HYERS-ULAM STABILITY OF K-FIBONACCI FUNCTIONAL EQUATION

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Abstract. Let denote by $F_{k,n}$ the $n$th $k$-Fibonacci number where $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$ with initial conditions $F_{k,0} = 0, F_{k,1} = 1$, we may derive a functional equation $f(k, x) = kf(k, x - 1) + f(k, x - 2)$. In this paper, we solve this equation and prove its Hyere-Ulam stability in the class of functions $f : \mathbb{N} \times \mathbb{R} \rightarrow X$, where $X$ is a real Banach space.

1. Introduction

The stability of functional equation originated from an equaton of Ulam [11] concerning the stability of group homomorphisms. Later, the result of Ulam was generated by Rassias [10]. Since then, the stability problems of functional equations have been extensively investigated by several mathematiciens (see [1-9]).

For any positive real number $k$, the $k$-Fibonacci sequence, say $\{F_{k,n}\}$, is defined recurrently by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for all $n \geq 2$ with initial conditions $F_{k,0} = 0, F_{k,1} = 1$. From this famous formula, we may derive a functional equation

$$f(k, x) = kf(k, x - 1) + f(k, x - 2).$$

A function $f : \mathbb{N} \times \mathbb{R} \rightarrow X$, will be called a $k$-Fibonacci function if it satisfies in (1.1), for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$, where $X$ is a real vector space.

Characteristic equation of $k$-Fibonacci sequences is $x^2 - kx - 1 = 0$. We denote the positive and negative roots of this function by $\gamma$, $\theta$ (respectively); i.e,

$$\gamma = \frac{k + \sqrt{k^2 + 4}}{2}, \quad \theta = \frac{k - \sqrt{k^2 + 4}}{2}$$

for any $x \in \mathbb{R}, k \in \mathbb{N}$.

2. General solution of $k$-Fibonacci equation

Let $X$ be a real vector space. In the following theorem, we investigate the general solution for equation of the form (1.1) which is strongly related to the $F_{k,n}$.

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Theorem 2.1. Let $X$ be a real vector space. A function $f : \mathbb{N} \times \mathbb{R} \to X$ is a $k$-Fibonacci function if and only if there exists a function $h : \mathbb{N} \times [-1,1) \to X$ such that

$$
\begin{align*}
f(k, x) &= \begin{cases} 
F_{k,\lfloor x \rfloor + 1}h(k, x - \lfloor x \rfloor) + F_{k,\lfloor x \rfloor}h(k, x - \lfloor x \rfloor - 1) & x \geq 0 \\
(-1)^{\lfloor x \rfloor}[F_{k,\lceil x \rceil - 1}h(k, x - \lfloor x \rfloor) - F_{k,\lceil x \rceil}h(k, x - \lfloor x \rfloor - 1)] & x < 0
\end{cases}
\end{align*}
$$

(2.1)

where $\lfloor x \rfloor$ stands for the largest integer number that does not exceed $x$.

Proof. From (1.1) we have

$$
f(k, x) = kf(k, x - 1) + f(k, x - 2).
$$

Since $\gamma + \theta = k$, $\gamma \theta = -1$, hence

$$
f(k, x) = (\gamma + \theta)f(k, x - 1) - \gamma \theta f(k, x - 2)
$$

$$
= \gamma f(k, x - 1) + \theta f(k, x - 1) - \gamma \theta f(k, x - 2)
$$

which implies that

$$
\begin{align*}
\begin{cases} 
f(k, x) - \gamma f(k, x - 1) = \theta[f(k, x - 1) - \gamma f(k, x - 2)] \\
f(k, x) - \theta f(k, x - 1) = \gamma[f(k, x - 1) - \theta f(k, x - 2)]
\end{cases}
\end{align*}
$$

(2.2)

By induction on $n$, it follows that

$$
\begin{align*}
\begin{cases} 
f(k, x) - \gamma f(k, x - 1) = \theta^n[f(k, x - n) - \gamma f(k, x - n - 1)] \\
f(k, x) - \theta f(k, x - 1) = \gamma^n[f(k, x - n) - \theta f(k, x - n - 1)]
\end{cases}
\end{align*}
$$

(2.3)

If we replace $x$ by $x + n$ ($n \geq 0$) in (2.3), divide the resulting equation by $\theta^n$ (resp. $\gamma^n$) and replace $n$ by $-m$ in the resulting equation, then we obtain a equation with $m$ in place of $n$, where $m \in \{0, -1, -2, ...\}$. Therefore, (2.3) is true for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Now by multiplying the first and second equations of (2.3) by $\theta$ and $-\gamma$ (respectively) and then adding with together, we get

$$
f(k, x) = \frac{\theta^{n+1} - \gamma^{n+1}}{\theta - \gamma} f(k, x - n) + \frac{\theta^n - \gamma^n}{\theta - \gamma} f(k, x - n - 1)
$$

(2.4)

for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. For $n = \lfloor x \rfloor$, $x \geq 0$ in (2.4) and using Binet’s formula

$$
F_{k,n} = \frac{\theta^n - \gamma^n}{\theta - \gamma},
$$

we have

$$
f(k, x) = F_{k,\lfloor x \rfloor + 1}f(k, x - \lfloor x \rfloor) + F_{k,\lfloor x \rfloor}f(k, x - \lfloor x \rfloor - 1)
$$

and if $x < 0$, then for $n = \lfloor x \rfloor = -[|x|]$, we have
\[ f(k, x) = \frac{\theta^{-[x]+1} - \gamma^{-[x]+1}}{\theta - \gamma} f(k, x - [x]) + \frac{\theta^{-[x]} - \gamma^{-[x]}}{\theta - \gamma} f(k, x - [x] - 1) \]
\[ = \frac{-1}{(\gamma \theta)^{|[x]|-1}} \frac{\theta^{|[x]|-1} - \gamma^{|[x]|-1}}{\theta - \gamma} f(k, x - [x]) + \frac{-1}{(\gamma \theta)^{|[x]|}} \frac{\theta^{|[x]|} - \gamma^{|[x]|}}{\theta - \gamma} f(k, x - [x] - 1) \]
\[ = (-1)^{|x|} F_{k,|[x]|-1} f(k, x - [x]) + (-1)^{1+|x|} F_{k,|[x]|} f(k, x - [x] - 1) \]
\[ = (-1)^{|x|} [F_{k,-[x]-1} f(k, x - [x]) - F_{k,-[x]} f(k, x - [x] - 1)]. \]

Since \(0 \leq x - [x] < 1\), and \(-1 \leq x - [x] - 1 < 0\), if we define a function \(h : \mathbb{N} \times [-1, 1) \rightarrow X\), by \(h := f|_{\mathbb{N} \times [-1, 1)}\), then \(f\) is a function of the form (2.1).

Now, let \(f\) be a function of the form (2.1), where \(h : \mathbb{N} \times [-1, 1) \rightarrow X\) is an arbitrary function, we want to show that
\[ f(k, x) = kf(k, x - 1) + f(k, x - 2) \]

and so \(f\) is a k-Fibonacci function.

If \(x \geq 2\), then \(x - 1 \geq 1, \ x - 2 \geq 0\).

and by (2.1) we have
\[ f(k, x) = F_{k,[x]+1} h(k, x - [x]) + F_{k,[x]} h(k, x - [x] - 1) \]
\[ f(k, x - 1) = F_{k,[x-1]+1} h(k, x - 1 - [x - 1]) + F_{k,[x-1]} h(k, x - 1 - [x - 1] - 1) \]

Since \((x - 1) - [x - 1] = x - [x]\), hence
\[ f(k, x - 1) = F_{k,[x]} h(k, x - [x]) + F_{k,[x]-1} h(k, x - [x] - 1), \]
\[ f(k, x - 2) = F_{k,[x]-1} h(k, x - [x]) + F_{k,[x]-2} h(k, x - [x] - 1). \]

Therefore
\[ kf(k, x - 1) + f(k, x - 2) = kf_{k,[x]} h(k, x - [x]) + kf_{k,[x]-1} h(k, x - [x] - 1) + F_{k,[x]-1} h(k, x - [x]) + F_{k,[x]-2} h(k, x - [x] - 1) \]
\[ = (kf_{k,[x]} + F_{k,[x]-1}) h(k, x - [x]) + (kf_{k,[x]-1} + F_{k,[x]-2}) h(k, x - [x] - 1) \]
\[ = F_{k,[x]} h(k, x - [x]) + F_{k,[x]} h(k, x - [x] - 1) = f(k, x). \]

If \(1 \leq x \leq 2\), then \(0 \leq x - 1 \leq 1, \ -1 \leq x - 2 \leq 0\) and by (2.1), we have
\[ f(k, x) = F_{k,[x]+1} h(k, x - [x]) + F_{k,[x]} h(k, x - [x] - 1) \]
\[ = F_{k,2} h(k, x - [x]) + F_{k,1} h(k, x - [x] - 1) \]
\[ = kh(k, x - [x]) + h(k, x - [x] - 1) \]
\[
f(k, x - 1) = F_{k, [x-1]+1} h(k, x - 1 - [x - 1]) + F_{k, [x-1]} h(k, x - 1 - [x - 1] - 1) \\
= F_{k, 1} h(k, x - [x]) + F_{k, 0} h(k, x - [x] - 1) \\
= h(k, x - [x])
\]

\[
f(k, x - 2) = (-1)^{[x-2]} [F_{k, ([x]-1)} h(k, x - [x]) - F_{k, 2-[x]} h(k, x - [x] - 1)] \\
= -[F_{k, 0} h(k, x - [x]) - F_{k, 1} h(k, x - [x] - 1)] \\
= h(k, x - [x] - 1).
\]

Hence
\[
kf(k, x - 1) + f(k, x - 2) = kh(k, x - [x]) + h(k, x - [x] - 1) = f(k, x).
\]

If \(0 \leq x < 1\), then \(-1 \leq x - 1 < 0\), \(-2 \leq x - 2 < -1\) and by (2.1), we have
\[
f(k, x) = F_{k, 1} h(k, x - [x]) + F_{k, 0} h(k, x - [x] - 1) = h(k, x - [x])
\]

\[
f(k, x - 1) = (-1)^{-1} [F_{k, 0} h(k, x - [x]) - F_{k, 1} h(k, x - [x] - 1)] = h(k, x - [x] - 1)
\]

\[
f(k, x - 2) = (-1)^{-2} [F_{k, 1} h(k, x - [x]) + F_{k, 2} h(k, x - [x] - 1)] = h(k, x - [x]) - kh(k, x - [x] - 1).
\]

Thus, we get
\[
kf(k, x - 1) + f(k, x - 2) = h(k, x - [x]) = f(k, x).
\]

Finally, if \(x < 0\), then we have
\[
f(k, x) = (-1)^{[x]} [F_{k, -[x]-1} h(k, x - [x]) - F_{k, -[x]} h(k, x - [x] - 1)]
\]

\[
f(k, x - 1) = (-1)^{[x-1]} [F_{k, -[x-1]-1} h(k, x - 1 - [x - 1]) - F_{k, -[x-1]} h(k, x - 1 - [x - 1] - 1)] \\
= (-1)^{[x-1]} [F_{k, -[x]} h(k, x - [x]) - F_{k, -[x]-1} h(k, x - [x] - 1)]
\]

\[
f(k, x - 2) = (-1)^{[x-2]} [F_{k, -[x-2]-1} h(k, x - 2 - [x - 2]) - F_{k, -[x-2]} h(k, x - 2 - [x - 2] - 1)] \\
= (-1)^{[x-2]} [F_{k, -[x]} h(k, x - [x]) - F_{k, -[x]+2} h(k, x - [x] - 1)].
\]

Therefore
\[
f(k, x) = kf(k, x - 1) + f(k, x - 2).
\]
3. HYERS-ULAM STABILITY OF K-FIBONACCI EQUATION

In the following theorem, we investigate the Hyers-Ulam stability for equations of the form (1.1).

**Theorem 3.1.** Let \((X, ||.||)\) be a real Banach space. If a function \(f : \mathbb{N} \times \mathbb{R} \to X\) satisfies the inequality
\[
||f(k, x) - kf(k, x - 1) - f(k, x - 2)|| \leq \varepsilon, \quad (3.1)
\]
for all \(x \in \mathbb{R}, k \in \mathbb{N},\) and for some \(\varepsilon > 0,\) then there exists a \(k\)-Fibonacci function \(G : \mathbb{N} \times \mathbb{R} \to X\) such that
\[
||f(k, x) - G(k, x)|| \leq \frac{\varepsilon}{2k} (k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}), \quad (3.2)
\]
for all \(x \in \mathbb{R}, k \in \mathbb{N}.\)

**Proof.** As \(\gamma + \theta = k, \gamma \theta = -1,\) we get from (3.1)
\[
||f(k, x) - (\gamma + \theta)f(k, x - 1) + \gamma \theta f(k, x - 2)|| \leq \varepsilon
\]
or
\[
||f(k, x) - \gamma f(k, x - 1) - \theta[f(k, x - 1) - \gamma f(k, x - 2)]|| \leq \varepsilon,
\]
for all \(x \in \mathbb{R}, k \in \mathbb{N}.\)

If we replace \(x\) by \(x - t\) and then multiplying the both sides of this inequality by \(|\theta|^t\), we get
\[
||\theta^t[f(k, x-t) - \gamma f(k, x-t-1)] - \theta^{t+1}[f(k, x-t-1) - \gamma f(k, x-t-2)]|| \leq |\theta|^t \varepsilon \quad (3.3)
\]
for all \(x \in \mathbb{R}, k \in \mathbb{N},\) and \(t \in \mathbb{Z}.\) Since
\[
||\sum_{t=0}^{n-1} \theta^t[f(k, x-t) - \gamma f(k, x-t-1)] - \theta^{t+1}[f(k, x-t-1) - \gamma f(k, x-t-2)]||
= ||f(k, x) - \gamma f(k, x - 1) - \theta^n[f(k, x - n) - \gamma f(k, x - n - 1)]||, \quad (3.3)
\]
and then multiplying the both sides of this inequality by \(\varepsilon,\) we get
\[
||f(k, x) - \gamma f(k, x - 1) - \theta^n[f(k, x - n) - \gamma f(k, x - n - 1)]|| \leq \sum_{t=0}^{n-1} |\theta|^t \varepsilon, \quad (3.4)
\]
for all \(x \in \mathbb{R}, k \in \mathbb{N},\) and \(t \in \mathbb{Z}.\)

From (3.3) for all \(x \in \mathbb{R}, k \in \mathbb{N},\) we have \(\{\theta^n[f(k, x - n) - \gamma f(k, x - n - 1)]\}\) is a Cauchy sequence \((|\theta| < 1).\) Therefore we can define \(G_1 : \mathbb{N} \times \mathbb{R} \to X\) by
\[
G_1(k, x) = \lim_{n \to \infty} \theta^n[f(k, x - n) - \gamma f(k, x - n - 1)].
\]

Since \(X\) is a Banach space, so it is complete and \(G_1\) is well defined function. We have
\[
kG_1(k, x - 1) + G_1(k, x - 2)
= k\theta^{-1}G_1(k, x) + \theta^{-2}G_1(k, x) = G_1(k, x)
\]
if \(n \to \infty,\) then from (3.4) we have
\[
||f(k, x) - \gamma f(k, x - 1) - G_1(k, x)|| \leq \frac{2 + k + \sqrt{k^2 + 4}}{2} \varepsilon, \quad (3.5)
\]
for all \( x \in \mathbb{R}, \ k \in \mathbb{N} \).

On the other hand, from (3.1), we have
\[
||f(k, x) - \theta f(k, x - 1) - \gamma [f(k, x - 1) - \theta f(k, x - 2)]|| \leq \varepsilon,
\]
for all \( x \in \mathbb{R}, \ k \in \mathbb{N} \).

Now if we replace \( x \) by \( x + t \) and then multiplying the both sides of this inequality by \( \gamma^{-t} \), we get
\[
||\gamma^{-t}[f(k, x+t) - \theta f(k, x+t-1)] - \gamma^{-t+1}[f(k, x+t-1) - \theta f(k, x+t-2)]|| \leq \gamma^{-t}\varepsilon, \quad (3.6)
\]
for all \( x \in \mathbb{R}, \ k \in \mathbb{N}, \) and \( t \in \mathbb{Z} \). Therefore
\[
||\gamma^{-n}[f(k, x+n) - \theta f(k, x+n-1)] - [f(k, x) - \theta f(k, x-1)]||
\leq \sum_{t=1}^{n} ||\gamma^{-t}[f(k, x+t) - \theta f(k, x + t - 1)] - \gamma^{-t+1}[f(k, x + t-1) - \theta f(k, x + t-2)]||
\leq \sum_{t=1}^{n} |\gamma^{-t}|\varepsilon, \quad (3.7)
\]
for all \( x \in \mathbb{R}, \ k \in \mathbb{N}, \) and \( t \in \mathbb{Z} \).

For all \( x \in \mathbb{R}, \ k \in \mathbb{N}(3.6), \) we have \( \{\gamma^{-n}[f(k, x+n) - \theta f(k, x+n-1)]\} \) is a Cauchy sequence and hence we can define \( G_{2} : \mathbb{N} \times \mathbb{R} \rightarrow X \) by
\[
G_{2}(k, x) = \lim_{n \to \infty} \gamma^{-n}[f(k, x+n) - \theta f(k, x+n-1)].
\]
Since \( X \) is a Banach space, so it is complete and \( G_{2} \) is well defined function. We have
\[
kG_{2}(k, x - 1) + G_{2}(k, x - 2)
= k\gamma^{-1}G_{2}(k, x) + \gamma^{-2}G_{2}(k, x) = G_{2}(k, x).
\]
If \( n \to \infty \), then from (3.7), we have
\[
||G_{2}(k, x) - f(k, x) - \theta f(k, x - 1)|| \leq \frac{2 - k + \sqrt{k^{2} + 4}}{2k} \varepsilon, \quad (3.8)
\]
for all \( x \in \mathbb{R}, \ k \in \mathbb{N} \).

For
\[
G(k, x) = \frac{\theta}{\theta - \gamma}G_{1}(k, x) - \frac{\gamma}{\theta - \gamma}G_{2}(k, x),
\]
we have
\[
||f(k, x) - G(k, x)|| = ||f(k, x) - \frac{\theta}{\theta - \gamma}G_{1}(k, x) - \frac{\gamma}{\theta - \gamma}G_{2}(k, x)||
= \frac{1}{|\theta - \gamma|}||(\theta - \gamma)f(k, x) - [\theta G_{1}(k, x) - \gamma G_{2}(k, x)]||
\leq \frac{1}{\gamma - \theta}||\theta[f(k, x) - \gamma f(k, x - 1) - G_{1}(k, x)]||
+ \frac{1}{\gamma - \theta}||\gamma[G_{2}(k, x) - f(k, x) - \theta f(k, x - 1)]||
\leq \frac{\varepsilon}{2k}(k + 1 - \frac{k^{2} - 3k - 2}{\sqrt{k^{2} + 4}}), \quad (By \ 3.5 \ and \ 3.8)
and it is easy to see $G$ is a $k$-Fibonacci function. \hfill \Box

In order to prove $G$ is also unique, we need the following lemma.

**Lemma 3.2.** Let $(X, ||.||)$ be a real normed space and $u, v \in X$ are given. If for all $n \in \mathbb{N}$ and for some $C \geq 0$ we have

$$||F_{k,n+1}u + F_{k,n}v|| \leq C$$

then,

$$\gamma u + v = 0.$$  

**Proof.** We have,

$$F_{k,n}||\gamma u + v|| = ||\gamma F_{k,n}u + F_{k,n}v + F_{k,n+1}u - F_{k,n+1}u||$$

$$\leq ||F_{k,n+1}u + F_{k,n}v|| + |F_{k,n+1} - \gamma F_n||u||$$

$$\leq C + |\gamma^{n+1} - \theta^{n+1}| + |\gamma^n - \theta^n| ||u|| \quad \text{(By Binet's formula)}$$

$$= C + |\theta|^n ||u||,$$

for all $n \in \mathbb{N}$, $k \in \mathbb{N}$. Since $|\theta| < 1$, if $n \to \infty$, then $F_{k,n} \to \infty$, and so $\gamma u + v = 0$. \hfill \Box

**Theorem 3.3.** The $k$-Fibonacci function in Theorem (3.1) is unique.

**Proof.** Let there exist $k$-Fibonacci functions, $G_1 : \mathbb{N} \times \mathbb{R} \rightarrow X$, and $G_2 : \mathbb{N} \times \mathbb{R} \rightarrow X$ satisfying

$$||f(k,x) - G_i(k,x)|| \leq \frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}), \quad (3.9)$$

for all $x \in \mathbb{R}$, $k \in \mathbb{N}, i \in \{1, 2\}$. Since $G_1$ and $G_2$ are $k$-Fibonacci function, by Theorem (2.1), there exist functions $g_i : \mathbb{N} \times [-1, 1] \rightarrow X$ ($i \in \{1, 2\}$) such that

$$G_i(k,x) = \begin{cases} 
F_{k,\lfloor x \rfloor + 1}g_i(k,x - \lfloor x \rfloor) + F_{k,\lfloor x \rfloor}g_i(k,x - \lfloor x \rfloor - 1) & x \geq 0 \\
(-1)^{\lfloor x \rfloor}[F_{k,\lfloor x \rfloor - 1}g_i(k,x - \lfloor x \rfloor) - F_{k,\lfloor x \rfloor}g_i(k,x - \lfloor x \rfloor - 1)] & x < 0 \end{cases}, \quad (3.10)$$

for $i \in 1, 2$.

Fix a $t$ in $[0, 1)$, from (3.9), we have

$$||G_1(k,n+t) - G_2(k,n+t)||$$

$$\leq ||G_1(k,n+t) - f(k,n+t)|| + ||f(k,n+t) - G_2(k,n+t)||$$

$$\leq 2\frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}),$$

for all $n \in \mathbb{Z}$, $k \in \mathbb{N}$.

by (3.10), we have

$$||F_{k,n+1}[g_1(k,t) - g_2(k,t)] + F_{k,n}[g_1(k,t-1) - g_2(k,t-1)||$$

$$= ||G_1(k,n+t) - G_2(k,n+t)|| \leq 2\frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}),$$

for all $n \in \mathbb{Z}$, $k \in \mathbb{N}$.
and

\[||F_{k,n-1}[g_1(k,t) - g_2(k,t)] - F_{k,n}[g_1(k,t-1) - g_2(k,t-1)]||
= ||G_1(k,-n+t) - G_2(k,-n+t)|| \leq \frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}),\]

for all \( n \in \mathbb{N}, k \in \mathbb{N}. \)

According to Lemma (3.2), we have

\[
\begin{cases}
\gamma[g_1(k,t) - g_2(k,t)] + [g_1(k,t-1) - g_2(k,t-1)] = 0 \\
-\gamma[g_1(k,t-1) - g_2(k,t-1)] + [g_1(k,t) - g_2(k,t)] = 0
\end{cases}
\]

or

\[
\begin{pmatrix}
\gamma & 1 \\
1 & -\gamma
\end{pmatrix}
\begin{pmatrix}
g_1(k,t) - g_2(k,t) \\
g_1(k,t-1) - g_2(k,t-1)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Since \(-\gamma^2 - 1 \neq 0\), hence
\[g_1(k,t) - g_2(k,t) = g_1(k,t-1) - g_2(k,t-1).\]

Since \(0 \leq t < 1\) is arbitrary, therefore \(g_1(k,t) = g_2(k,t)\), for any \(0 \leq t < 1\), and from (3.10), we have \(G_1(k,x) = G_2(k,x)\), for all \(x \in \mathbb{R}\.\)

\[\square\]

References