = d\left(\frac{L}{3} \cdot 3 \cdot \frac{\gamma}{L} x\right) = \frac{L}{3} d\left(3 \cdot \frac{\gamma}{L} x\right) = \frac{L}{3} \left(3 \cdot \frac{\gamma}{L} d(x)\right) = \frac{L}{3} d(3 \cdot \frac{\gamma}{L} x) \]
for all \( x \in A \). Hence \( d : A \to A \) is a \( \mathbb{C} \)-linear mapping. A similar argument shows that \( \delta, \epsilon \) are \( \mathbb{C} \)-linear.

By (2.2) and (2.4), we get
\[ d(u^*) = \lim_{m \to \infty} \frac{f(2^m u^*)}{2^m} = \lim_{m \to \infty} \frac{f(2^m u)}{2^m} = \lim_{m \to \infty} \frac{f(2^m u)^*}{2^m} = d(u)^*, \]
\[ \delta(u^*) = \lim_{m \to \infty} \frac{g(2^m u^*)}{2^m} = \lim_{m \to \infty} \frac{g(2^m u)}{2^m} = \lim_{m \to \infty} \frac{g(2^m u)^*}{2^m} = \delta(u)^*, \]
\[ \epsilon(u^*) = \lim_{m \to \infty} \frac{h(2^m u^*)}{2^m} = \lim_{m \to \infty} \frac{h(2^m u)}{2^m} = \lim_{m \to \infty} \frac{h(2^m u)^*}{2^m} = \epsilon(u)^*, \]
for all \( u \in U(A) \). Since \( d : A \to A \) is \( \mathbb{C} \)-linear and each \( x \in A \) is a finite linear combination of unitary elements (see Theorem 4.17 of [15]), i.e., \( x = \sum_{j=1}^l \lambda_j u_j \) (\( \lambda_j \in \mathbb{C}, u_j \in U(A) \)),
\[ d(x^*) = d\left(\sum_{j=1}^l \lambda_j u_j^*\right) = \sum_{j=1}^l \lambda_j d(u_j^*) = \sum_{j=1}^l \lambda_j d(u_j)^* \]
\[ = \left(\sum_{j=1}^l \lambda_j d(u_j)\right)^* = d\left(\sum_{j=1}^l \lambda_j u_j\right)^* = d(x)^* \]
for all \( x \in A \). By the same method, one can obtain that \( \delta(x^*) = \delta(x)^* \) and \( \epsilon(x^*) = \epsilon(x)^* \) for all \( x \in A \). Setting \( x_1 = x_2 = \ldots = x_n = 0 \) in the inequality (2.6), we get
\[ \|C_{f,g,h}(u, b)\| \leq \varphi(0, 0, \ldots, 0, u, b), \]
that is,
\[ \frac{1}{2^m} \|f([2^m u]) - f(2^m u) - [2^m u, f(2^m b)] - [\delta(2^m u) - \epsilon(2^m b)] \|
- \|\epsilon(2^m u) - \delta(2^m b)\| \leq \frac{1}{2^m} \varphi(0, 0, \ldots, 0, 2^m u, 2^m b) \]
\[ \leq \frac{1}{2^m} \varphi(0, 0, \ldots, 0, 2^m u, 2^m b) \]
for all \( b \in A \) and all \( u \in U(A) \). Since the right hand side tends to zero as \( m \to \infty \), we have
\[ d([u, b]) = [d(u, b)] + [u, d(b)] + [\delta(u), \epsilon(b)] + [\epsilon(u), \delta(b)] \]
for all $b \in \mathcal{A}$ and all $u \in U(\mathcal{A})$. Since $d : \mathcal{A} \to \mathcal{A}$ is $\mathbb{C}$–linear and each $a \in \mathcal{A}$ is $a = \sum_{j=1}^{l} \lambda_j u_j \ (\lambda_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$,

$$d([a, b]) = d(\sum_{j=1}^{l} [\lambda_j u_j, b]) = \sum_{j=1}^{l} \lambda_j d([u_j, b])$$

$$= \sum_{j=1}^{l} \lambda_j ([u_j, d(b)] + [\delta(u_j), \epsilon(b)] + [\epsilon(u_j), \delta(b)])$$

$$= [d(\sum_{j=1}^{l} \lambda_j u_j), b] + [\delta(\sum_{j=1}^{l} \lambda_j u_j), \epsilon(b)] + [\epsilon(\sum_{j=1}^{l} \lambda_j u_j), \delta(b)]$$

$$= [d(a), b] + [a, d(b)] + [\delta(a), \epsilon(b)] + [\epsilon(a), \delta(b)]$$

for all $a, b \in \mathcal{A}$. Hence the $\mathbb{C}$–linear mapping $d : \mathcal{A} \to \mathcal{A}$ is a Lie $\ast - (\delta, \epsilon)$–double derivation, as desired. \hfill \Box

**Corollary 2.4.** If $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings for which exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\max\{\|D_\mu f(x_1, x_2, ..., x_n) - C_{f,g,h}(u, b), D_\mu g(x_1, x_2, ..., x_n) - C_{f,g,h}(u, b)\|\}$$

$$\leq \theta(1 + \|b\|^p + \sum_{i=1}^{n} \|x_i\|^p), \quad (2.16)$$

$$\max\{f(2^m u^*) - f(2^m u^*), g(2^m u^*) - g(2^m u^*), h(2^m u^*) - h(2^m u^*)\}$$

$$\leq \theta(n + 2)2^{np} \quad (2.17)$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(\mathcal{A})$, $m = 0, 1, ...$, and all $a, b \in \mathcal{A}$, then there exist unique $\mathbb{C}$–linear $\ast$–mappings $d, \delta, \epsilon : \mathcal{A} \to \mathcal{A}$ such that

$$\max\{\|f(x) - d(x)\|, \|g(x) - \delta(x)\|, \|h(x) - \epsilon(x)\|\} \leq \frac{2\theta}{2n-1} + \frac{2\theta}{2n-1(1 - 2p-1)} \|x\|^p \quad (2.18)$$

for all $x \in \mathcal{A}$. Moreover, $d : \mathcal{A} \to \mathcal{A}$ is a Lie $\ast - (\delta, \epsilon)$–double derivation on $\mathcal{A}$.

**Proof.** Define $\varphi(x_1, x_2, ..., x_n, u, b) := \theta(1 + \|b\|^p + \sum_{i=1}^{n} \|x_i\|^p)$ for all $u \in U(\mathcal{A})$ and $b, x_i \in \mathcal{A} (i = 1, ..., n)$, and apply Theorem 2.3. \hfill \Box

**Corollary 2.5.** Suppose that $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings satisfying (2.3) and (2.4). If there exists a function $\varphi^{n+2} : \mathcal{A} \to [0, \infty)$ such that

$$\tilde{\varphi}(x) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, ..., 0, 0, 0, 0\right) < \infty,$$

$$\lim_{j \to -\infty} 2^j \varphi\left(\frac{x_1}{2^j}, \frac{x_2}{2^j}, ..., \frac{a}{2^j}, \frac{b}{2^j}\right) = 0$$

for all $a, b, x_i \in \mathcal{A} (i = 1, ..., n)$, then there exist unique $\mathbb{C}$–linear $\ast$–mappings $d, \delta, \epsilon : \mathcal{A} \to \mathcal{A}$ such that

$$\max\{\|f(x) - d(x)\|, \|g(x) - \delta(x)\|, \|h(x) - \epsilon(x)\|\} \leq \frac{1}{2n} \tilde{\varphi}(x)$$
for all \( x \in A \). Moreover, \( d : A \to A \) is a Lie \( \ast - (\delta, \epsilon) - \) double derivation on \( A \).

**Proof.** By the same method as in the proof of Theorem 2.3, one can obtain that
\[
d(x) = \lim_{m \to \infty} 2^m f\left(\frac{x}{2^m}\right),
\]
\[
\delta(x) = \lim_{m \to \infty} 2^m g\left(\frac{x}{2^m}\right),
\]
\[
\epsilon(x) = \lim_{m \to \infty} 2^m h\left(\frac{x}{2^m}\right)
\]
for all \( x \in A \). The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

**Corollary 2.6.** If \( f, g, h : A \to A \) are mappings for which exist constants \( \theta \geq 0 \) and \( p > 1 \) satisfying (2.16) and (2.17). Then there exist unique \( C - \) linear \( \ast - \) mappings \( d, \delta, \epsilon : A \to A \) such that
\[
\max\{\|f(x) - d(x)\|, \|g(x) - \delta(x)\|, \|h(x) - \epsilon(x)\|\} \leq \frac{2\theta}{2^{n-1}} + \frac{2\theta}{2^{n-1}(21-p-1)}\|x\|^p
\]
for all \( x \in A \). Moreover, \( d : A \to A \) is a Lie \( \ast - (\delta, \epsilon) - \) double derivation on \( A \).

**References**


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