NEW INEQUALITIES FOR A CLASS OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities related to Chebyshev’s functional in the case of two differentiable functions.

1. Introduction and Basic Definitions

Let us consider

\[ T(f, g) := \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \left( \int_a^b f(x) \, dx \right) \left( \frac{1}{b-a} \int_a^b g(x) \, dx \right) \tag{1.1} \]

where \( f \) and \( g \) are two integrable functions on \([a, b]\) [4].

The relation (1.1) has evoked the interest of many researchers and several inequalities related to this functional have appeared in the literature, to mention a few, see [1, 2, 6, 7] and the references cited therein.

The main aim of this paper is to establish some new inequalities for (1.1) by using the Riemann-Liouville fractional integrals. We give our results in the case of differentiable functions.

We shall introduce the following spaces which are used throughout this paper.

Let \( C([0, \infty[) \) the space of all continuous functions from \([0, \infty[\] into \( \mathbb{R} \) and let \( L_\infty([0, \infty[) \) the space of essentially bounded functions \( f(x) \) on \([0, \infty[\), with the norm \( \|f\|_\infty := \inf \{C \geq 0, |f(x)| \leq C; \text{for almost every } x \in [0, \infty[\} \).

For the Riemann-Liouville integrals, we give the following definitions and properties.

Definition 1.1. A real valued function \( f(t), t \geq 0 \) is said to be in the space \( C_\mu, \mu \in \mathbb{R} \) if there exists a real number \( p > \mu \) such that \( f(t) = t^p f_1(t) \), where \( f_1(t) \in C([0, \infty[) \).

Definition 1.2. A function \( f(t), t \geq 0 \) is said to be in the space \( C_\mu^n, \mu \in \mathbb{R} \), if \( f^{(n)}(t) \in C_\mu \)

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Definition 1.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_{\mu}, (\mu \geq -1)$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0,$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0,$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t).$$

For more details, one can consult [8].

2. Main Results

Theorem 2.1. Let $f$ and $g$ be two differentiable functions on $[0, \infty[$ such that $f', g' \in L_\infty([0, \infty])$. Then for all $t > 0, \alpha > 0$, we have:

$$\left| \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha fg(t) - J^\alpha f(t) J^\alpha g(t) \right|$$

$$\leq \|f'\|_\infty \|g'\|_\infty \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha t^2 - (J^\alpha t)^2 \right].$$

Proof. Let $f$ and $g$ be two functions satisfying the conditions of Theorem 2.1. Define

$$H(\tau, \rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)); \quad \tau, \rho \in (0, t), t > 0.$$ (2.2)

Multiplying (2.2) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$; $\tau \in (0, t)$ and integrating the resulting identity with respect to $\tau$ from 0 to $t$, we can state that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} H(\tau, \rho) d\tau$$

$$= J^\alpha fg(t) - f(\rho) J^\alpha g(t) - g(\rho) J^\alpha f(t) + f(\rho) g(\rho) \frac{t^\alpha}{\Gamma(\alpha+1)}.$$ (2.3)

Now, multiplying (2.3) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$; $\rho \in (0, t)$ and integrating the resulting identity with respect to $\rho$ over $(0, t)$, we can write

$$\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} H(\tau, \rho) d\tau d\rho$$

$$= 2 \left( \frac{t^\alpha}{\Gamma(\alpha+1)} J^\alpha fg(t) - J^\alpha f(t) J^\alpha g(t) \right).$$ (2.4)

On the other hand, we have

$$H(\tau, \rho) = \int_\tau^\rho \int_\tau^\rho f'(y) g'(z) dy dz.$$ (2.5)
Since \( f', g' \in L_\infty([0, \infty[) \), then we can write
\[
|H(\tau, \rho)| \leq \left| \int_\tau^\rho f'(y)dy \right| \left| \int_\tau^\rho g'(z)dz \right| \leq \|f'\|_\infty \|g'\|_\infty (\tau - \rho)^2. \tag{2.6}
\]
Consequently,
\[
\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t - \tau)^{\alpha-1}(t - \rho)^{\alpha-1}|H(\tau, \rho)|d\tau d\rho
\leq \frac{\|f'\|_\infty \|g'\|_\infty}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t - \tau)^{\alpha-1}(t - \rho)^{\alpha-1}(\tau^2 - 2\tau \rho + \rho^2)d\tau d\rho. \tag{2.7}
\]
Thus, we obtain the following estimate
\[
\frac{1}{\Gamma^2(\alpha)} \int_0^t \int_0^t (t - \tau)^{\alpha-1}(t - \rho)^{\alpha-1}|H(\tau, \rho)|d\tau d\rho
\leq \|f'\|_\infty \|g'\|_\infty \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta t^2 - 2(J^\alpha t)^2 + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha t^2 \right]. \tag{2.8}
\]
By the relations (2.4), (2.8) and using the properties of the modulus, we get the desired inequality (2.1).

**Remark 2.2.** Applying Theorem 2.1 for \( \alpha = 1 \), we obtain (Corollary 6.2 of [7] on \([0, t] \)):
\[
\left| t \int_0^t f(\tau)g(\tau)d\tau - \left( \int_0^t f(\tau)d\tau \right) \left( \int_0^t g(\tau)d\tau \right) \right| \leq t^4/12.
\]

Our next result is the following theorem, in which we use two real positive parameters.

**Theorem 2.3.** Let \( f \) and \( g \) be two differentiable functions on \([0, \infty[ \) such that \( f', g' \in L_\infty([0, \infty[) \). Then for all \( t > 0, \alpha > 0, \beta > 0 \), we have
\[
\left| \frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta fg(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha fg(t) - J^\alpha f(t) J^\beta g(t) - J^\beta f(t) J^\alpha g(t) \right|
\leq \|f'\|_\infty \|g'\|_\infty \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta t^2 - 2(J^\alpha t)^2 + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha t^2 \right]. \tag{2.9}
\]

**Proof.** The relation (2.3) implies that
\[
\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t - \tau)^{\alpha-1}(t - \rho)^{\beta-1}H(\tau, \rho)d\tau d\rho
= \frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta fg(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha fg(t) - J^\alpha f(t) J^\beta g(t) - J^\beta f(t) J^\alpha g(t). \tag{2.10}
\]
On the other hand, the relation (2.6) implies that
\[
\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t - \tau)^{\alpha-1}(t - \rho)^{\beta-1}|H(\tau, \rho)|d\tau d\rho
\leq \|f'\|_\infty \|g'\|_\infty \int_0^t \int_0^t (t - \tau)^{\alpha-1}(t - \rho)^{\beta-1}(\tau - \rho)^2d\tau d\rho. \tag{2.11}
\]
Using (2.10) and (2.11), we get the inequality (2.9). \( \Box \)
Remark 2.4. Applying Theorem 2.3 for \( \alpha = \beta \) we obtain Theorem 2.1.

The following results have some applications in the perturbed quadrature rules (see, for example, [3, 5]).

**Theorem 2.5.** Let \( f \) and \( g \) be two differentiable functions on \([0, \infty[\) with \( g'(t) \neq 0, t \in [0, \infty[\). If there exists a constant \( M > 0 \) such that \( \frac{f'(t)}{g'(t)} \leq M \), then for all \( \alpha > 0, \beta > 0 \), we have

\[
\left| \frac{t^{\alpha}}{\Gamma(\alpha + 1)} J^\beta f g(t) + \frac{t^{\beta}}{\Gamma(\beta + 1)} J^\alpha f g(t) - J^\alpha f(t) J^\beta g(t) - J^\beta f(t) J^\alpha g(t) \right| \leq M \left[ \frac{t^{\beta}}{\Gamma(\beta + 1)} J^\alpha g^2(t) + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} J^\beta g^2(t) - 2 J^\alpha g(t) J^\beta g(t) \right].
\]

(2.12)

**Proof.** Let \( f \) and \( g \) be two functions satisfying the conditions of Theorem 2.5. Then for every \( \tau, \rho \in [0, t]; \tau = \rho, t > 0 \) there exists a \( c \) between \( \tau \) and \( \rho \) so that

\[
\frac{f(\tau) - f(\rho)}{g(\tau) - g(\rho)} = \frac{f'(c)}{g'(c)}.
\]

Hence for every \( \tau, \rho \in [0, t]; t > 0 \), we have

\[
|f(\tau) - f(\rho)| \leq M |g(\tau) - g(\rho)|.
\]

(2.13)

This implies that

\[
|H(\tau, \rho)| \leq M \left( g(\tau) - g(\rho) \right)^2, \tau, \rho \in [0, t].
\]

(2.14)

Then, it follows that

\[
\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t - \tau)^{\alpha - 1} (t - \rho)^{\beta - 1} |H(\tau, \rho)| d\tau d\rho
\]

\[
\leq \frac{M}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t - \tau)^{\alpha - 1} (t - \rho)^{\beta - 1} \left( g^2(\tau) - 2g(\tau)g(\rho) + g^2(\rho) \right) d\tau d\rho.
\]

(2.15)

Therefore,

\[
\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t - \tau)^{\alpha - 1} (t - \rho)^{\beta - 1} |H(\tau, \rho)| d\tau d\rho
\]

\[
\leq M \left[ \frac{t^{\beta}}{\Gamma(\beta + 1)} J^\alpha g^2(t) + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} J^\beta g^2(t) - 2 J^\alpha g(t) J^\beta g(t) \right].
\]

(2.16)

Theorem 2.5 is thus proved.

**Corollary 2.6.** Let \( f \) and \( g \) be two differentiable functions on \([0, \infty[; \) with \( g'(t) \neq 0, t \in [0, \infty[\). If there exists a constant \( M > 0 \) such that \( \frac{f'(t)}{g'(t)} \leq M \), then for all \( \alpha > 0 \), we have:
\[ \left| \frac{t^\alpha}{\Gamma(\alpha + 1)} J^\alpha f g(t) - J^\alpha f(t) J^\alpha g(t) \right| \]

\[ \leq M \left[ \frac{t^\alpha}{\Gamma(\alpha + 1)} J^\alpha g^2(t) - (J^\alpha g(t))^2 \right]. \]

(2.17)

**Proof.** We apply Theorem 2.5 for \( \alpha = \beta \). \( \square \)

**Remark 2.7.** Applying Theorem 2.5 for \( \alpha = \beta = 1 \), we obtain (Corollary 4.2 of [7] on \([0, t]\)):

\[ \left| t \int_0^t f(\tau) g(\tau) d\tau - \left( \int_0^t f(\tau) d\tau \right) \left( \int_0^t g(\tau) d\tau \right) \right| \]

\[ M \leq \left[ t \int_0^t g^2(\tau) d\tau - \left( \int_0^t g(\tau) d\tau \right)^2 \right]. \]

(2.18)

**References**


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