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STABILITY OF GENERALIZED QCA–FUNCTIONAL EQUATION IN P–BANACH SPACES

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

Abstract. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability for the quartic, cubic and additive functional equation

$f(x+ky)+f(x-ky) = k^2f(x+y)+k^2f(x-y)+(k^2-1)[k^2f(y)+k^2f(-y)-2f(x)]$

$(k \in \mathbb{Z} - \{0, \pm 1\})$ in p−Banach spaces.

1. Introduction and preliminaries

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [15] in 1940 and affirmatively solved by Hyers [6]. The result of Hyers was generalized by Aoki [1] for approximate additive function and by Rassias [12] for approximate linear functions by allowing the difference Cauchy equation $\|f(x_1 + x_2) - f(x_1) - f(x_2)\|$ to be controlled by $\varepsilon(\| x_1 \|^p + \| x_2 \|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the generalized Ulam-Rassias stability or Hyers-Ulam-Rassias stability (see [7, 13, 9]). In 1994, a generalization of Rassias [5] theorem was obtained by Găvruta , who replaced $\varepsilon(\| x_1 \|^p + \| x_2 \|^p)$ by a general control function $\varphi(x_1, x_2)$.

Jun and Kim [8] introduced the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.1}$$

and established the general solution and the generalized Hyers–Ulam–Rassias stability for functional equation (1.1). They proved that a function f between two real vector spaces X and Y is a solution of (1.1) if and only if there exists a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. The function C is given by

$$C(x, y, z) = \frac{1}{24}(f(x + y + z) + f(x - y - z) - f(x + y - z) - f(x - y + z))$$

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for all $x, y, z \in X$. Obviously, the function $f(x) = bx^3$ satisfies functional equation (1.1), so it is natural to call (1.1) the cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Lee et. al. [10] considered the following functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \tag{1.2}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1.2) if and only if there exists a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$ such that $f(x) = B_2(x, x)$ for all x. The bi-quadratic function B_2 is given by

$$B_2(x, y) = \frac{1}{12}(f(x + y) + f(x - y) - 2f(x) - 2f(y))$$

for all $x, y \in X$. It is easy to show that the function $f(x) = cx^4$ satisfies the functional equation (1.2), which is called the quartic functional equation .

We consider some basic concepts concerning p–normed spaces.

Definition 1.1. (See [2, 14]). Let X be a real linear space. A function $\| \, . \, \| : X \to \mathbb{R}$ is a quasi-norm (valuation) if it satisfies the following conditions:

(QN_1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(QN_2) $\|\lambda . \, x\| = |\lambda| . \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;

(QN_3) There is a constant $M \geq 1$: $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

Then $(X, \| \, . \, \|)$ is called a quasi-normed space. The smallest possible M is called the modulus of concavity of $\| \, . \, \|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\| \, . \, \|$ is called a p–norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p–Banach space.

By the Aoki-Rolewicz Theorem [14], each quasi-norm is equivalent to some p–norm (see also [2]). Since it is much easier to work with p–norms, henceforth we restrict our attention mainly to p–norms.

Najati and Moghimi [11], have obtained the generalized Hyers-Ulam-Rassias stability for a mixed type of quadratic and additive functional equation. In addition Eshaghi Gordji and Khodaei [3], established the general solution and investigated the Hyers-Ulam-Rassias stability for a mixed type of cubic, quadratic and additive functional equation, with $f(0) = 0$,

$$f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(x) \tag{1.3}$$

in quasi-Banach spaces, where k is nonzero integer numbers with $k \neq \pm 1$. Obviously, the function $f(x) = ax + bx^2 + cx^3$ is a solution of the functional equation (1.3). In this paper, we investigate the generalized Hyers-Ulam stability for the quartic, cubic and additive functional equation:

$$\begin{aligned} f(x + ky) + f(x - ky) = {} & k^2 f(x + y) + k^2 f(x - y) \\ & + (k^2 - 1)[k^2 f(y) + k^2 f(-y) - 2f(x)] \end{aligned} \tag{1.4}$$

($k \in \mathbb{Z} - \{0, \pm 1\}$) in p–Banach spaces. It is easy to see that the function $f(x) = ax + bx^3 + cx^4$ is a solution of the functional equation (1.4). Eshaghi et. al. [4] investigated the general solution of the functional equation (1.4).

2. Main result

In the rest of this paper, we will assume that X be a p–normed space and Y be a p–Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$,

$$D_f(x, y) := f(x + ky) + f(x - ky) - k^2 f(x + y) - k^2 f(x - y) \\ - (k^2 - 1)[k^2 f(y) + k^2 f(-y) - 2f(x)]$$

for all $x, y \in X$.

Lemma 2.1. *(See [11]) Let $0 < p \leq 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then*

$$(\sum_{i=1}^{n} x_i)^p \leq \sum_{i=1}^{n} {x_i}^p.$$

Lemma 2.2. *(See [4]) Let V_1 and V_2 be real vector spaces. If an odd function $f : V_1 \to V_2$ satisfies (1.4), then the function $g : V_1 \to V_2$ defined by $g(x) = f(2x) - 8f(x)$ is additive.*

Theorem 2.3. *Let $\ell \in \{-1, 1\}$ be fixed, and $\varphi_a : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 2^{n\ell} \varphi_a(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}) = 0 \tag{2.1}$$

for all $x, y \in X$, and

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{\imath p\ell} \varphi_a^p(\frac{u}{2^{\imath\ell}}, \frac{y}{2^{\imath\ell}}) < \infty \tag{2.2}$$

for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ and all $y \in \{x, 2x, 3x\}$. (denoted $(\varphi(x, y))^p$ by $\varphi^p(x, y)$). Suppose that an odd function $f : X \to Y$ satisfies the inequality

$$\|D_f(x, y)\| \leq \varphi_a(x, y) \tag{2.3}$$

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ in (2.3) for the case $\ell = 1$. Then the limit

$$A(x) := \lim_{n\to\infty} 2^{n\ell}[f(\frac{x}{2^{n\ell-1}}) - 8f(\frac{x}{2^{n\ell}})] \tag{2.4}$$

exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2}[\widetilde{\psi}_a(x)]^{\frac{1}{p}} \tag{2.5}$$

for all $x \in X$, *where*

$$\widetilde{\psi}_a(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{\imath p\ell} \Big\{ \frac{1}{k^{2p}(k^2-1)^p} \Big[(4k^2-3)^p \varphi_a^p(\frac{x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + k^{2p}\varphi_a^p(\frac{2x}{2^{\imath\ell}}, \frac{2x}{2^{\imath\ell}}) + (2k^2)^p\varphi_a^p(\frac{2x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + (2k^2)^p\varphi_a^p(\frac{x}{2^{\imath\ell}}, \frac{2x}{2^{\imath\ell}}) + \varphi_a^p(\frac{x}{2^{\imath\ell}}, \frac{3x}{2^{\imath\ell}}) + 2^p\varphi_a^p(\frac{(k+1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + 2^p\varphi_a^p(\frac{(k-1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + \varphi_a^p(\frac{(2k+1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + \varphi_a^p(\frac{(2k-1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) \Big] \Big\}. \quad (2.6)$$

Proof. Let $\ell = 1$. It follows from (2.3) and using the oddness of f that

$$\|f(ky+x) - f(ky-x) - k^2f(x+y) - k^2f(x-y) + 2(k^2-1)f(x)\| \leq \varphi_a(x,y) \quad (2.7)$$

for all $x, y \in X$. Letting $y = x$ in (2.7), we have

$$\|f((k+1)x) - f((k-1)x) - k^2f(2x) + 2(k^2-1)f(x)\| \leq \varphi_a(x,x) \quad (2.8)$$

for all $x \in X$. It follows from (2.8) that

$$\|f(2(k+1)x) - f(2(k-1)x) - k^2f(4x) + 2(k^2-1)f(2x)\| \leq \varphi_a(2x,2x) \quad (2.9)$$

for all $x \in X$. Replacing x and y by $2x$ and x in (2.7), respectively, we get

$$\|f((k+2)x) - f((k-2)x) - k^2f(3x) - k^2f(x) + 2(k^2-1)f(2x)\| \leq \varphi_a(2x,x) \quad (2.10)$$

for all $x \in X$. Setting $y = 2x$ in (2.7), gives

$$\|f((2k+1)x) - f((2k-1)x) - k^2f(3x) - k^2f(-x) + 2(k^2-1)f(x)\| \leq \varphi_a(x,2x) \quad (2.11)$$

for all $x \in X$. Putting $y = 3x$ in (2.7), we obtain

$$\|f((3k+1)x) - f((3k-1)x) - k^2f(4x) - k^2f(-2x) + 2(k^2-1)f(x)\| \leq \varphi_a(x,3x) \quad (2.12)$$

for all $x \in X$. Replacing x by $(k+1)x$ and y by x in (2.7), we get

$$\|f((2k+1)x) - f(-x) - k^2f((k+2)x) - k^2f(kx) + 2(k^2-1)f((k+1)x)\| \leq \varphi_a((k+1)x,x) \quad (2.13)$$

for all $x \in X$. Replacing x and y by $(k-1)x$ and x in (2.7), respectively, one gets

$$\|f((2k-1)x) - f(x) - k^2f((k-2)x) - k^2f(kx) + 2(k^2-1)f((k-1)x)\| \leq \varphi_a((k-1)x,x) \quad (2.14)$$

for all $x \in X$. Replacing x and y by $(2k+1)x$ and x in (2.7), respectively, we obtain

$$\|f((3k+1)x) - f(-(k+1)x) - k^2f(2(k+1)x) - k^2f(2kx) + 2(k^2-1)f((2k+1)x)\| \leq \varphi_a((2k+1)x,x) \quad (2.15)$$

for all $x \in X$. Replacing x and y by $(2k-1)x$ and x in (2.7), respectively, we have

$$\begin{aligned} \|f((3k-1)x) - f(-(k-1)x) - k^2 f(2(k-1)x) - k^2 f(2kx) \\ + 2(k^2-1)f((2k-1)x)\| \leq \varphi_a((2k-1)x, x) \end{aligned} \tag{2.16}$$

for all $x \in X$. It follows from (2.8), (2.10), (2.11), (2.13) and (2.14) that

$$\begin{aligned} \|f(3x) - 4f(2x) + 5f(x)\| \leq \frac{1}{k^2(k^2-1)} \,[2(k^2-1)\varphi_a(x,x) + k^2\varphi_a(2x,x) \\ + \varphi_a(x,2x) + \varphi_a((k+1)x,x) + \varphi_a((k-1)x,x)\,] \end{aligned} \tag{2.17}$$

for all $x \in X$. And, from (2.8), (2.9), (2.11), (2.12), (2.15) and (2.16), we conclude that

$$\begin{aligned} \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \leq \frac{1}{k^2(k^2-1)} \,[\varphi_a(x,x) + k^2\varphi_a(2x,2x) \\ + 2(k^2-1)\varphi_a(x,2x) + \varphi_a(x,3x) + \varphi_a((2k+1)x,x) + \varphi_a((2k-1)x,x)\,] \end{aligned} \tag{2.18}$$

for all $x \in X$. Finally, by using (2.17) and (2.18), we obtain that

$$\begin{aligned} \|f(4x) - 10f(2x) + 16f(x)\| \leq \frac{1}{k^2(k^2-1)} \,&[(4k^2-3)\varphi_a(x,x) \\ &+ 2k^2\varphi_a(2x,x) + 2k^2\varphi_a(x,2x) + \varphi_a(x,3x) \\ &+ 2\varphi_a((k+1)x,x) + k^2\varphi_a(2x,2x) \\ &+ 2\varphi_a((k-1)x,x) + \varphi_a((2k+1)x,x) \\ &+ \varphi_a((2k-1)x,x)\,] \end{aligned} \tag{2.19}$$

for all $x \in X$, and let

$$\begin{aligned} \psi_a(x) = \frac{1}{k^2(k^2-1)} \,&[(4k^2-3)\varphi_a(x,x) + k^2\varphi_a(2x,2x) + 2k^2\varphi_a(2x,x) \\ &+ 2k^2\varphi_a(x,2x) + \varphi_a(x,3x) + 2\varphi_a((k+1)x,x) \\ &+ 2\varphi_a((k-1)x,x) + \varphi_a((2k+1)x,x) \\ &+ \varphi_a((2k-1)x,x)\,] \end{aligned} \tag{2.20}$$

for all $x \in X$. Thus (2.19) means that

$$\|f(4x) - 10f(2x) + 16f(x)\| \leq \psi_a(x) \tag{2.21}$$

for all $x \in X$. Let $g : X \to Y$ be a function defined by $g(x) := f(2x) - 8f(x)$ for all $x \in X$. From (2.21), we conclude that

$$\|g(2x) - 2g(x)\| \leq \psi_a(x) \tag{2.22}$$

for all $x \in X$. If we replace x in (2.22) by $\frac{x}{2^{n+1}}$ and multiply both sides of (2.22) by 2^n, we see that

$$\|2^{n+1}g(\frac{x}{2^{n+1}}) - 2^n g(\frac{x}{2^n})\| \leq 2^n\psi_a(\frac{x}{2^{n+1}}) \tag{2.23}$$

for all $x \in X$ and all non-negative integers n. Hence

$$\|2^{n+1}g(\frac{x}{2^{n+1}}) - 2^m g(\frac{x}{2^m})\|^p \leq \sum_{\imath=m}^{n} \|2^{\imath+1}g(\frac{x}{2^{\imath+1}}) - 2^\imath g(\frac{x}{2^\imath})\|^p \leq \sum_{\imath=m}^{n} 2^{\imath p}\psi_a^p(\frac{x}{2^{\imath+1}}) \tag{2.24}$$

for all non-negative integers n and m with $n \geq m$ and all $x \in X$. Since $0 < p \leq 1$, so by Lemma 2.1 and (2.20), we get

$$\begin{aligned}\psi_a^p(x) \leq & \frac{1}{k^{2p}(k^2-1)^p} \,[(4k^2-3)^p\varphi_a^p(x,x) + k^{2p}\varphi_a^p(2x,2x) + (2k^2)^p\varphi_a^p(2x,x) \\ & + (2k^2)^p\varphi_a^p(x,2x) + \varphi_a^p(x,3x) + 2^p\varphi_a^p((k+1)x,x) \\ & + 2^p\varphi_a^p((k-1)x,x) + \varphi_a^p((2k+1)x,x) + \varphi_a^p((2k-1)x,x)\,]\end{aligned} \tag{2.25}$$

for all $x \in X$. Therefore it follows from (2.1), (2.2) and (2.25) that

$$\sum_{\imath=1}^{\infty} 2^{\imath p}\psi_a^p(\frac{x}{2^\imath}) < \infty, \qquad \lim_{n\to\infty} 2^n\psi_a(\frac{x}{2^n}) = 0 \tag{2.26}$$

for all $x \in X$. It follows from (2.24) and (2.26) that the sequence $\{2^n g(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n g(\frac{x}{2^n})\}$ converges for all $x \in X$. Therefore, one can define a function $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n g(\frac{x}{2^n}) \tag{2.27}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.24), we get

$$\|g(x) - A(x)\|^p \leq \sum_{\imath=0}^{\infty} 2^{\imath p}\psi_a^p(\frac{x}{2^{\imath+1}}) = \frac{1}{2^p}\sum_{\imath=1}^{\infty} 2^{\imath p}\psi_a^p(\frac{x}{2^\imath}) \tag{2.28}$$

for all $x \in X$. Therefore (2.5) follows from (2.25) and (2.28). Now we show that A is additive. It follows from (2.23), (2.26) and (2.27) that

$$\begin{aligned}\|A(2x) - 2A(x)\| = \lim_{n\to\infty} \|2^n g(\frac{x}{2^{n-1}}) - 2^{n+1}(\frac{x}{2^n})\| &= 2\lim_{n\to\infty} \|2^{n-1} g(\frac{x}{2^{n-1}}) - 2^n g(\frac{x}{2^n})\| \\ &\leq \lim_{n\to\infty} 2^n\psi_a(\frac{x}{2^n}) = 0\end{aligned}$$

for all $x \in X$. So

$$A(2x) = 2A(x) \tag{2.29}$$

for all $x \in X$. On the other hand it follows from (2.1), (2.3) and (2.27) that

$$\begin{aligned}\|D_A(x,y)\| &= \lim_{n\to\infty} 2^n\|D_g(\frac{x}{2^n},\frac{y}{2^n})\| = \lim_{n\to\infty} 2^n\|D_f(\frac{x}{2^{n-1}},\frac{y}{2^{n-1}}) - 8D_f(\frac{x}{2^n},\frac{y}{2^n})\| \\ &\leq \lim_{n\to\infty} 2^n\{\|D_f(\frac{x}{2^{n-1}},\frac{y}{2^{n-1}})\| + 8\|D_f(\frac{x}{2^n},\frac{y}{2^n})\|\} \\ &\leq \lim_{n\to\infty} 2^n\{\varphi_a(\frac{x}{2^{n-1}},\frac{y}{2^{n-1}}) + 8\varphi_a(\frac{x}{2^n},\frac{y}{2^n})\} = 0\end{aligned}$$

for all $x, y \in X$. Hence the function A satisfies (1.4). Thus by Lemma 2.2, the function $x \rightsquigarrow A(2x) - 8A(x)$ is additive. Therefore (2.29) implies that the function A is additive.

To prove the uniqueness property of A, let $A' : X \to Y$ be another additive function satisfying (2.5). Since

$$\lim_{n\to\infty} 2^{np}\sum_{\imath=1}^{\infty} 2^{\imath p}\varphi_a^p(\frac{u}{2^{n+\imath}},\frac{y}{2^{n+\imath}}) = \lim_{n\to\infty}\sum_{\imath=n+1}^{\infty} 2^{\imath p}\varphi_a^p(\frac{u}{2^\imath},\frac{y}{2^\imath}) = 0$$

for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ and all $y \in \{x, 2x, 3x\}$. Hence

$$\lim_{n\to\infty} 2^{np}\widetilde{\psi}_a(\frac{x}{2^n}) = 0 \tag{2.30}$$

for all $x \in X$. It follows from (2.5) and (2.30) that

$$\|A(x) - A^{'}(x)\|^p = \lim_{n\to\infty} 2^{np}\|g(\frac{x}{2^n}) - A^{'}(\frac{x}{2^n})\|^p \leq \frac{1}{2^p}\lim_{n\to\infty} 2^{np}\widetilde{\psi}_a(\frac{x}{2^n}) = 0$$

for all $x \in X$. So $A = A^{'}$.

For $\ell = -1$, we can prove the theorem by a similar technique. □

Corollary 2.4. *Let ϵ, r, s be non-negative real numbers such that $r, s > 1$ or $r, s < 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$\|D_f(x,y)\| \leq \begin{cases} \epsilon, & r=s=0; \\ \epsilon\|x\|^r, & r>0,\ s=0; \\ \epsilon\|y\|^s, & r=0,\ s>0; \\ \epsilon(\|x\|^r + \|y\|^s), & r,\ s>0. \end{cases} \tag{2.31}$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{\epsilon}{k^2(k^2-1)} \begin{cases} \delta_a, & r=s=0; \\ \alpha_a\ \|x\|^r, & r>0,\ s=0; \\ \beta_a\ \|x\|^s, & r=0,\ s>0; \\ (\alpha_a^p\ \|x\|^{rp} + \beta_a^p\ \|x\|^{sp})^{\frac{1}{p}}, & r,\ s>0. \end{cases}$$

for all $x \in X$, where

$$\delta_a = \{\ \frac{1}{2^p - 1}\ [(4k^2-3)^p + 2^{p+1}(k^{2p}+1) + k^{2p} + 3]\ \}^{\frac{1}{p}},$$

$$\alpha_a = \{\ \frac{1}{|2^p - 2^{rp}|}\ [(4k^2-3)^p + (2k+1)^{rp} + (2k-1)^{rp} + 2^p(k+1)^{rp} + 2^p(k-1)^{rp} + k^{2p}(2^{(r+1)p} + 2^{rp} + 2^p) + 1]\ \}^{\frac{1}{p}},$$

$$\beta_a = \{\ \frac{1}{|2^p - 2^{sp}|}\ [(4k^2-3)^p + k^{2p}(2^{(s+1)p} + 2^{sp} + 2^p) + 3^{sp} + 2^{p+1} + 2]\ \}^{\frac{1}{p}}.$$

Proof. It follows from Theorem 2.3 by putting $\varphi(x,y) := \epsilon(\|x\|^r + \|y\|^s)$ for all $x, y \in X$. □

Lemma 2.5. (*See* [4]) *Let V_1 and V_2 be real vector spaces. If an odd function $f : V_1 \to V_2$ satisfies (1.4), then the function $h : V_1 \to V_2$ defined by $h(x) = f(2x) - 2f(x)$ is cubic.*

Theorem 2.6. *Let $\ell \in \{-1, 1\}$ be fixed, and $\varphi_c : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 8^{n\ell}\varphi_c(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}) = 0 \tag{2.32}$$

for all $x, y \in X$ and

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} 8^{\imath p\ell}\varphi_c^p(\frac{u}{2^{\imath\ell}}, \frac{y}{2^{\imath\ell}}) < \infty \tag{2.33}$$

for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ *and all* $y \in \{x, 2x, 3x\}$. *Suppose that an odd function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|D_f(x,y)\| \leq \varphi_c(x,y) \tag{2.34}$$

for all $x, y \in X$. *Then the limit*

$$C(x) := \lim_{n\to\infty} 8^{n\ell}[f(\frac{x}{2^{n\ell-1}}) - 2f(\frac{x}{2^{n\ell}})] \tag{2.35}$$

exists for all $x \in X$ *and* $C : X \to Y$ *is a unique cubic function satisfying*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8}[\widetilde{\psi}_c(x)]^{\frac{1}{p}} \tag{2.36}$$

for all $x \in X$, *where*

$$\begin{aligned}\widetilde{\psi}_c(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 8^{\imath p\ell} \Big\{ \frac{1}{k^{2p}(k^2-1)^p} \Big[& (4k^2-3)^p\varphi_c^p(\frac{x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + k^{2p}\varphi_c^p(\frac{2x}{2^{\imath\ell}}, \frac{2x}{2^{\imath\ell}}) \\ & + (2k^2)^p\varphi_c^p(\frac{2x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + (2k^2)^p\varphi_c^p(\frac{x}{2^{\imath\ell}}, \frac{2x}{2^{\imath\ell}}) + \varphi_c^p(\frac{x}{2^{\imath\ell}}, \frac{3x}{2^{\imath\ell}}) \\ & + 2^p\varphi_c^p(\frac{(k+1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + 2^p\varphi_c^p(\frac{(k-1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) \\ & + \varphi_c^p(\frac{(2k+1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) + \varphi_c^p(\frac{(2k-1)x}{2^{\imath\ell}}, \frac{x}{2^{\imath\ell}}) \Big] \Big\}.\end{aligned} \tag{2.37}$$

Proof. Let $\ell = 1$. Similar to the proof of Theorem 2.3, we have

$$\|f(4x) - 10f(2x) + 16f(x)\| \leq \psi_c(x), \tag{2.38}$$

for all $x \in X$, where

$$\begin{aligned}\psi_c(x) = \frac{1}{k^2(k^2-1)} & [(4k^2-3)\varphi_c(x,x) + k^2\varphi_c(2x,2x) + 2k^2\varphi_c(2x,x) \\ & + 2k^2\varphi_c(x,2x) + \varphi_c(x,3x) + 2\varphi_c((k+1)x,x) \\ & + 2\varphi_c((k-1)x,x) + \varphi_c((2k+1)x,x) \\ & + \varphi_c((2k-1)x,x)]\end{aligned} \tag{2.39}$$

for all $x \in X$. Letting $h : X \to Y$ be a function defined by $h(x) := f(2x) - 2f(x)$. Then, we see that

$$\|h(2x) - 8h(x)\| \leq \psi_c(x) \tag{2.40}$$

for all $x \in X$. If we replace x in (2.40) $\frac{x}{2^{n+1}}$ and multiply both sides of (2.40) by 8^n, we get

$$\|8^{n+1}h(\frac{x}{2^{n+1}}) - 8^n h(\frac{x}{2^n})\| \leq 8^n\psi_c(\frac{x}{2^{n+1}}) \tag{2.41}$$

for all $x \in X$ and all non-negative integers n. Hence

$$\begin{aligned}\|8^{n+1}h(\frac{x}{2^{n+1}}) - 8^m h(\frac{x}{2^m})\|^p & \leq \sum_{\imath=m}^{n} \|8^{\imath+1}h(\frac{x}{2^{\imath+1}}) - 8^\imath h(\frac{x}{2^\imath})\|^p \\ & \leq \sum_{\imath=m}^{n} 8^{\imath p}{\psi_c}^p(\frac{x}{2^{\imath+1}})\end{aligned} \tag{2.42}$$

for all non-negative integers n and m with $n \geq m$ and all $x \in X$. Since $0 < p \leq 1$, so by Lemma 2.1 and (2.39), we get

$$\begin{aligned}\psi_c^p(x) \leq & \frac{1}{k^{2p}(k^2-1)^p} \ [(4k^2-3)^p\varphi_c^p(x,x) + k^{2p}\varphi_c^p(2x,2x) + (2k^2)^p\varphi_c^p(2x,x) \\ & + (2k^2)^p\varphi_c^p(x,2x) + \varphi_c^p(x,3x) + 2^p\varphi_c^p((k+1)x,x) \\ & + 2^p\varphi_c^p((k-1)x,x) + \varphi_c^p((2k+1)x,x) \\ & + \varphi_c^p((2k-1)x,x)\ ]\end{aligned} \tag{2.43}$$

for all $x \in X$. Therefore it follows from (2.32), (2.33) and (2.43) that

$$\sum_{\imath=1}^{\infty} 2^{\imath p}\psi_c^p(\frac{x}{2^\imath}) < \infty, \qquad \lim_{n\to\infty} 2^n\psi_c(\frac{x}{2^n}) = 0 \tag{2.44}$$

for all $x \in X$. Therefore we conclude from (2.42) and (2.44) that the sequence $\{8^n h(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{8^n h(\frac{x}{2^n})\}$ converges for all $x \in X$. So one can define the function $C : X \to Y$ by

$$C(x) = \lim_{n\to\infty} 8^n h(\frac{x}{2^n}) \tag{2.45}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.42), we get

$$\|h(x) - C(x)\|^p \leq \sum_{\imath=0}^{\infty} 8^{\imath p}{\psi_c}^p(\frac{x}{2^{\imath+1}}) = \frac{1}{8^p}\sum_{\imath=1}^{\infty} 8^{\imath p}{\psi_c}^p(\frac{x}{2^\imath}) \tag{2.46}$$

for all $x \in X$. Therefore, (2.36) follows from (2.43) and (2.46). Now we show that C is cubic. It follows from (2.41), (2.44) and (2.45) that

$$\begin{aligned}\|C(2x) - 8C(x)\| = \lim_{n\to\infty} \|8^n h(\frac{x}{2^{n-1}}) - 8^{n+1}h(\frac{x}{2^n})\| &= 8 \lim_{n\to\infty} \|8^{n-1}h(\frac{x}{2^{n-1}}) - 8^n h(\frac{x}{2^n})\| \\ &\leq \lim_{n\to\infty} 8^n\psi_c(\frac{x}{2^n}) = 0\end{aligned}$$

for all $x \in X$. So

$$C(2x) = 8C(x) \tag{2.47}$$

for all $x \in X$. On the other hand it follows from (2.32), (2.34) and (2.45) that

$$\begin{aligned}\|D_C(x,y)\| &= \lim_{n\to\infty} 8^n\|D_h(\frac{x}{2^n}, \frac{y}{2^n})\| = \lim_{n\to\infty} 8^n\|D_f(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}) - 2D_f(\frac{x}{2^n}, \frac{y}{2^n})\| \\ &\leq \lim_{n\to\infty} 8^n\{\|D_f(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}})\| + 2\|D_f(\frac{x}{2^n}, \frac{y}{2^n})\|\} \\ &\leq \lim_{n\to\infty} 8^n\{\varphi_c(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}) + 2\varphi_c(\frac{x}{2^n}, \frac{y}{2^n})\} = 0\end{aligned}$$

for all $x, y \in X$. Hence the function C satisfies (1.4). By Lemma 2.5, the function $x \rightsquigarrow C(2x) - 2C(x)$ is cubic. Hence, (2.47) implies that function C is cubic.

To prove the uniqueness of C, let $C' : X \to Y$ be another additive function satisfying (2.36). Since

$$\lim_{n\to\infty} 8^{np}\sum_{\imath=1}^{\infty} 8^{\imath p}{\varphi_c}^p(\frac{u}{2^{n+\imath}}, \frac{y}{2^{n+\imath}}) = \lim_{n\to\infty} \sum_{\imath=n+1}^{\infty} 8^{\imath p}{\varphi_c}^p(\frac{u}{2^\imath}, \frac{y}{2^\imath}) = 0$$

for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ and all $y \in \{x, 2x, 3x\}$. Hence

$$\lim_{n\to\infty} 8^{np}\widetilde{\psi}_c(\frac{x}{2^n}) = 0 \tag{2.48}$$

for all $x \in X$. It follows from (2.36) and (2.48) that

$$\|C(x) - C^{'}(x)\| = \lim_{n\to\infty} 8^{np}\|h(\frac{x}{2^n}) - C^{'}(\frac{x}{2^n})\|^p \leq \frac{1}{8^p}\lim_{n\to\infty} 8^{np}\widetilde{\psi}_c(\frac{x}{2^n}) = 0$$

for all $x \in X$. So $C = C^{'}$.

For $\ell = -1$, we can prove the theorem by a similar technique. □

Corollary 2.7. *Let ϵ, r, s be non-negative real numbers such that $r, s > 3$ or $r, s < 3$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (2.31) for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfying*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{\epsilon}{k^2(k^2-1)} \begin{cases} \delta_c, & r=s=0; \\ \alpha_c\ \|x\|^r, & r>0,\ s=0; \\ \beta_c\ \|x\|^s, & r=0,\ s>0; \\ (\alpha_c^p\ \|x\|^{rp} + \beta_c^p\ \|x\|^{sp})^{\frac{1}{p}}, & r,\ s>0. \end{cases}$$

for all $x \in X$, where

$$\delta_c = \{\ \frac{1}{8^p - 1}\ [(4k^2-3)^p + 2^{p+1}(k^{2p}+1) + k^{2p} + 3]\ \}^{\frac{1}{p}},$$

$$\alpha_c = \{\ \frac{1}{|8^p - 2^{rp}|}\ [(4k^2-3)^p + (2k+1)^{rp} + (2k-1)^{rp} + 2^p(k+1)^{rp} + 2^p(k-1)^{rp} + k^{2p}(2^{(r+1)p} + 2^{rp} + 2^p) + 1]\ \}^{\frac{1}{p}},$$

$$\beta_c = \{\ \frac{1}{|8^p - 2^{sp}|}\ [(4k^2-3)^p + k^{2p}(2^{(s+1)p} + 2^{sp} + 2^p) + 3^{sp} + 2^{p+1} + 2]\ \}^{\frac{1}{p}}.$$

Theorem 2.8. *Let $\ell \in \{-1, 1\}$ be fixed, and $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \{(\frac{1-\ell}{2})2^{n\ell}\varphi(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}) + (\frac{1+\ell}{2})8^{n\ell}\varphi(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}})\} = 0 \tag{2.49}$$

for all $x, y \in X$ and

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} \{(\frac{1-\ell}{2})2^{\imath p\ell}\varphi^p(\frac{u}{2^{\imath\ell}}, \frac{y}{2^{\imath\ell}}) + (\frac{1+\ell}{2})8^{\imath p\ell}\varphi^p(\frac{u}{2^{\imath\ell}}, \frac{y}{2^{\imath\ell}})\} < \infty \tag{2.50}$$

for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ and all $y \in \{x, 2x, 3x\}$. Suppose that an odd function $f : X \to Y$ satisfies the inequality $\|D_f(x, y)\| \leq \varphi(x, y)$ for all $x, y \in X$. Then there exist a unique cubic function $C : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\|f(x) - C(x) - A(x)\| \leq \frac{1}{48}\ (4[\widetilde{\psi}_a(x)]^{\frac{1}{p}} + [\widetilde{\psi}_c(x)]^{\frac{1}{p}}) \tag{2.51}$$

for all $x \in X$, *where*

$$\begin{aligned}\widetilde{\psi}_a(x) :=& \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 2^{\imath p\ell} \Big\{\frac{1}{k^{2p}(k^2-1)^p} \; [\; (4k^2-3)^p\varphi^p(\frac{x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + k^{2p}\varphi^p(\frac{2x}{2^{\imath\ell}},\frac{2x}{2^{\imath\ell}}) \\ &+ (2k^2)^p\varphi^p(\frac{2x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + (2k^2)^p\varphi^p(\frac{x}{2^{\imath\ell}},\frac{2x}{2^{\imath\ell}}) + \varphi^p(\frac{x}{2^{\imath\ell}},\frac{3x}{2^{\imath\ell}}) \\ &+ 2^p\varphi^p(\frac{(k+1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + 2^p\varphi^p(\frac{(k-1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) \\ &+ \varphi^p(\frac{(2k+1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + \varphi^p(\frac{(2k-1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) \;] \Big\},\end{aligned} \tag{2.52}$$

$$\begin{aligned}\widetilde{\psi}_c(x) :=& \sum_{\imath=\frac{1+\ell}{2}}^{\infty} 8^{\imath p\ell} \Big\{\frac{1}{k^{2p}(k^2-1)^p} \; [\; (4k^2-3)^p\varphi^p(\frac{x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + k^{2p}\varphi^p(\frac{2x}{2^{\imath\ell}},\frac{2x}{2^{\imath\ell}}) \\ &+ (2k^2)^p\varphi^p(\frac{2x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + (2k^2)^p\varphi^p(\frac{x}{2^{\imath\ell}},\frac{2x}{2^{\imath\ell}}) + \varphi^p(\frac{x}{2^{\imath\ell}},\frac{3x}{2^{\imath\ell}}) \\ &+ 2^p\varphi^p(\frac{(k+1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + 2^p\varphi^p(\frac{(k-1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) \\ &+ \varphi^p(\frac{(2k+1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) + \varphi^p(\frac{(2k-1)x}{2^{\imath\ell}},\frac{x}{2^{\imath\ell}}) \;] \Big\}.\end{aligned} \tag{2.53}$$

Proof. Let $\ell = 1$. By Theorem 2.3 and 2.6, there exist an additive function $A_0 : X \to Y$ and a cubic function $C_0 : X \to Y$ such that

$$\|f(2x) - 8f(x) - A_0(x)\| \le \frac{1}{2}[\widetilde{\psi}_a(x)]^{\frac{1}{p}}, \qquad \|f(2x) - 2f(x) - C_0(x)\| \le \frac{1}{8}[\widetilde{\psi}_c(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore, it follows from the last inequality that

$$\|f(x) + \frac{1}{6}A_0(x) - \frac{1}{6}C_0(x)\| \le \frac{1}{48} \; (4[\widetilde{\psi}_a(x)]^{\frac{1}{p}} + [\widetilde{\psi}_c(x)]^{\frac{1}{p}})$$

for all $x \in X$. So we obtain (2.51) by letting $A(x) = -\frac{1}{6}A_0(x)$ and $C(x) = \frac{1}{6}C_0(x)$ for all $x \in X$. To prove the uniqueness property of A and C, let $A_1, C_1 : X \to Y$ be another additive and cubic functions satisfying (2.51). Let $A^{'} = A - A_1$ and $C^{'} = C - C_1$. So

$$\begin{aligned}\|A^{'}(x) + C^{'}(x)\| &\le \{\|f(x) - A(x) - C(x)\| + \|f(x) - A_1(x) - C_1(x)\|\} \\ &\le \frac{1}{24} \; (4[\widetilde{\psi}_a(x)]^{\frac{1}{p}} + [\widetilde{\psi}_c(x)]^{\frac{1}{p}})\end{aligned} \tag{2.54}$$

for all $x \in x$. Since

$$\lim_{n\to\infty} 2^{np}\widetilde{\psi}_a(\frac{x}{2^n}) = \lim_{n\to\infty} 8^{np}\widetilde{\psi}_c(\frac{x}{2^n}) = 0$$

for all $x \in X$, so if we replace x in (2.54) $\frac{x}{2^n}$ and multiply both sides of (2.54) by 8^n, we get

$$\lim_{n\to\infty} 8^n\|A^{'}(\frac{x}{2^n}) + C^{'}(\frac{x}{2^n})\| = 0$$

for all $x \in X$. Therefore $C^{'} = 0$. So it follows from (2.54) that

$$\|A^{'}(x)\| \le \frac{5}{24}[\widetilde{\psi}_a(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore $A^{'} = 0$.

For $\ell = -1$, we can prove the theorem by a similar technique. □

Corollary 2.9. *Let ϵ, r, s be non-negative real numbers such that $r, s > 3$ or $1 < r, s < 3$ or $r, s < 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (2.31) for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that*

$$\|f(x) - A(x) - C(x)\| \leq \frac{\epsilon}{6k^2(k^2-1)} \begin{cases} \delta_a + \delta_c, & r = s = 0; \\ (\alpha_a + \alpha_c)\ \|x\|^r, & r > 0,\ s = 0; \\ (\beta_a + \beta_c)\ \|x\|^s, & r = 0,\ s > 0; \\ \gamma_a(x) + \gamma_c(x), & r,\ s > 0. \end{cases}$$

for all $x \in X$, where $\delta_a, \delta_c, \alpha_a, \alpha_c, \beta_a$ and β_c are defined as in Corollaries 2.4 and 2.7 and

$$\gamma_a(x) = \{\alpha_a^p\ \|x\|^{rp} + \beta_a^p\ \|x\|^{sp}\}^{\frac{1}{p}}, \qquad \gamma_c(x) = \{\alpha_c^p\ \|x\|^{rp} + \beta_c^p\ \|x\|^{sp}\}^{\frac{1}{p}}$$

for all $x \in X$.

Lemma 2.10. (*See* [4]) *Let V_1 and V_2 be real vector spaces. If an even function $f : V_1 \to V_2$ satisfies (1.4), then f is quartic.*

Theorem 2.11. *Let $\ell \in \{-1, 1\}$ be fixed, and $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} k^{4n\ell}\varphi(\frac{x}{k^{n\ell}}, \frac{y}{k^{n\ell}}) = 0 \tag{2.55}$$

for all $x, y \in X$ and

$$\tilde{\psi}_e(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} k^{4\imath p\ell}\varphi^p(0, \frac{x}{k^{\imath\ell}}) < \infty \tag{2.56}$$

for all $x \in X$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\| \leq \varphi(x, y) \tag{2.57}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n\to\infty} k^{4n\ell} f(\frac{x}{k^{n\ell}}) \tag{2.58}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\| \leq \frac{1}{2k^4}[\tilde{\psi}_e(x)]^{\frac{1}{p}} \tag{2.59}$$

for all $x \in X$.

Proof. For $\ell = 1$, setting $x = 0$ in (2.57) and then use $f(0) = 0$ and evenness of f, we obtain that

$$\|2f(ky) - 2k^4 f(y)\| \leq \varphi(0, y) \tag{2.60}$$

for all $y \in X$. Replacing y by x in (2.60) and divide both sides of (2.60) by 2, we get

$$\|f(kx) - k^4 f(x)\| \leq \frac{1}{2}\varphi(0, x) \tag{2.61}$$

for all $x \in X$. Let $\psi_e(x) = \frac{1}{2}\varphi(0, x)$ for all $x \in X$, then by (2.61), we get

$$\|f(kx) - k^4 f(x)\| \leq \psi_e(x) \tag{2.62}$$

for all $x \in X$. If we replace x in (2.62) by $\frac{x}{k^{n+1}}$ and multiply both sides of (2.62) by k^{4n}, then we have

$$\|k^{4(n+1)} f(\frac{x}{k^{n+1}}) - k^{4n} f(\frac{x}{k^n})\| \leq k^{4n}\psi_e(\frac{x}{k^{n+1}}) \tag{2.63}$$

for all $x \in X$ and all non-negative integers n. Hence

$$\begin{aligned} \|k^{4(n+1)} f(\frac{x}{k^{n+1}}) - k^{4m} f(\frac{x}{k^m})\|^p &\leq \sum_{\imath=m}^{n} \|k^{4(\imath+1)} f(\frac{x}{k^{\imath+1}}) - k^{4\imath} f(\frac{x}{k^\imath})\|^p \\ &\leq \sum_{\imath=m}^{n} k^{4\imath p} {\psi_e}^p(\frac{x}{k^{\imath+1}}) \end{aligned} \tag{2.64}$$

for all non-negative integers n and m with $n \geq m$ and all $x \in X$. Since ${\psi_e}^p(x) = \frac{1}{2^p}\varphi^p(0, x)$ for all $x \in X$, therefore by (2.56) we have

$$\sum_{\imath=1}^{\infty} k^{4\imath p} {\psi_e}^p(\frac{x}{k^\imath}) < \infty \tag{2.65}$$

for all $x \in X$. Therefore we conclude from (2.64) and (2.65) that the sequence $\{k^{4n} f(\frac{x}{k^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{k^{4n} f(\frac{x}{k^n})\}$ converges for all $x \in X$. So one can define the function $Q : X \to Y$ by (2.58) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.64), we get

$$\|f(x) - Q(x)\|^p \leq \sum_{\imath=0}^{\infty} k^{4\imath p} {\psi_e}^p(\frac{x}{k^{\imath+1}}) = \frac{1}{k^{4p}} \sum_{\imath=1}^{\infty} k^{4\imath p} {\psi_e}^p(\frac{x}{k^\imath}) \tag{2.66}$$

for all $x \in X$. Therefore (2.59) follows from (2.56) and (2.66). Now we show that Q is quartic. It follows from (2.55), (2.57) and (2.58) that

$$\|D_Q(x, y)\| = \lim_{n\to\infty} k^{4n}\|D_f(\frac{x}{k^n}, \frac{y}{k^n})\| \leq \lim_{n\to\infty} k^{4n}\varphi(\frac{x}{k^n}, \frac{y}{k^n}) = 0$$

for all $x, y \in X$. Therefore the function $Q : X \to Y$ satisfies (1.4). Since f is an even function, then (2.58) implies that the function $Q : X \to Y$ is even. By Lemma 2.10, the function $x \rightsquigarrow Q(x)$ is quartic.

To prove the uniqueness of Q, let $Q' : X \to Y$ be another quartic function satisfying (2.59). Since

$$\lim_{n\to\infty} k^{4np} \sum_{\imath=1}^{\infty} k^{4\imath p}\varphi^p(0, \frac{x}{k^{\imath+n}}) = \lim_{n\to\infty} \sum_{\imath=n+1}^{\infty} k^{4\imath p}\varphi^p(0, \frac{x}{k^\imath}) = 0$$

for all $x \in X$, then $\lim_{n\to\infty} k^{4np}\tilde{\psi}_e(\frac{x}{k^n}) = 0$ for all $x \in X$. Therefore it follows from (2.59) and the last equation that

$$\|Q(x) - Q'(x)\|^p = \lim_{n\to\infty} k^{4np}\|f(\frac{x}{k^n}) - Q'(\frac{x}{k^n})\|^p \leq \frac{1}{(2k^4)^p} \lim_{n\to\infty} k^{4np}\tilde{\psi}_e(\frac{x}{k^n}) = 0$$

for all $x \in X$. Hence $Q = Q'$. For $\ell = -1$, we can prove the theorem by a similar technique. □

Corollary 2.12. *Let ϵ, r, s be non-negative real numbers such that $r, s > 2$ or $0 \leq r, s < 2$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x, y)\| \leq \epsilon(\|x\|^r + \|y\|^s) \tag{2.67}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ satisfies

$$\|f(x) - Q(x)\| \leq \frac{\epsilon}{2}\left(\frac{1}{|k^{4p} - k^{sp}|}\ \|x\|^{sp}\right)^{\frac{1}{p}}$$

for all $x \in X$.

Now, we are ready to prove the main theorem concerning the stability problem for the equation (1.4).

Theorem 2.13. *Let $\varphi : X \times X \to [0, \infty)$ be a function which satisfies (2.55) for all $x, y \in X$ and (2.56) for all $x \in X$, and satisfies (2.49) for all $x, y \in X$ and (2.50) for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ and all $y \in \{x, 2x, 3x\}$. Suppose that a function $f : X \to Y$ satisfies the inequality (2.57) for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ in (2.57) for the case f is even. Then there exist a unique cubic function $C : X \to Y$, a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

$$\begin{aligned}\|f(x) - C(x) - Q(x) - A(x)\| \leq{}& \frac{1}{4k^4}\{[\widetilde{\psi}_e(x) + \widetilde{\psi}_e(-x)]^{\frac{1}{p}}\} \\ &+ \frac{1}{96}\{4[\widetilde{\psi}_a(x) + \widetilde{\psi}_a(-x)]^{\frac{1}{p}} + [\widetilde{\psi}_c(x) + \widetilde{\psi}_c(-x)]^{\frac{1}{p}}\}\end{aligned} \tag{2.68}$$

for all $x \in X$, where $\widetilde{\psi}_e(x)$, $\widetilde{\psi}_a(x)$ and $\widetilde{\psi}_c(x)$ are defined as in equations (2.52), (2.53) and (2.56).

Proof. Assume that $\varphi : X \times X \to [0, \infty)$ satisfies (2.55) for all $x, y \in X$ and (2.56) for all $x \in X$. Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$, then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\|D_{f_e}(x, y)\| \leq \tilde{\varphi}(x, y)$$

for all $x, y \in X$, where $\tilde{\varphi}(x, y) := \frac{1}{2}(\varphi(x, y) + \varphi(-x, -y))$. So

$$\lim_{n\to\infty} k^{4n\ell}\tilde{\varphi}\left(\frac{x}{k^{n\ell}}, \frac{y}{k^{n\ell}}\right) = 0$$

for all $x, y \in X$. Since

$$\tilde{\varphi}^p(x, y) \leq \frac{1}{2^p}(\varphi^p(x, y) + \varphi^p(-x, -y))$$

for all $x, y \in X$, then

$$\sum_{\imath=\frac{1+\ell}{2}}^{\infty} k^{4\imath p\ell}\tilde{\varphi}^p\left(0, \frac{x}{k^{\imath\ell}}\right) < \infty$$

for all $x \in X$. Hence, from Theorem 2.11, there exist a unique quartic function $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2k^4}\ [\tilde{\widetilde{\psi}}_e(x)]^{\frac{1}{p}} \tag{2.69}$$

for all $x \in X$, where

$$\tilde{\tilde{\psi}}_e(x) := \sum_{\imath=\frac{1+\ell}{2}}^{\infty} k^{4\imath p\ell} \tilde{\varphi}^p(0, \frac{x}{k^{\imath\ell}})$$

for all $x \in X$. It is clear that

$$\tilde{\tilde{\psi}}_e(x) \leq \frac{1}{2^p} \, [\widetilde{\psi}_e(x) + \widetilde{\psi}_e(-x)]$$

for all $x \in X$. Therefore it follows from (2.69) that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{4k^4} \, [\widetilde{\psi}_e(x) + \widetilde{\psi}_e(-x)]^{\frac{1}{p}} \tag{2.70}$$

for all $x \in X$.

Also, let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$, by using the above method and Theorem 2.8, it follows that there exist a unique cubic function $C : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\|f_o(x) - C(x) - A(x)\| \leq \frac{1}{96} \, (4[\widetilde{\psi}_a(x) + \widetilde{\psi}_a(-x)]^{\frac{1}{p}} + [\widetilde{\psi}_c(x) + \widetilde{\psi}_c(-x)]^{\frac{1}{p}}) \tag{2.71}$$

for all $x \in X$. Hence (2.68) follows from (2.70) and (2.71). Now, if $\varphi : X \times X \to [0, \infty)$ satisfies (2.49) for all $x, y \in X$ and (2.50) for all $u \in \{x, 2x, (k-1)x, (k+1)x, (2k-1)x, (2k+1)x\}$ and all $y \in \{x, 2x, 3x\}$, we can prove the theorem by a similar technique. □

Corollary 2.14. *Let ϵ, r, s be non-negative real numbers such that $r, s > 3$ or $2 < r, s < 3$ or $1 < r, s < 2$ or $r, s < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality (2.67) for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ for the case f is even. Then there exist a unique cubic function $C : X \to Y$, a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that*

$$\begin{aligned}\|f(x) - C(x) - Q(x) - A(x)\| &\leq \frac{\epsilon}{6k^2(k^2-1)}(\lambda_a(x) + \lambda_c(x)) \\ &+ \frac{\epsilon}{2} \, (\frac{1}{|k^{4p} - k^{sp}|} \, \|x\|^{sp})^{\frac{1}{p}}\end{aligned}$$

for all $x \in X$, where $\lambda_a(x)$ and $\lambda_c(x)$ are defined as in Corollary 2.9.

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