ON THE FIXED POINT OF ORDER 2

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Abstract. This paper deals with a new type of fixed point, i.e; ”fixed point of order 2” which is introduced in a metric space and some results are achieved.

1. Introduction

In 1922, Banach proved the following famous fixed point theorem [1]. Let \((X, d)\) be a complete metric space, \(T : X \rightarrow X\) be a contraction, there exists a unique fixed point \(x_0 \in X\) of \(T\). This theorem, called the Banach contraction principle is a forceful tool in nonlinear analysis. This principle has many applications and is extended by several authors: Caristi [2], Edelstein [4], Ekeland [5, 6], Khan [9], Meir and Keeler [12], Nadler [13] and others. These theorems are also extended; see [3, 7, 8, 10, 15, 16, 17, 18] and others.

Many expressions and generalizations of Banach fixed point theorem were derived in recent years. The results presented in this paper extend properly the Banach contraction principle.

As we have experience with zero’s of a map of order 2, we want to introduce a fixed point of order 2 for a map. Our idea goes back to special case in \(R\), which if a real map on \(R\) has a fixed point of order 2 means that this map is tangent to axis \(y = x\). Therefore, the derivative of map (if exists) is equal to 1 at this point.

2. Main Results

Definition 2.1. Suppose that \((X, d)\) is a metric space, \(T : X \rightarrow X\) is a function and \(x_0 \in X\) is a fixed point for \(T\). We call \(x_0\) is a fixed point of order 2 if it is not alone point and the following satisfies:

\[
\lim_{x \to x_0} \frac{d(Tx, x_0)}{d(x, x_0)} = 1.
\]

We remember the following definitions. We will show that for the case \((a)\) there is not fixed point of order 2 but in two other cases there is fixed point of order 2.

Definition 2.2. Suppose that \((X, d)\) is a metric space, \(T : X \rightarrow X\) is a function.
• (a) $T$ is a contraction, if there exist $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y$ in $X$.
• (b) $T$ is a contractive mapping, if $d(Tx,Ty) < d(x,y)$ for all $x,y$ in $X$ which $x \neq y$.
• (c) $T$ is a non-expansive mapping, if $d(Tx,Ty) \leq d(x,y)$ for all $x,y$ in $X$.

In the following we consider first some properties for fixed point of order 2.

**Proposition 2.3.** If $x_0 \in X$ is a fixed point of order 2 for $T$ on $X$. Then $T$ is continuous at $x_0$.

**Proof.**
$$\lim_{n \to \infty} d(Tx, x_0) = \lim_{x \to x_0} d(Tx, x_0) = \lim_{x \to x_0} d(x, x_0)\lim_{x \to x_0} d(x, x_0) = 0.$$

**Proposition 2.4.** Let $(X, d)$ be a metric spaces and $T : X \to X$ be a function such that $x_0 \in X$ is a fixed point for $f$, not alone point for $X$ and a alone point for $T(X)$. Then $x_0$ is not fixed point of order 2 for $T$.

**Proof.** According to assume the $x_0$ is a alone point for $T(X)$. There is a neighborhood of $x_0$, like $N(x_0)$ such that $N(x_0) \cap T(X) = \{x_0\}$ and each $x \in N(x_0)$ implies that $d(Tx, x_0) = 0$. Therefore, $\lim_{x \to x_0} d(Tx, x_0) = 0$, i.e; $x_0$ is not a fixed point of order 2 for $T$.

**Proposition 2.5.** Suppose that $x_0 \in X$ be a fixed point for $T_i : X \to X$ which $i = 1, \ldots, n$ ($n \in N$) and also $\lim_{x \to x_0} d(x, x_0) = \lambda_i$. Then $x_0$ is a fixed point of order 2 for $T_1T_2\ldots T_n$ if and only $\lambda_1\lambda_2\ldots\lambda_n = 1$.

**Proof.** $T_i$ is continuous at $x_0$ for all $i = 1, \ldots, n$, by a simple change of variable, that
$$\lim_{x \to x_0} \frac{d(T_k(T_{k+1}\ldots T_n)x_0)}{d(T_{k+1}\ldots T_nx_0)} = \lim_{x \to x_0} \frac{d(T_k t, x_0)}{d(t, x_0)}$$
and the last limit is equal with $\lambda_k$ for $k = 1, \ldots, n$. Hence,
$$\lim_{x \to x_0} \frac{d(T_1T_2\ldots T_n x_0)}{d(x, x_0)} = \lim_{x \to x_0} \frac{d(T_1(T_2\ldots T_n)x_0)}{d(T_2\ldots T_n, x_0)}\frac{d(T_2(T_3\ldots T_n), x_0)}{d(T_3\ldots T_n, x_0)} \ldots \frac{d(T_n x_0)}{d(x, x_0)} = \lambda_1\lambda_2\ldots\lambda_n.$$

**Proposition 2.6.** Let $x_0 \in X$ be a fixed point for $T_i : X \to X$ for $i = 1, \ldots, n$ and $n \in N$.

• (a) If $x_0$ is fixed point order 2 for all $T_i$, then $x_0$ is fixed point of order 2 for $T_1T_2\ldots T_n$.
• (b) If $x_0$ is fixed point order 2 for $T_1T_2$ and $T_2$, then $x_0$ is fixed point of order 2 for $T_1$.

**Proof.** (a) By proposition 2.3.
(b) $x_0$ is fixed point order 2 for $T_1T_2$ and $T_2$. Thus, $\lim_{x \to x_0} \frac{d(T_1T_2 x_0)}{d(x, x_0)} = 1$, $\lim_{x \to x_0} \frac{d(T_2 x_0)}{d(x, x_0)} = 1$. Since $T$ is continuous at $x_0$ for $t = T_2 x$.

$$1 = \lim_{x \to x_0} \frac{d(T_1T_2 x_0)}{d(x, x_0)} = \lim_{x \to x_0} \frac{d(T_1T_2 x, x_0)}{d(x, x_0)} = \lim_{x \to x_0} \frac{d(T_1 t, x_0)}{d(t, x_0)}.$$
Proposition 2.7. Suppose that $x_0$ is not alone point and is a fixed point for $T_i : X \to X$ for $i = 1,...,n$ and $n \in N$.

- (a) If $T_i$ be a contractive mapping or non-expansive mapping for all $i = 1,...,n$ and $\lim_{x \to x_0} \frac{d(T_i x, x_0)}{d(x, x_0)} = \lambda_i$. Then $x_0 \in X$ is a fixed point of order 2 for $T_1 T_2...T_n$ if and only if $x_0$ is a fixed point of order 2 for all $T_i$.
- (b) If $\lim_{x \to x_0} \frac{d(T_i x, x_0)}{d(x, x_0)} = \lambda$ then $x_0$ is a fixed point of order 2 for $T_1$ if and only if $x_0$ be a fixed point of order 2 for $T_i^n$, where $n$ is arbitrary positive integer.
- (c) If $T_i$ be a contractive mapping or non-expansive mapping, then $x_0$ is a fixed point of order 2 for $T_1$ if and only if there exist $n \in N$ such that $x_0$ be a fixed point of order 2 for $T_i^n$.

Proof. (a) Let $T_i$ be a contractive mapping for all $i = 1,...,n$. If $x_0$ is a fixed point of order 2 for all $T_i$ then by proposition 2.5, $x_0$ is a fixed point of order 2 for $T_1 T_2...T_n$. Now assume that $x_0$ is a fixed point of order 2 for $T_1 T_2...T_n$, then by proposition 2.4, $1 = \lim_{x \to x_0} \frac{d(T_1 T_2...T_n x, x_0)}{d(x, x_0)} = \lambda_1 \lambda_2...\lambda_n$. But all $T_i$ are contractive mappings so $\frac{d(T_1 x, x_0)}{d(x, x_0)} < 1$ which implies that $\lambda_i \leq 1$ for all $i = 1,...,n$. Hence, $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$. Proof for non-expansive is similar.

(b) By proposition 2.4, $\lim_{x \to x_0} \frac{d(T_i^n x, x_0)}{d(x, x_0)} = \lambda^n$. Then $\lambda^n = 1$ if and only if $\lambda = 1$ because $\lambda \geq 0$.

(c) Let $T_1$ be a contractive mapping and there exists $n \in N$ such that $x_0$ is a fixed point of order 2 for $T_1^n$. $T_1$ is a contractive mapping, so

$$d(T_1^n x, x_0) < ... < d(T_1 x, x_0) < d(x, x_0)$$

Therefore, $\lim_{x \to x_0} \frac{d(T_1^n x, x_0)}{d(x, x_0)} = 1$.

Proposition 2.8. Suppose that $(X,d)$ is a metric space, $T : X \to X$ is a function and $x_0 \in X$ is a fixed point for $T$. If $T$ is a contraction then $x_0$ is not a fixed point of order 2 for $T$.

Proof. Since $T$ is a contractive mapping so there exists $\alpha \in [0,1)$ such that $d(T x, T y) \leq \alpha d(x, y)$ for all $x, y \in X$. Therefore, $\frac{d(T x, x_0)}{d(x, x_0)} \leq \alpha < 1$ and $x_0$ can not be a fixed point of order 2 for $T$.

Proposition 2.9. Suppose that $x_0 \in X$ be a fixed point of order 2 for $T : X \to X$, where $T$ is one-to-one and $g$ is left inverse of $T$. Then $x_0$ is also a fixed point of order 2 for $g$. 
**Proof.** It is clear that \( x_0 \) is a fixed point for \( g \). On the other hand, since \( T \) is continuous at \( x_0 \) for \( t = Tx \) so

\[
1 = \lim_{x \to x_0} \frac{d(Tx, x_0)}{d(x, x_0)} = \lim_{x \to x_0} \frac{d(g(Tx), x_0)}{d(g(x), x_0)} = \lim_{t \to x_0} \frac{d(T(t), x_0)}{d(g(t), x_0)} = \lim_{t \to x_0} \frac{d(t, x_0)}{d(g(t), x_0)} = \lim_{t \to x_0} \frac{1}{d(T(t), x_0)}.
\]

Therefore, \( \lim_{t \to x_0} \frac{d(T(t), x_0)}{d(t, x_0)} = 1 \).

In the following we give another condition for the fixed point of order 2.

**Proposition 2.10.** Suppose that \( x_0 \) is not alone point and is a fixed point for \( T : X \to X \).

1. (a) If \( \lim_{x \to x_0} \frac{d(Tx, x)}{d(x, x_0)} = 0 \) then \( x_0 \) is a fixed point of order 2 for \( T \).
2. (b) If \( \lim_{x \to x_0} \frac{d(Tx, x)}{d(Tx, x_0)} = 0 \) then \( x_0 \) is a fixed point of order 2 for \( T \).

**Proof.**

(a) Using \( |d(Tx, x_0) - d(x, x_0)| \leq d(Tx, x) \),

\[
1 - \frac{d(Tx, x)}{d(x, x_0)} \leq \frac{d(Tx, x_0)}{d(x, x_0)} \leq 1 + \frac{d(Tx, x)}{d(x, x_0)},
\]

\( \lim_{x \to x_0} \frac{d(Tx, x_0)}{d(x, x_0)} = 1 \).

(b) Using \( |d(Tx, x_0) - d(x, x_0)| \leq d(Tx, x) \),

\[
1 - \frac{d(Tx, x)}{d(Tx, x_0)} \leq \frac{d(x, x_0)}{d(Tx, x_0)} \leq 1 + \frac{d(Tx, x)}{d(Tx, x_0)}.
\]

This shows that \( \lim_{x \to x_0} \frac{d(x, x_0)}{d(Tx, x_0)} = 1 \). Therefore, \( \lim_{x \to x_0} \frac{d(Tx, x_0)}{d(x, x_0)} = 1 \).

**Proposition 2.11.** Suppose that \( x_0 \) is a fixed point for \( T : X \to X \) and \( \varphi : X \to R^+ \) is a real valued function.

1. (a) If \( x_0 \) be a fixed point of order 2 for \( T \) then \( \lim_{x \to x_0} \frac{d(Tx, x)}{d(x, x_0)} \leq 2 \).
2. (b) If \( d(Tx, x) \leq \varphi(x) - \varphi(Tx) \leq d(x, x_0) \) for all \( x \) in \( X \) then \( x_0 \) is a fixed point of order 2 for \( T \) if and only if \( \lim_{x \to x_0} \frac{d(Tx, x)}{d(x, x_0)} = 0 \).

**Proof.**

(a) From the inequality \( d(Tx, x) \leq d(Tx, x_0) + d(x, x_0) \),

\[
\frac{d(Tx, x)}{d(x, x_0)} \leq \frac{d(Tx, x_0)}{d(x, x_0)} + 1.
\]
Therefore, \( \lim_{x \to x_0} \frac{d(Tx,x)}{d(x,x_0)} \leq 2 \).

(b) From inequality \( d(Tx,x) \leq \varphi(x) - \varphi(Tx) \leq d(x,x_0) \),
\[
d(x,Tx) + d(Tx,T^2x) + \ldots + d(T^{n-1}x,T^n x) \leq \sum_{i=1}^{n} \varphi(T^{i-1}x) - \varphi(T^i x) = \varphi(x) - \varphi(T^n x)
\]
and
\[
\frac{d(T^{n-1}x,T^n x)}{d(x,x_0)} = \frac{d(T^{n-1}x,T^n x)}{d(x,x_0)} \frac{d(T^{n-2}x,T^n x)}{d(x,x_0)} \ldots \frac{d(T^2x,T^n x)}{d(x,x_0)} \frac{d(Tx,T^n x)}{d(x,x_0)} \frac{d(x,T^n x)}{d(x,x_0)}
\]
since \( \lim_{x \to x_0} \frac{d(T^{n-1}x,T^n x)}{d(x,x_0)} = \lim_{x \to x_0} \frac{d(Tx,x)}{d(x,x_0)} = 1 \) which \( k = 1,2,\ldots,n-1 \), so \( \lim_{x \to x_0} \frac{d(T^{n-k+1}x,x)}{d(x,x_0)} = 1 \). From inequality \( d(Tx,x) \leq \varphi(x) - \varphi(Tx) \). It is clear that \( \varphi(T^n x) \) is strict decreasing.
\[
n \lim_{x \to x_0} \frac{d(Tx,x)}{d(x,x_0)} \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(T^n x)}{d(x,x_0)} \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(T^n x)}{\varphi(x) - \varphi(Tx)} \leq \lim_{x \to x_0} \varphi(x) - \varphi(T^n x) = 1.
\]
Hence, \( \lim_{x \to x_0} \frac{d(Tx,x)}{d(x,x_0)} \leq \frac{1}{n} \). Since \( n \) is arbitrary positive integer, \( \lim_{x \to x_0} \frac{d(Tx,x)}{d(x,x_0)} = 0 \).

In the following we prove common fixed point of order 2.

**Proposition 2.12.** Suppose that \( (X,d) \) is a metric space, \( f,g : X \to X \) are two function and \( x_0 \in X \) is a fixed point for \( f \) such that \( f,g \) satisfies
\[
\begin{align*}
d(f(x),g(x)) & \leq d(f(x),x) \leq \varphi(x) - \varphi(f(x)) \leq d(x,x_0) \\
n \lim_{x \to x_0} \frac{d(f(x),x_0)}{d(x,x_0)} & \geq 1
\end{align*}
\]
for all \( x \) in \( X \). Then \( x_0 \) is a common fixed point of order 2 for \( f,g \).

**Proof.** First we show that \( x_0 \) is a fixed point of order 2 for \( f \). From inequality \( d(f(x),x) \leq \varphi(x) - \varphi(f(x)) \leq d(x,x_0) \) we have
\[
d(x,f(x)) + d(f(x),f^2(x)) + \ldots + d(f^{n-1}(x),f^n(x)) \leq \sum_{i=1}^{n} \varphi(f^{i-1}(x)) - \varphi(f^i(x)) = \varphi(x) - \varphi(f^n(x))
\]
and
\[
\frac{d(f^{n-1}(x), f^n(x))}{d(x, x_0)} = \frac{d(f^{n-1}(x), f^n(x))}{d(f^{n-1}(x), f^{n-2}(x_0))} \frac{d(f^{n-2}(x), f^{n-3}(x_0))}{d(f^{n-2}(x), f^{n-3}(x_0))} \cdots \frac{d(f(x), f(x_0))}{d(f(x), x_0)} \cdot \frac{d(f(x), x_0)}{d(x, x_0)}.
\]

Since \(\lim_{x \to x_0} \frac{d(f^{n-1}(x), f^n(x))}{d(f^{n-1}(x), x_0)} = \lim_{x \to x_0} \frac{d(f(x), x)}{d(x, x_0)}\) and \(\lim_{x \to x_0} \frac{d(f^{n-k}(x), x_0)}{d(f^{n-k}(x), x_0)} = \lambda\) which

\[
\lim_{x \to x_0} \frac{d(f(x), x)}{d(x, x_0)} (1 + \lambda + \lambda^2 + \ldots + \lambda^n) \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(f^n(x))}{d(x, x_0)}.
\]

But \(\varphi(f^n(x))\) is strict decreasing so,

\[
\lim_{x \to x_0} \frac{d(f(x), x)}{d(x, x_0)} (1 + \lambda + \lambda^2 + \ldots + \lambda^n) \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(f^n(x))}{d(x, x_0)} \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(f^n(x))}{\varphi(x) - \varphi(f^n(x))} \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(x)}{\varphi(x) - \varphi(x)} = 1,
\]

and also \(\lim_{x \to x_0} \frac{d(f(x), x)}{d(x, x_0)} \leq \frac{1}{(1+\lambda+\lambda^2+\ldots+\lambda^n)}\), but \(\lambda \geq 1\) and \(n\) is arbitrary positive integer. Then \(\lim_{x \to x_0} \frac{d(f(x), x)}{d(x, x_0)} = 0\) and proposition 2.9 implies that \(x_0\) is a fixed point of order 2 for \(f\). Now, we show that \(x_0\) is a fixed point of order 2 for \(g\). It is clear that \(x_0\) is a fixed point for \(g\), because \(d(f(x), g(x)) \leq d(f(x), x)\) and \(x_0\) is fixed point for \(f\). From inequality \(d(f(x), g(x)) \leq d(f(x), x)\) and triangle inequality,

\[
0 \leq \frac{d(g(x), x)}{d(x, x_0)} \leq \frac{d(g(x), f(x))}{d(x, x_0)} + \frac{d(f(x), x)}{d(x, x_0)} \leq 2 \frac{d(f(x), x)}{d(x, x_0)}.
\]

Therefore, \(\lim_{x \to x_0} \frac{d(g(x), x)}{d(x, x_0)} = 0\) and \(x_0\) is a fixed point of order 2 for \(g\).

**References**


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