On a More Accurate Multiple Hilbert-Type Inequality

Qiliang Huang, Bicheng Yang∗
Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, P. R. China.

Dedicated to the Memory of Charalambos J. Papaioannou
(Communicated by Th. M. Rassias)

Abstract
By using Euler-Maclaurin’s summation formula and the way of real analysis, a more accurate multiple Hilbert-type inequality and the equivalent form are given. We also prove that the same constant factor in the equivalent inequalities is the best possible.

Foundation item: This work is supported by the National Natural Science Foundation of China (No.61370186).

Keywords: Multiple Hilbert-Type Inequality, Weight Coefficient, Euler-Maclaurin’s Summation Formula.

2010 MSC: 26D15.

1. Introduction
If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} b_n^q < \infty \), then a new inequality with the homogeneous kernel of degree 1 is given as (cf. [12])

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min\{m,n\} a_m b_n < \sum_{n=1}^{\infty} (na_n)^p \left( \frac{1}{p} \right)^\frac{1}{p} \sum_{n=1}^{\infty} (nb_n)^q \left( \frac{1}{q} \right)^\frac{1}{q},
\]  

(1.1)

∗Corresponding author

Email addresses: qlhuang@yeah.net (Qiliang Huang), bcyang@gdei.edu.cn (Bicheng Yang)

Received: March 2013 Revised: February 2014
where the constant factor $pq$ is the best possible. Hilbert-type inequalities including \([1.1]\) are important in analysis and its applications (cf. \([1, 5, 13]\)).

By introducing another pair of conjugate exponents $(r, s)(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ and a parameter $0 < \lambda \leq \min\{r, s\}$, (1) has been extended as (cf. \([12]\)):

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m, n\})^{\lambda} a_m b_n < \frac{rs}{\lambda} \left\{ \sum_{n=1}^{\infty} n^{p\left(1+\frac{1}{r}\right)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q\left(1+\frac{1}{s}\right)-1} b_n^q \right\}^{\frac{1}{q}},
$$

(1.2)

where the constant factor $\frac{rs}{\lambda}$ is the best possible. For $\lambda = 1, r = p, s = q$, (1.2) reduces to (1.1).

Recently, by introducing $\alpha \geq \sqrt{\frac{21}{12} - \frac{3}{4}} = -0.3681^+, 0 < \lambda \leq 1$, Yang gave a more accurate best extension of (1.2) and the equivalent form as (cf. \([7]\)):

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m, n\} + \alpha) \lambda a_m b_n < \frac{rs}{\lambda} \left\{ \sum_{n=1}^{\infty} (n + \alpha)^{p\left(1+\frac{1}{r}\right)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (n + \alpha)^{q\left(1+\frac{1}{s}\right)-1} b_n^q \right\}^{\frac{1}{q}},
$$

(1.3)

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m, n\} + \alpha)^{\lambda} a_m b_n < \frac{rs}{\lambda} \left\{ \sum_{n=1}^{\infty} (n + \alpha)^{p\left(1+\frac{1}{r}\right)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (n + \alpha)^{q\left(1+\frac{1}{s}\right)-1} b_n^q \right\}^{\frac{1}{q}},
$$

(1.4)

For $\alpha = 0$, inequality (1.3) reduces to (1.2). Another more accurate Hilbert-type inequalities were given by \([3, 13, 10, 9, 11, 15]\). Yang and Huang also considered the multiple Hilbert-type integral inequality (cf. \([8-3]\)). Recently, Huang gave a more accurate multiple Hilbert’s inequality (cf. \([2]\)).

In this paper, by using Euler-Maclaurin’s summation formula and the way of real analysis, a more accurate multiple Hilbert-type inequality and the equivalent form are given, which are the best extensions of (1.3) and (1.4).

2. Some lemmas

**Lemma 2.1.** If $n \in \mathbb{N}\setminus\{1\}, p_i, r_i > 1 (i = 1, \cdots, n), \sum_{i=1}^{n} \frac{1}{p_i} = 1, \sum_{i=1}^{n} \frac{1}{r_i} = 1, 0 < \lambda \leq 1, \alpha \geq \sqrt{\frac{12}{27} - \frac{3}{4}}$, then

$$
A := \prod_{i=1}^{n} \left[ (m_i + \alpha)^{\frac{1}{r_i}+1}(p_i-1) \prod_{j=1, j \neq i}^{n} \frac{1}{(m_j + \alpha)^{1+(\lambda/r_j)}} \right]^{\frac{1}{p_i}} = 1.
$$

(2.1)

**Proof.** We find

$$
A = \prod_{i=1}^{n} \left[ (m_i + \alpha)^{\frac{1}{r_i}+1}(p_i-1)+\frac{1}{r_i} \prod_{j=1}^{n} \frac{1}{(m_j + \alpha)^{1+(\lambda/r_j)}} \right]^{\frac{1}{r_i}}
$$
Maclaurin's summation formula (cf. [13]), we obtain

\[ \prod_{i=1}^{n} \left( (m_i + \alpha)^{\frac{1}{r_i}+1} \right)^{\frac{1}{r_i}} = \prod_{i=1}^{n} \left( (m_j + \alpha)^{1+\lambda/(r_j)} \right)^{\frac{1}{r_j}}, \]

and then (2.1) is valid. \( \square \)

**Lemma 2.2.** If \( n \in \mathbb{N}, r > 1, \frac{1}{r} + \frac{1}{s} = 1, 0 < \lambda \leq 1, \alpha \geq \frac{\sqrt{3}}{12} - \frac{3}{4}, \) then

\[ \frac{rs}{\lambda} \left[ 1 - \frac{1}{r} \left( \frac{1 + \alpha}{n + \alpha} \right)^{\frac{1}{r}} \right] < \sum_{n=1}^{\infty} \frac{(\min\{n, m\} + \alpha)^{\lambda}}{(n + \alpha)^{1/s}(m + \alpha)^{1+\lambda/(r)}} < \frac{rs}{\lambda}. \]  

(2.2)

**Proof.** For \( x \in (-\alpha, \infty) \), setting \( f(x) := \frac{(\min\{n, x\} + \alpha)^{\lambda}}{(n + \alpha)^{1/s}(x + \alpha)^{1+\lambda/(r)}} \), \( f_1(x) := (n + \alpha)^{-\frac{1}{2}}(x + \alpha)^{-\frac{1}{2}-1} \), \( f_2(x) := (n + \alpha)^{\frac{1}{2}}(x + \alpha)^{-\frac{1}{2}-1} \). We find \((-1)^i f_{j(i)}(x) > 0, f_{j(i)}(\infty) = 0 (i = 0, 1, 2, 3, 4; j = 1, 2) \). By Euler-Maclaurin’s summation formula (cf. [13]), we obtain

\[ \sum_{m=1}^{n} f_1(m) < \int_{1}^{n} f_1(x)dx + \frac{1}{2} [f_1(1) + f_1(n)] + \frac{1}{12} f'_1(x) \bigg|_1^n, \]
\[ \sum_{m=n}^{\infty} f_2(m) < \int_{n}^{\infty} f_2(x)dx + \frac{1}{2} f_2(n) - \frac{1}{12} f_2'(n). \]

\( \square \)

For \( f_1(n) = f_2(n) \), we have the following:

\[ \sum_{m=1}^{\infty} \frac{(\min\{n, m\} + \alpha)^{\lambda}}{(n + \alpha)^{1/s}(m + \alpha)^{1+\lambda/(r)}} = \sum_{m=1}^{n} f_1(m) + \sum_{m=n}^{\infty} f_2(m) - f_1(n) \]

\[ < \int_{1}^{\infty} f(x)dx + \frac{1}{2} f_1(1) - \frac{1}{12} f'_1(1) + \frac{1}{12} (f'_1(n) - f_2(n)) \]
\[ = \int_{-\alpha}^{\infty} f(x)dx - \left[ \int_{-\alpha}^{1} f_1(x)dx - \frac{1}{2} f_1(1) + \frac{1}{12} f'_1(1) - \frac{1}{12} (f'_1(n) - f'_2(n)) \right], \]

\[ \int_{-\alpha}^{\infty} f(x)dx = \int_{-\alpha}^{1} f_1(x)dx + \int_{1}^{\infty} f_2(x)dx = \frac{s}{\lambda} + \frac{r}{\lambda} = \frac{rs}{\lambda}, \]
\[ = \frac{s}{\lambda} (n + \alpha)^{-\frac{1}{2}}(1 + \alpha)^{\frac{1}{2}} - \frac{1}{2} (n + \alpha)^{-\frac{1}{2}}(1 + \alpha)^{-\frac{1}{2}} - \frac{\lambda}{12} (n + \alpha)^{-2} \]
\[ = \frac{(\frac{1}{s} + \frac{1}{n + \alpha})^{\lambda/s}(s/\lambda)}{12(1 + \alpha)^2} \]
\[ \times \left\{ (\frac{\lambda}{s})^2 - [6(1 + \alpha) + 1](\frac{\lambda}{s}) + 12(1 + \alpha)^2 - \frac{\lambda^2}{s} (\frac{1 + \alpha}{n + \alpha})^{(1+\lambda/3)} \right\}. \]
Proof. Moreover, it follows (decreasing in $n$, $\omega$)

As the assumption of Lemma 1, define the weight coefficients $\omega_i(m_i) = \omega_\lambda(m_i, r_i; r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$ as

$$\omega_i(m_i) : = \frac{1}{(m_i + \alpha)^{\lambda/\alpha}} \sum_{m_{n+1} = 1}^{\infty} \cdots \sum_{m_{i+1} = 1}^{\infty} \sum_{m_{i-1} = 1}^{\infty} \sum_{m_1 = 1}^{\infty} \times (\min_{1 \leq k \leq n} \{m_k\} + \alpha)^\lambda \prod_{j=1}^{n} \frac{1}{(m_j + \alpha)^{1+(\lambda/r_j)}}$$  \hspace{1cm} (2.3)

($i = 1, \ldots, n$), then we have

$$\frac{1}{\lambda^{n-1}} \prod_{j=1}^{n} r_j \left[1 - O\left(\frac{1}{(m_n + \alpha)^{\lambda/r_n}}\right)\right] < \omega_n(m_n) = \frac{1}{(m_n + \alpha)^{\lambda/r_n}}$$

$$\times \sum_{m_{n+1} = 1}^{\infty} \cdots \sum_{m_{i+1} = 1}^{\infty} \sum_{m_{i-1} = 1}^{\infty} \sum_{m_1 = 1}^{\infty} \frac{\min_{1 \leq k \leq n} \{m_k\} + \alpha}{\prod_{j=1}^{n-1} (m_j + \alpha)^{1+(\lambda/r_j)}} < \frac{1}{\lambda^{n-1}} \prod_{j=1}^{n} r_j.$$  \hspace{1cm} (2.4)

Moreover, it follows

$$\omega_i(m_i) < \frac{1}{\lambda^{n-1}} \prod_{j=1}^{n} r_j (i = 1, \ldots, n).$$  \hspace{1cm} (2.5)

**Proof.** Proof. We prove (2.4) by mathematical induction. For $n = 2$, by (2.2), setting $n = m_2, m = m_1, r = r_1, s = r_2$, we have (2.4). Assuming that for $n \geq 2$, (2.4) are valid, then for $n + 1$, setting $m_0 + \alpha = \min_{2 \leq k \leq n+1} \{m_k\} + \alpha, s_1 := (1 - \frac{1}{r_1})^{-1}$, then by (2.3), we have the following:

$$\omega_{n+1}(m_{n+1}) = \frac{1}{(m_{n+1} + \alpha)^{\lambda/r_{n+1}}} \sum_{m_{n+1} = 1}^{\infty} \cdots \sum_{m_2 = 1}^{\infty} (m_{n+1} + \alpha)^{\lambda/s_1} \sum_{m_1 = 1}^{\infty} \frac{(\min_{2 \leq k \leq n+1} \{m_k\} + \alpha)^\lambda}{(m_{n+1} + \alpha)^{1+(\lambda/r_1)}}$$

$$< \frac{r_1 s_1}{\lambda (m_{n+1} + \alpha)^{\lambda/r_{n+1}}} \sum_{m_{n+1} = 1}^{\infty} \cdots \sum_{m_2 = 1}^{\infty} \frac{\min_{2 \leq k \leq n+1} \{m_k\} + \alpha}{\prod_{j=2}^{n} (m_j + \alpha)^{1+(\lambda/r_j)}}.$$  \hspace{1cm} (2.6)
Setting $\tilde{\lambda} = \frac{\lambda}{s_1}, \tilde{r}_j = \frac{r_j}{s_1}$ in (2.6), since $\sum_{j=2}^{n+1} \tilde{r}_j^{-1} = 1, 0 < \tilde{\lambda} \leq 1$, by the assumption of induction, it follows that

$$\omega_{n+1}(m_{n+1}) < \frac{r_1 s_1}{\lambda} \frac{1}{\lambda^{n-1}} \prod_{j=2}^{n+1} \tilde{r}_j = \frac{1}{\lambda^n} \prod_{j=1}^{n+1} r_j. \quad (2.7)$$

By (2.3) and the assumption of induction, we still have

$$\omega_{n+1}(m_{n+1}) > \frac{r_1 s_1}{\lambda} \left( \frac{1}{(m_{n+1} + \alpha) \lambda^{r_{n+1}}} \right) \sum_{m_{n+1}} \cdots \sum_{m_2} \left( \min_{2 \leq k \leq n+1} \{m_k\} + \alpha \right)^{\lambda/s_1} \prod_{j=2}^{n+1} (m_j + \alpha)^{1+(\lambda/r_j)} \left[ 1 - \frac{1}{r_1} \left( \frac{1 + \alpha}{m_{j_0} + \alpha} \right)^{n_1} \right]$$

$$= \frac{r_1 s_1}{\lambda} \left( \frac{1}{\lambda^{n-1}} \prod_{j=2}^{n+1} \tilde{r}_j \right) \left( 1 - O \left( \frac{1}{(m_{n+1} + \alpha) \lambda^{r_{n+1}}} \right) \right) - \frac{\beta}{(m_{n+1} + \alpha) \lambda^{r_{n+1}}},$$

$$= \frac{1}{\lambda^n} \prod_{j=1}^{n+1} r_j \left[ 1 - O \left( \frac{1}{(m_{n+1} + \alpha) \lambda^{r_{n+1}}} \right) \right], \quad (2.8)$$

where $\beta = \frac{(1+\alpha)^{\lambda/s_1}}{\lambda} \prod_{j=2}^{n+1} \sum_{m_{j+1}} \cdots \sum_{m_2} \left( \min_{2 \leq k \leq n+1} \{m_k\} + \alpha \right)^{\lambda/s_1} \in \mathbb{R}$. By (2.7) and (2.8), (2.4) are valid for $n+1$. By mathematical induction, (2.4) are valid for $n \in \mathbb{N} \setminus \{1\}$.

Setting $\tilde{m}_n = m_i, \tilde{r}_n = r_i, \tilde{m}_j = m_{j+1}, \tilde{r}_j = r_{j+1} (i = 1, \ldots, n-1), \tilde{m}_j = m_j, \tilde{r}_j = r_j (j = 1, \ldots, i-1)$, then we have the following:

$$\omega_i(m_i) = \omega_{\lambda}(\tilde{m}_n, \tilde{r}_n, \tilde{r}_1, \cdots, \tilde{r}_{n-1}) < \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} \tilde{r}_i = \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i.$$

Hence (2.5) is valid. □

3. Main Results

Theorem 3.1. Suppose that $n \in \mathbb{N} \setminus \{1\}, p_i, r_i > 1 (i = 1, \ldots, n), \sum_{i=1}^{n} \frac{1}{p_i} = 1, \sum_{i=1}^{n} \frac{1}{r_i} = 1, \frac{1}{q_n} = 1 - \frac{1}{p_n}, 0 < \lambda \leq 1, \alpha > \frac{\sqrt{21}}{12} - \frac{3}{4}$. If $a_{m_i}^{(i)} \geq 0, 0 < \sum_{m_i=1}^{\infty} (m_i + \alpha)^{p_i(1+\frac{\lambda}{r_i})^{-1}} (a_{m_i}^{(i)})^{p_i} < \infty (i = 1, \ldots, n)$, then we have the following equivalent inequalities:

$$I : = \sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda} \prod_{i=1}^{n} a_{m_i}^{(i)}$$

$$< \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \left( \sum_{m_i=1}^{\infty} (m_i + \alpha)^{p_i(1+\frac{\lambda}{r_i})^{-1}} (a_{m_i}^{(i)})^{p_i} \right)^{\frac{1}{r_i}}, \quad (3.1)$$

$$J : = \left\{ \sum_{m_n=1}^{\infty} \left( \sum_{m_{n-1}=1}^{\infty} \frac{1}{m_n + \alpha} \frac{1}{(q_n \lambda/r_n)} \right) \prod_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda}$$

$$\times \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right\} \leq \frac{r_n}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \left( \sum_{m_i=1}^{\infty} (m_i + \alpha)^{p_i(1+\frac{\lambda}{r_i})^{-1}} (a_{m_i}^{(i)})^{p_i} \right)^{\frac{1}{r_i}}. \quad (3.2)$$
Lemma 3.2. Proof. We have proven the theorem for $n = 2$ (cf. [7]). In the following, we prove the theorem for $n \geq 3$. □

Since $\frac{1}{p_n} + \frac{1}{q_n} = 1$, by (2.1), (2.4) and Hölder’s inequality (cf. [4]), we find

$$
\left[ \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda} \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right]^{q_n}
\leq \left\{ \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda} \prod_{j=1}^{n} \frac{1}{(m_j + \alpha)^{1+(\lambda/r_j)}} \right\}^{q_n}
\leq \left( \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \right) \left( m_n + \alpha \right)^{1+\frac{q_n}{p_n}} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda}
\times \prod_{i=1}^{n-1} \left[ (m_i + \alpha)^{\frac{\lambda}{r_i}+1+(\lambda/r_i)} \prod_{j=1}^{n} \frac{1}{(m_j + \alpha)^{1+(\lambda/r_j)}} \right] \left( a_{m_i}^{(i)} \right)^{q_n},
$$

$$
J \leq \left( \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \right) \left( m_n + \alpha \right)^{1+\frac{q_n}{p_n}} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda}
\times \prod_{i=1}^{n-1} \left[ (m_i + \alpha)^{\frac{\lambda}{r_i}+1+(\lambda/r_i)} \prod_{j=1}^{n} \frac{1}{(m_j + \alpha)^{1+(\lambda/r_j)}} \right] \left( a_{m_i}^{(i)} \right)^{q_n},
$$

(3.3)

For $n \geq 3$, since $\sum_{i=1}^{n-1} \frac{q_n}{p_n} = 1$, by Hölder’s inequality again in (3.3), we find

$$
J \leq \left( \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \right) \prod_{i=1}^{n-1} \left\{ \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{m_k\} + \alpha \right)^{\lambda} \right\}
$$
Then by (3.4), it follows that (3.2) is valid, which is equivalent to (3.1).

Proof. (3.2) is the best possible. Then (3.2) is valid, which is equivalent to (3.1).

0 naturally valid. Suppose \( \tilde{r}_n = \left( \sum_{m_n=1}^\infty (m_n + \alpha)^{\frac{1}{r_n}} \right) < \infty \). Then by (3.4), it follows that (3.2) is valid, which is equivalent to (3.3). Then by (3.3), it follows that (3.2) is valid, which is equivalent to (3.1).

\[ \tilde{I} : = \sum_{m_n=1}^\infty \cdots \sum_{m_1=1}^\infty \left( \min_{1 \leq k \leq n} \{ m_k \} + \alpha \right)^\lambda \prod_{i=1}^n a_{m_i}^{(i)} = \sum_{m_n=1}^\infty \left( \frac{1}{(m_n + \alpha)^{\frac{1}{r_n}}} \right) \]

Then (3.3) is valid, which is equivalent to (3.1). □

**Theorem 3.3.** As the assumption of Theorem 1, the same constant factor \( \frac{1}{\lambda_n - r_i} \prod_{i=1}^n r_i \) in (3.1) and (3.2) is the best possible.

Proof. For \( 0 < \varepsilon < \frac{q \lambda}{r_n} \), setting \( \tilde{r}_i = (1 + \frac{1}{r_n}) - \varepsilon, \tilde{a}_{m_i}^{(i)} = (m_i + \alpha)^{\frac{1}{r_i}} - \varepsilon, \) we have \( \tilde{r}_i > 1(i = 1, \cdots, n), \sum_{i=1}^n \frac{1}{r_i} = 1 \). Then by (2.4), we find

\[ \tilde{I} : = \sum_{m_n=1}^\infty \cdots \sum_{m_1=1}^\infty \left( \min_{1 \leq k \leq n} \{ m_k \} + \alpha \right)^\lambda \prod_{i=1}^n \tilde{a}_{m_i}^{(i)} = \sum_{m_n=1}^\infty \left( \frac{1}{(m_n + \alpha)^{\frac{1}{r_n}}} \right) \]
that the constant factor 

In virtue of (3.6) and (3.7), it follows,

by (3.5) that the constant factor in (3.1) is not the best possible.

Remark 3.4. For \( k \) in (3.4), we have a multiple best extension of (1.2) as

\[
\begin{align*}
\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left( \min_{1 \leq k \leq n} \{ m_k \} \right)^{\lambda} & \prod_{i=1}^{n} a_{m_i}^{(i)} \\
< & \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i \left( \sum_{m_i=1}^{\infty} m_i p_i(1+\frac{1}{r_i})^{-1} (a_{m_i}^{(i)}) \right)^{\frac{1}{p_i}}.
\end{align*}
\] (3.8)

References


