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LIE $^*-$DOUBLE DERIVATIONS ON LIE C^*-ALGEBRAS

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

ABSTRACT. A unital C^* − algebra $\mathcal{A}$, endowed with the Lie product $[x, y] = xy - yx$ on $\mathcal{A}$, is called a Lie C^* − algebra. Let $\mathcal{A}$ be a Lie C^* − algebra and $g, h : \mathcal{A} \to \mathcal{A}$ be $\mathbb{C}$ − linear mappings. A $\mathbb{C}$ − linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a Lie (g, h) − double derivation if $f([a, b]) = [f(a), b] + [a, f(b)] + [g(a), h(b)] + [h(a), g(b)]$ for all $a, b \in \mathcal{A}$. In this paper, our main purpose is to prove the generalized Hyers - Ulam - Rassias stability of Lie $*$ - double derivations on Lie C^* - algebras associated with the following additive mapping:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,..,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1)$$

for a fixed positive integer n with $n \geq 2$.

1. Introduction and preliminaries

It seems that the stability problem was first studied by D.H. Hyers [11], which was raised by S.M. Ulam [31] *For what metric groups G is it true that an $\epsilon-$automorphism of G is necessarily near to a strict automorphism?* An answer has been given in the following way. Let E_1, E_2 be two real Banach spaces and $f : E_1 \to E_2$ be a mapping. In 1941, Hyers [11] gave an answer to the problem above as follows: if there exists an $\epsilon \geq 0$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \epsilon$ for every $x \in E_1$. This result is called the *Hyers – Ulam stability* of the additive Cauchy equation $g(x + y) = g(x) + g(y)$. In 1978, Th.M. Rassias [26] introduced a new functional inequality that we call Cauchy − Rassias inequality and succeeded to extend the result of Hyers by weakening the condition for the Cauchy difference to be unbounded: if there exist an $\epsilon \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

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for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2 - 2^p|}\|x\|^p$$

for every $x \in E_1$ (see [12, 13, 27, 28, 29]). This stability phenomenon of this kind is called the *Hyers – Ulam – Rassias stability.* In 1991, Z. Gajda [9] solved the problem for $1 < p$, which was raised by Rassias. In fact, the result of Rassias is valid for $1 < p$; moreover, Gajda gave an example that a similar stability result does not hold for $p = 1$. Another example was given by Th.M. Rassias and P. Šemrl [30]. J.M. Rassias [23] followed the innovative approach of Rassias' theorem [26] in which he replaced the factor $\|x\|^p + \|y\|^q$ by $\|x\|^p.\|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.
In 1994, a generalization of the Rassias' theorem was obtained by Găvruta as follows [10].
Suppose (G,+) is an abelian group, E is a Banach space, and that the so-called admissible control function $\varphi : G \times G \to \mathbb{R}$ satisfies

$$\tilde{\varphi}(x, y) := 2^{-1} \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to E$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$ for all $x, y \in G$.
Let $\mathcal{A}$ be a subalgebra of an algebra $\mathcal{B}, \mathcal{X}$ and be a $\mathcal{B}$ – module $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $f : \mathcal{A} \to \mathcal{B}$ is called σ – derivation (see [17, 18]) if

$$f(ab) = f(a)\sigma(b) + \sigma(a)f(b) \tag{1.1}$$

for all $a, b \in \mathcal{A}$.
Clearly, if $\sigma = id$, the identity mapping on $\mathcal{A}$, then a σ – derivation an ordinary derivation. On the other hand, each homomorphism f is a $\frac{f}{2}$ – derivation. Thus, the theory of σ – derivations combines the theory of derivations and homomorphisms. If $g : \mathcal{A} \to \mathcal{A}$ is an ordinary derivation and $\sigma : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then $f = g\sigma$ is a σ – derivation. Although, a σ – derivation is not necessarily of the form $g\sigma$, but it seems that the generalized Leibniz rule, $f(ab) = f(a)\sigma(b) + \sigma(a)f(b)$, comes from this observation.
M. Mirzavaziri and E. Omidvar Tehrani [16] took ideas from above fact, and considered two derivations g, h to find a similar rule, for $f = gh$. In this case, they saw that f satisfies

$$f(ab) = f(a)b + af(b) + g(a)h(b) + h(a)g(b) \tag{1.2}$$

for all $a, b \in \mathcal{A}$. They said that a linear mapping $f : \mathcal{A} \to \mathcal{A}$ is a (g, h) – double derivation if satisfies (1.2). Moreovre, by a f – double derivation they called a (f, f) – derivaion and proved that if $\mathcal{A}$ is a C^* – algebra, $f : \mathcal{A} \to \mathcal{A}$ is a $*$ - linear mapping and $g : \mathcal{A} \to \mathcal{A}$ is a continuous f – double derivation then f is continuous.
A unital C^* – algebra $\mathcal{A}$, endowed with the Lie product $[x, y] = xy - yx$ on $\mathcal{A}$, is called a Lie C^* – algebra. Let $\mathcal{A}$ be a Lie C^* – algebra and $g, h : \mathcal{A} \to \mathcal{A}$ be $\mathbb{C}$ – linear mappings. A $\mathbb{C}$ – linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a Lie (g, h) – double derivation if $f([a, b]) = [f(a), b] + [a, f(b)] + [g(a), h(b)] + [h(a), g(b)]$ for all $a, b \in \mathcal{A}$.

M. Eshaghi Gordji, H. Khodaei, R. Saadati and Gh. Sadeghi [8] finded the general n − dimensional additive functional equation as follows:

$$\sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i=1,i\neq i_1,..,i_{n-k+1}}^{n}x_i - \sum_{r=1}^{n-k+1}x_{i_r})+f(\sum_{i=1}^{n}x_i)=2^{n-1}f(x_1) \quad (1.3)$$

for a fixed positive integer n with $n \geq 2$, and investigated stability of functional equation (1.3) in random normed spaces via fixed point method.
In this paper, our main purpose is to prove the generalized Hyers − Ulam − Rassias stability of Lie $*$ − double derivations on Lie C^* − algebras associated with the functional equation (1.3).
Throughout this paper, assume that $\mathcal{A}$ is a Lie C^* − algebra and $U(\mathcal{A}) = \{u \in \mathcal{A}\,|uu^* = u^*u = e\}$.

2. Main results

For given mappings $f, g, h : \mathcal{A} \to \mathcal{A}$, we define the difference operators $D_\mu f : \mathcal{A}^n \to \mathcal{A}$ and $C_{f,g,h} : \mathcal{A}^2 \to \mathcal{A}$ by

$$D_\mu f(x_1,...,x_n) := \sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i=1,i\neq i_1,..,i_{n-k+1}}^{n}\mu x_i - \sum_{r=1}^{n-k+1}\mu x_{i_r})+f(\sum_{i=1}^{n}\mu x_i)=2^{n-1}f(\mu x_1)$$

and

$$C_{f,g,h}(a,b) := f([a,b]) - [f(a),b] - [a,f(b)] - [g(a),h(b)] - [h(a),g(b)]$$

for all $\mu \in \mathbb{T}^1 := \{\lambda : \ |\lambda| = 1\}$ and all $a, b, x_i \in \mathcal{A}$ $(i = 1, 2, ..., n)$.
Throughout this section, assume that $f(0) = g(0) = h(0) = 0$.
We are going to investigate the generalized Hyers − Ulam − Rassias stability of Lie $*$ − double derivations on Lie C^* − algebras for functional equation (1.3).

Definition 2.1. Let $\mathcal{A}$ be a Lie C^* − algebra and $g, h : \mathcal{A} \to \mathcal{A}$ be $\mathbb{C}$ − linear mappings. A $\mathbb{C}$ − linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a Lie (g, h) − double derivation if $f([a, b]) = [f(a), b] + [a, f(b)] + [g(a), h(b)] + [h(a), g(b)]$ for all $a, b \in \mathcal{A}$.

We will use the following lemma in this paper.

Lemma 2.2. [8] *A function $f : \mathcal{A} \to \mathcal{A}$ with $f(0) = 0$ satisfies the functional equation* (1.3) *if and only if $f : \mathcal{A} \to \mathcal{A}$ is additive.*

Theorem 2.3. *If $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings for which there exists function $\varphi : \mathcal{A}^{n+2} \to [0, \infty)$ such that*

$$\tilde{\varphi}(x) := \sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^j x, 2^j x, ..., 0, 0, 0) < \infty, \quad (2.1)$$

$$\lim_{j\to\infty}\frac{1}{2^j}\varphi(2^jx_1,2^jx_2,...,2^jx_n,2^ja,2^jb)=0, \tag{2.2}$$

$$\begin{aligned}&\max\{\|D_\mu f(x_1,x_2,...,x_n)-C_{f,g,h}(u,b),D_\mu g(x_1,x_2,...,x_n)\\&\quad -C_{f,g,h}(u,b),D_\mu h(x_1,x_2,...,x_n)-C_{f,g,h}(u,b)\|\}\\&\quad\le\varphi(x_1,x_2,...,x_n,u,b),\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\max\{f(2^mu^*)-f(2^mu)^*,g(2^mu^*)-g(2^mu)^*,h(2^mu^*)-h(2^mu)^*\}\\&\quad\le\varphi(2^mu,2^mu,...,2^mu,2^mu,2^mu)\end{aligned} \tag{2.4}$$

for all $\mu\in\mathbb{T}^1:=\{\lambda\in\mathbb{C};\ |\lambda|=1\}$, *all* $u\in U(\mathcal{A})$, $m=0,1,...,$ *and all* $a,b,x_i\in\mathcal{A}$ $(i=1,2,...,n)$. *Then there exist unique* $\mathbb{C}$ – *linear* $*$ – *mappings* $d,\delta,\epsilon:\mathcal{A}\to\mathcal{A}$ *such that*

$$\max\{\|f(x)-d(x)\|,\|g(x)-\delta(x)\|,\|h(x)-\epsilon(x)\|\}\le\frac{1}{2^{n-1}}\tilde{\varphi}(x) \tag{2.5}$$

for all $x\in\mathcal{A}$. *Moreover,* $d:\mathcal{A}\to\mathcal{A}$ *is a Lie* $*$ – (δ,ϵ) – *double derivation on* $\mathcal{A}$.

Proof. It follows from the inequality (2.3) that

$$\|D_\mu f(x_1,x_2,...,x_n)-C_{f,g,h}(u,b)\|\le\varphi(x_1,x_2,...,x_n,u,b), \tag{2.6}$$

$$\|D_\mu g(x_1,x_2,...,x_n)-C_{f,g,h}(u,b)\|\le\varphi(x_1,x_2,...,x_n,u,b), \tag{2.7}$$

$$\|D_\mu h(x_1,x_2,...,x_n)-C_{f,g,h}(u,b)\|\le\varphi(x_1,x_2,...,x_n,u,b) \tag{2.8}$$

for all $a,x_i\in\mathcal{A}$ $(i=1,2,...,n)$, all $u\in U(\mathcal{A})$ and all $\mu\in\mathbb{T}^1$. Let $\mu=1$. We use the relation

$$1+\sum_{k=1}^{n-k}\binom{n-k}{k}=\sum_{k=0}^{n-k}\binom{n-k}{k}=2^{n-k} \tag{2.9}$$

for all $n>k$ and put $x_1=x_2=x$ and $b=u=x_i=0$ $(i=3,...,n)$ in (2.6). Then we obtain

$$\|2^{n-2}f(2x)-2^{n-1}f(x)\|\le\varphi(x,x,...,0,0,0) \tag{2.10}$$

for all $x\in\mathcal{A}$. So

$$\|\frac{f(2x)}{2}-f(x)\|\le\frac{1}{2^{n-1}}\varphi(x,x,...,0,0,0) \tag{2.11}$$

for all $x\in\mathcal{A}$. By induction on m, we shall show that

$$\|\frac{f(2^mx)}{2^m}-f(x)\|\le\frac{1}{2^{n-1}}\sum_{j=0}^{m-1}\frac{1}{2^j}\varphi(2^jx,2^jx,0,...,0,0,0) \tag{2.12}$$

for all $x\in\mathcal{A}$. It follows from (2.1) and (2.12) that the sequence $\{\frac{f(2^mx)}{2^m}\}$ is a Cauchy sequence for all $x\in\mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{\frac{f(2^mx)}{2^m}\}$ converges. Therefore, one can define the function $d:\mathcal{A}\to\mathcal{A}$ by

$$d(x):=\lim_{m\to\infty}\frac{f(2^mx)}{2^m}$$

for all $x \in \mathcal{A}$. In the inequality (2.6), assume that $b = u = 0$ and $\mu = 1$. Then By (2.2),

$$\begin{aligned}\|D_1 d(x_1, ..., x_n)\| &= \lim_{m\to\infty} \frac{1}{2^m}\|D_1 f(2^m x_1, ..., 2^m x_n)\| \\ &\leq \lim_{m\to\infty} \frac{1}{2^m}\varphi(2^m x_1, ..., 2^m x_n, 0, 0) = 0\end{aligned}$$

for all $x_1, ..., x_n \in \mathcal{A}$. So $D_1 d(x_1, ..., x_n) = 0$. By Lemma 2.2, the function $d : \mathcal{A} \to \mathcal{A}$ is additive. Moreover, passing the limit $m \to \infty$ in (2.12), we get the inequality (2.5). Now, let $d' : \mathcal{A} \to \mathcal{A}$ be another additive function satisfying (1.3) and (2.5). So

$$\begin{aligned}\|d(x) - d'(x)\| &= \frac{1}{2^m}\|d(2^m x) - d'(2^m x)\| \\ &\leq \frac{1}{2^m}(\|d(2^m x) - f(2^m x)\| + \|d'(2^m x) - f(2^m x)\|) \\ &\leq \frac{2}{2^m 2^{n-1}}\widetilde{\varphi}(2^m x)\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x \in \mathcal{A}$. So we can conclude that $d(x) = d'(x)$ for all $x \in \mathcal{A}$. This proves the uniqueness of d.
A similar argument shows that there exist unique additive mappings $\delta, \epsilon : \mathcal{A} \to \mathcal{A}$ satisfying (2.5). The additive mappings $\delta, \epsilon : \mathcal{A} \to \mathcal{A}$ are by

$$\delta(x) := \lim_{m\to\infty} \frac{g(2^m x)}{2^m} \tag{2.13}$$

and

$$\epsilon(x) := \lim_{m\to\infty} \frac{h(2^m x)}{2^m} \tag{2.14}$$

for all $x \in \mathcal{A}$.
Let $\mu \in \mathbb{T}^1$. Set $x_1 = x$ and $u = b = x_i = 0$ $(i = 2, ..., n)$ in (2.6). Then by the relation (2.9), we get

$$\|2^{n-1} f(\mu x) - 2^{n-1}\mu f(x)\| \leq \varphi(x, 0, ..., 0, 0, 0) \tag{2.15}$$

for all $x \in \mathcal{A}$. So that

$$\|2^{-m}(f(2^m \mu x) - \mu f(2^m x))\| \leq \frac{2^{-m}}{2^{n-1}}\varphi(2^m x, 0, ..., 0, 0, 0)$$

for all $x \in \mathcal{A}$. Since the right hand side tends to zero as $m \to \infty$, we have

$$d(\mu x) = \lim_{m\to\infty} \frac{f(2^m \mu x)}{2^m} = \lim_{m\to\infty} \frac{\mu f(2^m x)}{2^m} = \mu d(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Obviously, $d(0x) = 0 = 0d(x)$.
Now, let $\gamma \in \mathbb{C}$ $(\gamma \neq 0)$ and L an integer greater than $4|\gamma|$. Then $|\frac{\gamma}{L}| < \frac{1}{4} < \frac{1}{3}$. By Theorem 1 of [14], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\gamma}{L} =$

$\mu_1 + \mu_2 + \mu_3$. Thus

$$\begin{aligned} d(\gamma x) &= d(\frac{L}{3}.3\frac{\gamma}{L}x) = L.d(\frac{1}{3}.3\frac{\gamma}{L}x) = \frac{L}{3}d(3\frac{\gamma}{L}x) \\ &= \frac{L}{3}d(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{L}{3}(d(\mu_1 x) + d(\mu_2 x) + d(\mu_3 x)) \\ &= \frac{L}{3}(\mu_1 + \mu_2 + \mu_3)d(x) = \frac{L}{3}.3\frac{\gamma}{L}d(x) \\ &= \gamma d(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence $d : \mathcal{A} \to \mathcal{A}$ is a $\mathbb{C}$ – linear mapping. A similar argument shows that δ, ϵ are $\mathbb{C}$ – linear.
By (2.2) and (2.4), we get

$$d(u^*) = \lim_{m\to\infty} \frac{f(2^m u^*)}{2^m} = \lim_{m\to\infty} \frac{f(2^m u)^*}{2^m} = (\lim_{m\to\infty} \frac{f(2^m u)}{2^m})^* = d(u)^*,$$

$$\delta(u^*) = \lim_{m\to\infty} \frac{g(2^m u^*)}{2^m} = \lim_{m\to\infty} \frac{g(2^m u)^*}{2^m} = (\lim_{m\to\infty} \frac{g(2^m u)}{2^m})^* = \delta(u)^*,$$

$$\epsilon(u^*) = \lim_{m\to\infty} \frac{h(2^m u^*)}{2^m} = \lim_{m\to\infty} \frac{h(2^m u)^*}{2^m} = (\lim_{m\to\infty} \frac{h(2^m u)}{2^m})^* = \epsilon(u)^*$$

for all $u \in U(\mathcal{A})$. Since $d : \mathcal{A} \to \mathcal{A}$ is $\mathbb{C}$ – linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see Theorem 4.17 of [15]), i.e., $x = \sum_{j=1}^{l} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$,

$$\begin{aligned} d(x^*) &= d(\sum_{j=1}^{l} \bar{\lambda}_j u_j^*) = \sum_{j=1}^{l} \bar{\lambda}_j d(u_j^*) = \sum_{j=1}^{l} \bar{\lambda}_j d(u_j)^* \\ &= (\sum_{j=1}^{l} \lambda_j d(u_j))^* = d(\sum_{j=1}^{l} \lambda_j u_j)^* = d(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$. By the same method, one can obtain that $\delta(x^*) = \delta(x)^*$ and $\epsilon(x^*) = \epsilon(x)^*$ for all $x \in \mathcal{A}$. Setting $x_1 = x_2 = ... = x_n = 0$ in the inequality (2.6), we get

$$\|C_{f,g,h}(u, b)\| \le \varphi(0, 0, ..., 0, u, b),$$

that is,

$$\begin{aligned} \frac{1}{2^{2m}}\|f([2^m u, 2^m b] &- [f(2^m u), 2^m b] - [2^m u, f(2^m b)] - [\delta(2^m u), \epsilon(2^m b)] \\ &- [\epsilon(2^m u), \delta(2^m b)])\| \le \frac{1}{2^{2m}}\varphi(0, 0, ..., 0, 2^m u, 2^m b) \\ &\le \frac{1}{2^m}\varphi(0, 0, ..., 0, 2^m u, 2^m b) \end{aligned}$$

for all $b \in \mathcal{A}$ and all $u \in U(\mathcal{A})$. Since the right hand side tends to zero as $m \to \infty$, we have

$$d([u, b]) = [d(u, b)] + [u, d(b)] + [\delta(u), \epsilon(b)] + [\epsilon(u), \delta(b)]$$

for all $b \in \mathcal{A}$ and all $u \in U(\mathcal{A})$. Since $d : \mathcal{A} \to \mathcal{A}$ is $\mathbb{C}$ − linear and each $a \in \mathcal{A}$ is $a = \sum_{j=1}^{l} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$,

$$\begin{aligned} d([a,b]) &= d(\sum_{j=1}^{l}[\lambda_j u_j, b]) = \sum_{j=1}^{l} \lambda_j d([u_j, b]) \\ &= \sum_{j=1}^{l} \lambda_j([d(u_j), b] + [u_j, d(b)] + [\delta(u_j), \epsilon(b)] + [\epsilon(u_j), \delta(b)]) \\ &= [d(\sum_{j=1}^{l} \lambda_j u_j), b] + [(\sum_{j=1}^{l} \lambda_j u_j), d(b)] + [\delta(\sum_{j=1}^{l} \lambda_j u_j), \epsilon(b)]) + [\epsilon(\sum_{j=1}^{l} \lambda_j u_j), \delta(b)]) \\ &= [d(a), b] + [a, d(b)] + [\delta(a), \epsilon(b)] + [\epsilon(a), \delta(b)] \end{aligned}$$

for all $a, b \in \mathcal{A}$. Hence the $\mathbb{C}$ − linear mapping $d : \mathcal{A} \to \mathcal{A}$ is a Lie $*$ − (δ, ϵ) − double derivation, as desired. □

Corollary 2.4. *If $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings for which exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} &\max\{\|D_\mu f(x_1, x_2, ..., x_n) - C_{f,g,h}(u, b), D_\mu g(x_1, x_2, ..., x_n) \\ &\quad - C_{f,g,h}(u, b), D_\mu h(x_1, x_2, ..., x_n) - C_{f,g,h}(u, b)\|\} \\ &\quad \leq \theta(1 + \|b\|^p + \sum_{i=1}^{n} \|x_i\|^p), \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\max\{f(2^m u^*) - f(2^m u)^*, g(2^m u^*) - g(2^m u)^*, h(2^m u^*) - h(2^m u)^*\} \\ &\leq \theta(n+2)2^{mp} \end{aligned} \tag{2.17}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(\mathcal{A})$, $m = 0, 1, ...,$ and all $a, b \in \mathcal{A}$, then there exist unique $\mathbb{C}$ – linear $*$ – mappings $d, \delta, \epsilon : \mathcal{A} \to \mathcal{A}$ such that

$$\max\{\|f(x)-d(x)\|, \|g(x)-\delta(x)\|, \|h(x)-\epsilon(x)\|\} \leq \frac{2\theta}{2^{n-1}} + \frac{2\theta}{2^{n-1}(1-2^{p-1})}\|x\|^p \tag{2.18}$$

for all $x \in \mathcal{A}$. Moreover, $d : \mathcal{A} \to \mathcal{A}$ is a Lie $*$ – (δ, ϵ) – double derivation on $\mathcal{A}$.

Proof. Define $\varphi(x_1, x_2, ..., x_n, u, b) := \theta(1 + \|b\|^p + \sum_{i=1}^{n} \|x_i\|^p)$ for all $u \in U(\mathcal{A})$ and $b, x_i \in \mathcal{A}$ $(i = 1, ..., n)$, and apply Theorem 2.3. □

Corollary 2.5. *Suppose that $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings satisfying (2.3) and (2.4). If there exists a function $\varphi^{n+2} : \mathcal{A} \to [0, \infty)$ such that*

$$\tilde{\varphi}(x) := \sum_{j=1}^{\infty} 2^j \varphi(\frac{x}{2^j}, \frac{x}{2^j}, 0, ..., 0, 0, 0) < \infty,$$

$$\lim_{j\to\infty} 2^j \varphi(\frac{x_1}{2^j}, \frac{x_2}{2^j}, ..., \frac{a}{2^j}, \frac{b}{2^j}) = 0$$

for all $a, b, x_i \in \mathcal{A}$ $(i = 1, ..., n)$, then there exist unique $\mathbb{C}$ – linear $*$ – mappings $d, \delta, \epsilon : \mathcal{A} \to \mathcal{A}$ such that

$$\max\{\|f(x) - d(x)\|, \|g(x) - \delta(x)\|, \|h(x) - \epsilon(x)\|\} \leq \frac{1}{2^n}\tilde{\varphi}(x)$$

for all $x \in \mathcal{A}$. Moreover, $d : \mathcal{A} \to \mathcal{A}$ is a Lie $*$ – (δ, ϵ) – double derivation on $\mathcal{A}$.

Proof. By the same method as in the proof of Theorem 2.3, one can obtain that

$$d(x) = \lim_{m\to\infty} 2^m f(\frac{x}{2^m}),$$
$$\delta(x) = \lim_{m\to\infty} 2^m g(\frac{x}{2^m}),$$
$$\epsilon(x) = \lim_{m\to\infty} 2^m h(\frac{x}{2^m})$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.6. *If $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings for which exist constants $\theta \geq 0$ and $p > 1$ satisfying (2.16) and (2.17). Then there exist unique $\mathbb{C}$ – linear $*$ – mappings $d, \delta, \epsilon : \mathcal{A} \to \mathcal{A}$ such that*

$$\max\{\|f(x) - d(x)\|, \|g(x) - \delta(x)\|, \|h(x) - \epsilon(x)\|\} \leq \frac{2\theta}{2^{n-1}} + \frac{2\theta}{2^{n-1}(2^{1-p} - 1)}\|x\|^p$$

for all $x \in \mathcal{A}$. Moreover, $d : \mathcal{A} \to \mathcal{A}$ is a Lie $*$ – (δ, ϵ) – double derivation on $\mathcal{A}$.

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