Int. J. Nonlinear Anal. Appl. In Press, 1–9
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.35770.5317

Multiple solutions for a class of second order differential equations with nonlinear derivative dependence

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(Communicated by Abdolrahman Razani)

Abstract

In this paper, using variational methods and critical point theory, we prove the existence of multiple solutions for a class of second order differential equations with nonlinear derivative dependence involving a positive parameter. Some recent results are extended and improved. Some examples are presented to demonstrate the application of our main results.

Keywords: Dirichlet problem; Nonlinear derivative dependence; Variational methods
2020 MSC: 34B15, 58E05

1 Introduction

In the present paper, we study the following system

$$\begin{aligned} &-(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x)=\lambda F_{u_i}(x,u_1,...,u_n)h_i(x,u_i'),\ \ x\in(a,b) \\ &u_i(a)=u_i(b)=0,\ \text{ for }\ i=1,...,n \end{aligned} \tag{1.1}$$

where $p_i \geq 1$, for $1 \leq i \leq n$, $\lambda > 0$, $a, b \in \mathbb{R}$ with $a < b$, $F : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to x, for every $(t_1, ..., t_n) \in \mathbb{R}^n$ continuously differentiable in $(t_1, ..., t_n)$, for almost every $x \in [a, b]$, and $F_{t_i}(x, 0, ..., 0) = 0$ for all $x \in [a, b]$, and for $1 \leq i \leq n$, $h_i : [a, b] \times \mathbb{R} \to [0, +\infty)$ is a bounded and continuous function with $m_i := inf_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x, t) > 0$ for $1 \leq i \leq n$. Here, F_{t_i} denotes the partial derivative of F with respect to t_i. In recent years, many authors have applied variational methods to study the existence of solutions for Laplacian equations and p-Laplacian equations; see, for example, [1, 2, 6, 7, 8, 9, 10, 11, 12] and the references therein. For example, Averna and Bonanno in [2] have proved the existence of at least three classical solutions for the problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(x,u)h(u'),\ \ x\in(a,b) \\ u(a)=u(b)=0. \end{cases}$$

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Received: October 2024 *Accepted:* December 202

In [7], the existence of at least three classical solutions was discussed for the following Dirichlet quasilinear elliptic system

$$\begin{aligned} &-(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x)=[\lambda F_{u_i}(x,u_1,...,u_n)+\eta G_{u_i}(x,u_1,...,u_n)]h_i(x,u_i'),\ \ x\in(a,b)\\ &u_i(a)=u_i(b)=0,\ \text{ for }\ i=1,...,n. \end{aligned}$$

In [8], the existence of at least one non-trivial classical solution for Dirichlet quasilinear system (1.1) was established by using variational methods for smooth functionals defined on reflexive Banach spaces taking some assumptions on the asymptotic behaviour of the nonlinear data. In [10], the existence of two solutions for the system (1.1) under some algebraic conditions with the classical Ambrosett-Rabinowitz condition on the nonlinear terms was investigated by using a consequence of the local minimum theorem due to Bonanno and, the mountain pass theorem.

For further information on the subject we refer the interested reader to the recent papers [5, 16, 17].

In the present paper, we use variational methods and critical point theory to obtain existence results for the system (1.1) under suitable conditions imposed on F, the potential function of f (see, the conditions (F_0), (F_1) and (F_2) of Theorem 3.1). In Theorem 3.1 we prove the existence of at least two solutions for the system (1.1), while in Theorem 3.2 we discuss the existence of infinitely many solutions for the same system.

The present paper is organized as follows. In Section 2, we recall some basic definitions and our main tools. In Section 3, we state and prove the main results of the paper. Then, we give two examples to illustrate our results.

2 Preliminaries and Basic Notation

First, we introduce some basic notations in this section. Consider X be the Cartesian product of n Sobolev spaces $W^{1,p_1}([a,b])...$, and $W^{1,p_n}([a,b])$, i.e., $X=W^{1,p_1}([a,b])\times...\times W^{1,p_n}([a,b])$, equipped with the norm

$$\|(u_1,...,u_n)\|=\sum_{i=1}^{n}\|u_i'\|_{p_i}$$

where

$$\|u_i'\|_{p_i}=(\int_a^b|u_i'(x)|^{p_i}dx)^{\frac{1}{p_i}},\quad i=1,...,n.$$

Since $p_i>1$ for $i=1,...,n$, X is compactly embedded in $(C([a,b]))^n$. By a classical solution of (1.1), we mean a function u $=(u_1,...,u_n)$ such that, for $i=1,...,n$, $u_i\in C^1[a,b]$, $u_i'\in AC[a,b]$, and $u_i(x)$ satisfies (1.1) a.e. on $[a,b]$.

Definition 2.1. A weak solution of the system (1.1) is a function $u=(u_1,...,u_n)\in X$ such that

$$\sum_{i=1}^{n}\int_a^b(\int_0^{u_i'(x)}\frac{(p_i-1)|\tau|^{p_i-2}}{h_i(x,\tau)}d\tau)v_i'(x)dx-\lambda\int_a^b\sum_{i=1}^{n}F_{u_i}(x,u_1,...,u_n)v_i(x)dx=0$$

for every $v=(v_1,...,v_n)\in X$.

Using standard methods, we see that a weak solution of the system (1.1) is indeed a classical solution (see [7, Lemma 2.2]). Let

$$\begin{aligned} &\overline{p}:=\max\{p_i:1\le i\le n\},\ \ \underline{p}:=\min\{p_i:1\le i\le n\},\\ &m_i:=\inf_{(x,t)\in[a,b]\times\mathbb{R}}h_i(x,t)>0,\quad for\ \ 1\le i\le n,\\ &M_i:=\inf_{(x,t)\in[a,b]\times\mathbb{R}}h_i(x,t)>0,\quad for\ \ 1\le i\le n,\\ &\overline{M}:=\max\{M_i:1\le i\le n\},\ \ \underline{M}:=\min\{m_i:1\le i\le n\}. \end{aligned}$$

Then, $\overline{M}\ge Mi\ge mi\ge\underline{M}>0$ for each $i=1,...,n$. Put

$$H_i(x,t)=\int_0^t(\int_0^\tau\frac{(p_i-1)|\delta|^{p_i-2}}{h_i(x,\delta)}d\delta)d\tau$$

for all $(x,t) \in [a,b] \times \mathbb{R}$, $1 \le i \le n$. Consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \sum_{i=1}^{n} \int_a^b H_i(x, u_i'(x))dx \tag{2.1}$$

and

$$\Psi(u) = \int_a^b F(x, u_1, ..., u_n)dx. \tag{2.2}$$

We let $J_\lambda : X \to \mathbb{R}$, that $J_\lambda = \Phi(u) - \lambda\Psi(u)$ for $u = (u_1, ..., u_n) \in X$. J_λ is the energy functional corresponding to system (1.1) for the parameter $\lambda > 0$, that is well defined for every $u \in X$. Obviously, $J_\lambda \in C^1(X, \mathbb{R})$ and J_λ is weakly lower semi-continuous. Therefore, we can infer that $u \in X$ is a weak solution of the system (1.1) if and only if it holds

$$J_\lambda'(u)v = \sum_{i=1}^{n} \int_a^b \left(\int_0^{u_i'(x)} \frac{(p_i-1)|\tau|^{p_i-2}}{h_i(x,\tau)} d\tau \right) v'(x)dt - \lambda \int_a^b \sum_{i=1}^{n} F_{u_i}(x, u_1(x), ..., u_n(x))v(x)dx = 0,$$

for all $v \in X$. Furthermore, J_λ is sequentially weakly lower semicontinuous (see [7]).

Definition 2.2. Let X be a real reflexive Banach space. If any sequence $\{u_k\} \subset X$ for which $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence. Then it is said that J satisfies Palais-Smale condition.

Now, let us give the main tools which we will use to prove our main results.

Theorem 2.3. [14, Theorem 4.10] Let $J \in C^1(X, \mathbb{R})$, and J satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and

$$\inf_{u \in \partial\Omega} J(u) > \max\{J(u_0), J(u_1)\},$$

then there exists a critical point u of J, i.e. $J'(u) = 0$ with $J(u) > \max\{J(u_0), J(u_1)\}$.

Theorem 2.4. [15, Theorem 9.12] Let E be an infinite dimensional real Banach space. Let $J \in C^1(E, \mathbb{R})$ be an even functional which satisfies the (PS)-condition, and $J(0) = 0$. Suppose that $E = V \bigoplus X$, where V is finite dimensional, and J satisfies that

(i_1) There exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \ge \alpha$ for all $u \in X$ with $\|u\| = \rho$;

(i_2) For any finite dimensional subspace $W \subset E$ there is $R = R(W)$ such that $J(u) \le 0$ on $W \setminus B_R$.

Then J possesses an unbounded sequence of critical values.

Theorem 2.5. [18, Theorem 38] For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \ne \emptyset, \min_{u \in M} F(u) = \alpha$ has a solution in case the following conditions hold:

(i_3) X is a real reflexive Banach space,

(i_4) M is bounded and weak sequentially closed,

(i_5) F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \le \lim_{n\to\infty} \inf F(u_n)$ holds.

For successful employment of Theorems 2.3 and 2.4 we refer to the papers [3, 19]. In the paper [20], Theorem 2.4 has been successfully applied to obtain the existence of infinitely many solutions for boundary value problems.

3 Main results

We take the following assumptions on the nonlinear term:

(F_0) there exist $T > 0$ and a constant $\nu > \frac{\overline{p}\overline{M}}{\underline{M}}$ such that

$$0 < \nu F(x,t) \leq \sum_{i=1}^{n} t_i F_{t_i}(x,t), \quad |t| > T;$$

(F_1) $F : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies the Carathèodory condition and

$$|F(x,t)| \leq c(1 + \sum_{i=1}^{n} |t_i|^{q_i - 1}) \quad for \quad t \in \mathbb{R}^n,$$

where $q_i > p_i$ for $i = 1, ...n$ and $x \in [a,b]$;

(F_2) $F(x,t) = o(\sum_{i=1}^{n} |t_i|^{p_i-1})$, $t \longrightarrow 0$, for $x \in [a,b]$ uniformly.

The main results of the present paper are the following.

Theorem 3.1. Assume that the assumptions (F_0), (F_1) and (F_2) hold. Then, if $F(x,t) \geq 0$ for all $(x,t) \in [a,b] \times \mathbb{R}^n$, the system (1.1) has at least two classical solutions.

Theorem 3.2. Assume that the assumption (F_0) holds. Then, if $F(x,t)$ is odd in t, the system (1.1) has infinitely many classical solutions.

First, we start with the following lemma.

Lemma 3.3. Assume that (F_0), holds. Then $J_\lambda(u)$ satisfies the (PS)-condition.

Proof . Assume that $u_k = ((u_{1k}), ..., (u_{nk}))$ and $\{u_k\} \subset X$ such that $\{J_\lambda(u_k)\}$ is bounded and $J'_\lambda(u_k) \to 0$ as $k \to +\infty$. Then, there exists a positive constant c_0 such that $|J_\lambda(u_k)| \leq c_0$. Since $0 < \underline{M} \leq h_i(x,t) \leq \overline{M}$ for each $(x,t) \in [a,b] \times \mathbb{R}$ and $i = 1, ..., n$, from (2.1) we see that

$$\frac{1}{\overline{M}} \sum_{i=1}^{n} \frac{\|u'_i\|_{p_i}^{p_i}}{p_i} \leq \Phi(u) \leq \frac{1}{\underline{M}} \sum_{i=1}^{n} \frac{\|u'_i\|_{p_i}^{p_i}}{p_i} \tag{3.1}$$

for all $u \in X$. Therefore, letting $\|u_k\| > 1$, by the assumption (F_0), we have

$$\begin{aligned}
c_0 + \|u_k\| \geq & \nu J_\lambda(u_k) - J'_\lambda(u_k)(u_k) \\
\geq & \left(\frac{\nu}{\overline{M}}\right) \sum_{i=1}^{n} \frac{\|(u'_{ik})\|_{p_i}^{p_i}}{p_i} + \lambda \int_a^b \left(\sum_{i=1}^{n} (F_{(u_{ik})}(x,u_k)(u_{ik}) - \nu F(x,u_k)) \right) dx \\
& - \sum_{1}^{n} \int_a^b (\int_0^{(u'_{ik})(x)} \frac{(p_i - 1)|\xi|^{p_i - 2}}{h_i(x,\xi)} d\xi)(u'_{ik})(x) dx \\
\geq & \left(\frac{\nu}{\overline{M}}\right) \sum_{i=1}^{n} \frac{\|(u'_{ik})\|_{p_i}^{p_i}}{p_i} - \left(\frac{1}{\underline{M}}\right) \sum_{i=1}^{n} \|(u'_{ik})\|_{p_i}^{p_i} \\
\geq & (\frac{\nu}{\overline{p}\overline{M}} - \frac{1}{\underline{M}}) \sum_{i=1}^{n} \|(u'_{ik})\|_{p_i}^{p_i}.
\end{aligned}$$

Due to assumption $\nu > \frac{\overline{p}\overline{M}}{\underline{M}}$, we infer that $\{u_k\}$ is bounded. By using the same argument given in [4, Lemma 2.4], it can easily be proven that $\{u_k\}$ converges strongly to u in X. Overall, this implies J_λ satisfies the (PS)-condition. $\square$

3.1 The proof of Theorem 3.1

Proof . By the definition of J_λ, it is clear that $J_\lambda(0) = 0$. Moreover, from Lemma 3.3 we know that J_λ satisfies the (PS)-condition. The rest of the proof is split into two steps:

Step 1. We will show that there exists $Z > 0$ such that the functional J has a local minimum $u_0 \in B_Z = \{u \in X; \|u\| < Z\}$. To do this, we will apply Mazur's lemma (see, e.g., [13]) which states that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit. Let $\{u_k\} \subseteq \overline{B}_Z$ and $u_k \rightharpoonup u$ as $k \to \infty$, then there exists a sequence of convex combinations

$$v_k = \sum_{j=1}^{m} a_{k_j} u_j, \qquad \sum_{j=1}^{m} a_{k_j} = 1, \qquad a_{k_j} \geq 0,\ j \in N$$

such that $v_k \to u$ in X. Since $\overline{B}_Z$ is a closed convex set, we have $\{v_k\} \subseteq \overline{B}_Z$ and $u \in \overline{B}_Z$. Noting that J is weak sequentially lower semi-continuous on $\overline{B}_Z$, and that X is a reflexive Banach space, we can infer by Theorem 2.5 that J has a local minimum $u_0 \in \overline{B}_Z$. Now, we assume that $J_\lambda(u_0) = \min_{u \in \overline{B}_Z} J_\lambda(u)$, and show that

$$J_\lambda(u_0) < \inf_{u \in \partial B_Z} J_\lambda(u).$$

Let $\varepsilon > 0$ be small enough such that $\varepsilon\lambda < \frac{1}{2\overline{p}\overline{M}}$. By the assumptions (F_1) and (F_2), we have

$$F(x,t) \leq \varepsilon \sum_{i=1}^{n} |t_i|^{p_i} + c \sum_{i=1}^{n} |t_i|^{q_i} \text{ for } (x,t) \in [a,b] \times \mathbb{R}^n. \tag{3.2}$$

Then, from (3.1) and (3.2) into account for every $u \in X$, we have:

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{\overline{M}} \sum_{i=1}^{n} \frac{\|u\|_{p_i}^{p_i}}{p_i} - \lambda\varepsilon \int_a^b \sum_{i=1}^{n} |u_i|^{p_i} dx - \lambda c \int_a^b \sum_{i=1}^{n} |u_i|^{q_i} dx \\ &\geq \frac{1}{\overline{p}\overline{M}} \sum_{i=1}^{n} \|u_i\|_{p_i}^{p_i} - \lambda\varepsilon \sum_{i=1}^{n} \|u_i\|_{p_i}^{p_i} - \lambda c \sum_{i=1}^{n} \|u_i\|_{L_{q_i}}^{q_i} \\ &\geq \frac{1}{2\overline{p}\overline{M}} \sum_{i=1}^{n} \|u_i\|_{p_i}^{p_i} - \lambda c \sum_{i=1}^{n} \|u_i\|_{L_{q_i}}^{q_i}, \text{ when } \|u\| < 1 \end{aligned}$$

Since $q_i > p_i$ therefore, there exist $r, \delta > 0$ such that $J_\lambda(u) \geq \delta > 0$ for every $\|u\| = r < 1$. If we let $Z = r$, then $J_\lambda(u) > 0 = J_\lambda(0) \geq J_\lambda(u_0)$ for $u \in \partial B_Z$. Hence $u_0 \in B_Z$ and $J'(u_0) = 0$.

Step 2. Since $J_\lambda(u_0) = \min_{u \in X} J_\lambda(u)$, we can let $Z > 0$ be sufficiently large such that $J_\lambda(u_0) \leq 0 < \inf_{u \in \partial B_Z} J_\lambda(u)$, where $B_Z = \{u \in X; \|u\| < Z\}$.
Now we will show that there exists $u_1 \in X$ with $\|u_1\| > Z$ such that $J_\lambda(u_1) < \inf_{\partial B_Z} J_\lambda(u)$. For this, let $e = (e_1, ..., e_n)(x) \in X$ and $u_1 = (\gamma e_1, ..., \gamma e_n), \gamma > 0$ and $\sum_{i=1}^{n} \|e_i'\| = 1$. By (F_0), there exist constants $a_1, a_2 > 0$ such that $F(x,t) \geq a_1 \sum_{i=1}^{n} |t_i|^{\nu} - a_2$, for all $x \in [a,b]$, $|t| \geq T$,where $t = (t_1, ..., t_n)$, we have

$$\begin{aligned} J_\lambda(u_1) &= (\Phi - \lambda\Psi)(\gamma e) \\ &\leq \frac{1}{\underline{p}\underline{M}} \sum_{i=1}^{n} \frac{\|\gamma e_i'\|_{p_i}^{p_i}}{p_i} - \lambda \int_a^b F(x, \gamma e) dx \\ &\leq \frac{1}{\underline{p}\underline{M}} \sum_{i=1}^{n} \frac{\gamma^{p_i} \|e_i'\|_{p_i}^{p_i}}{p_i} - \lambda a_1 \gamma^{\nu} \sum_{i=1}^{n} \int_a^b |e_i(x)|^{\nu} dx + \lambda a_2 (b-a) \end{aligned}$$

Since $\nu > \frac{\overline{p}\overline{M}}{\underline{M}} \geq p_i$ there exists sufficiently large γ such that $\gamma > Z > 0$ which means $J_\lambda(\gamma e) < 0$. Hence, $\inf_{\partial B_Z} J_\lambda(u) > \max\{J_\lambda(u_0), J_\lambda(u_1)\}$. Then, Theorem 2.3 assures the existence of the second critical point u^*. Therefore, u_0, u^* are two critical points of J_λ which are two nontrivial weak solutions of the system (1.1), and they are classical solutions. $\square$

The following example illustrates Theorem 3.1.

Example 3.4. Let $p_1 = p_2 = 3$, then $\overline{p} = \underline{p} = 3$. We consider $F(t_1, t_2) = t_1^8 + t_2^8$ for $t_1, t_2 \in \mathbb{R}$, $h_1(x, t_1) = 1 + \sin^2 t_1$ and $h_2(x, t_2) = 1 + \cos^4 t_2$ for all $x \in [0, 2]$, $t_1, t_2 \in \mathbb{R}$. Therefore, $m_1 = 1$, $m_2 = 1$, $M_1 = 2$ and $M_2 = 2$, then $\underline{M} = 1$ and $\overline{M} = 2$. Moreover, $F(x, t_1, t_2) = o(|t_1^2| + |t_2^2|), (t_1, t_2) \to (0, 0)$, and by choosing $q_1 = q_2 = 10$ and $c = 2$ we observe that $|F(x, t)| < c(1 + \sum_{i=1}^2 |t|^{q_i - 1})$ for all $(t_1, t_2) \in \mathbb{R}^2$, and $F(x, t_1, t_2) \geq 0$ for all $(t_1, t_2) \in \mathbb{R}^2$. By choosing $\nu = 8$, that $\nu > \frac{\overline{p}\overline{M}}{\underline{M}}$ we have $8F(x, t) \leq \sum_{i=1}^2 t_i F_{t_i}(x, t_1, t_2)$, so we see that all conditions (F_0), (F_1), and (F_2) are satisfied. Also, the functions F, h_1 and h_2 are continuous functions, therefore, by using of Theorem 3.1 the following system

$$\begin{aligned}
&-2|u_1'(x)|u_1''(x) = \lambda F_{u_1}(x, u_1, u_2) h_1(x, u_1'), \ \ x \in (0, 2)\\
&-2|u_2'(x)|u_2''(x) = \lambda F_{u_2}(x, u_1, u_2) h_2(x, u_2'), \ \ x \in (0, 2)\\
&u_1(0) = u_1(2) = 0, \ \ u_2(0) = u_2(2) = 0.
\end{aligned}$$

has at least two nontrivial classical solutions.

A consequence of Theorem 3.1 is the following corollary.

Corollary 3.5. Suppose that the following conditions are satisfied:

(A_0) there exist $T' > 0$ and a constant $\nu' > \frac{\overline{p}\overline{M}}{\underline{M}}$ such that

$$0 < \nu' F(x, t) - \sum_{i=1}^n t_i F_{t_i}(x, t) \leq \sum_{i=1}^n |t_i|^{\tau_i}, \ \ |t| > T'.$$

where $\tau_i \leq p_i$ for every $1 \leq i \leq n$;

(A_1) there exist constant $K > 0$ and $\eta_i > p_i$ for every $1 \leq i \leq n$, such that

$$F(x, t) \leq K \sum_{i=1}^n |t_i|^{\eta_i}$$

when $\|t\| \to 0$;

(A_2) there exist constants $K', R > 0$ and $\vartheta_i > p_i$ for every $1 \leq i \leq n$, such that

$$F(x, t) \geq K' \sum_{i=1}^n |t_i|^{\vartheta_i}$$

when $\|t\| > R$.

If $F(x, t) > 0$, then the system (1.1) has two weak solutions.

3.2 The proof of Theorem 3.2

Proof . From the definitions of the functionals Φ and Ψ, it is clear that J_λ is even and $J_\lambda(0) = 0$. The rest of the proof is split into two steps:

Step 1. Since its proof is straightforward, we only depict briefly how J_λ satisfies condition (i_1) in Theorem 2.4. Since, J_λ is coercive and also satisfies (PS)-condition, by the minimization theorem [14, Theorem 4.4], the functional J_λ has a minimum critical point $u \in X$ with $J_\lambda(u) \geq \alpha > 0$ and $\|u\| = \rho$ for $\rho > 0$ small enough.

Step 2. Now, we will show that J_λ satisfies condition (i_2) in Theorem 2.4. Let $W \subset X$ be a finite dimensional subspace. Any non-zero vector $u \in W$ has a unique representation $u = \theta w$, where $w = (w_1, ...w_n)$, $\theta = \|u\|$ and $\sum_{i=1}^n \|w_i'\| = 1$. Then, similar to Step 2 in the proof of Theorem 3.1, it follows

$$\begin{aligned}
J_\lambda(\theta w) &= (\Phi - \lambda\Psi)(\theta w)\\
&\leq \frac{1}{\underline{pM}} \sum_{i=1}^n \frac{\|\theta w'\|_{p_i}^{p_i}}{p_i} - \lambda \int_a^b F(x, \theta w) dx\\
&\leq \frac{1}{\underline{pM}} \sum_{i=1}^n \frac{\theta^{p_i} \|w_i'\|_{p_i}^{p_i}}{p_i} - \lambda a_1 \theta^\nu \sum_{i=1}^n \int_a^b |w_i(x)|^\nu dx + \lambda a_2 (b - a)
\end{aligned}$$

since $\nu > \frac{\overline{p}\overline{M}}{\underline{M}} \geq p_i$, the above inequality implies that there exists θ_0 such that $\|\theta w\| > \rho$ and $J_\lambda(\theta w) < 0$ for every $\theta \geq \theta_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that for all $u \in W \setminus B_R$, that is, when $\|u\| \geq R$, we have $J_\lambda(u) \leq 0$. According to Theorem 2.4, the functional $J_\lambda(u)$ possesses infinitely many critical points, i.e., the system (1.1) has infinitely many weak solutions, and they are classical solutions. □

The following example illustrates Theorem 3.2.

Example 3.6. Consider the system

$$\begin{aligned}
& -u_1''(x) = \lambda F_{u_1}(x,u_1,u_2,u_3)h_1(x,u_1'), \quad x \in (0,1) \\
& -u_2''(x) = \lambda F_{u_2}(x,u_1,u_2,u_3)h_2(x,u_2'), \quad x \in (0,1) \\
& -2|u_3'(x)|u_3''(x) = \lambda F_{u_3}(x,u_1,u_2,u_3)h(x,u_3'), \;\; x \in (0,1) \\
& u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0, \\
& u_3(0) = u_3(1) = 0
\end{aligned} \tag{3.3}$$

where $F(t_1,t_2,t_3) = e^x(t_1^{11} + t_2^{11} + t_3^{11})$ for $t_1,t_2,t_3 \in \mathbb{R}$, $h_1(x,t_1) = x^2 + 1 + \sin^4 t_1$, $h_2(x,t_2) = 2x^2 + 1 + \cos^4 t_2$ and $h_3(x,t_3) = 2\cos^2 t_3 + 1$ for all $x \in [0,1]$, $t_1,t_2,t_3 \in \mathbb{R}$. We see that since where $p_1 = p_2 = 2$ and $p_3 = 3$ then $\underline{p} = 2$ and $\overline{p} = 3$. Also we observe that $m_1 = m_2 = m_3 = 1$, $M_1 = 3$, $M_2 = 4$ and $M_3 = 2$ then $\underline{M} = 1$ and $\overline{M} = 4$. Moreover, $F(x,t_1,t_2,t_3)$ is odd about for all $t = (t_1,t_2,t_3) \in \mathbb{R}^3$. By choosing $\nu = 11$, that $\nu > \frac{\overline{p}\overline{M}}{\underline{M}}$ we have $11F(x,t) \leq \sum_{i=1}^3 t_i F_{t_i}(x,t_1,t_2)$, so we see that condition (F_0) is satisfied. Also, the functions F, F_{t_1}, F_{t_2}, h_1, h_2 and h_3 are continuous functions, hence, by applying Theorem 3.2 the system (3.3) has many infinitely nontrivial classical solutions.

As a consequence of Theorem (3.2), we can show the following result.

Corollary 3.7. Suppose that the assumption (A_0) and (A_2) hold. Then,

If $F(x,t)$ about t is odd, the system (1.1) has infinitely many classical solutions.

As an application of the results, we let the problem

$$\begin{gathered}
-(p-1)|u_i'(x)|^{p-2}u_i''(x) = \lambda\beta(x)f(u)h(u'), \;\; x \in (a,b) \\
u(a) = u(b) = 0,
\end{gathered} \tag{3.4}$$

where $p > 1$, $\lambda > 0$, $\beta \in L^1([a,b])$ such that $\beta(x) \geq 0$ a.e. $x \in [a,b]$, $\beta \not\equiv 0$, $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function and $h : \mathbb{R} \to]0,+\infty[$ is a bounded and continuous function with $m := \inf_{t\in\mathbb{R}} h(t) > 0$ and $M := \sup_{t\in\mathbb{R}} h(t)$. We introduce the functions $F : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(t) = \int_0^t f(\xi)d\xi \;\;\; for \;\; all \;\; t \in \mathbb{R} \tag{3.5}$$

and

$$H(t) = \int_0^t \Big(\int_0^\tau \frac{(p_i-1)|\delta|^{p_i-2}}{h(\delta)} d\delta\Big) d\tau$$

for $t \in \mathbb{R}$.

Now, as consequences of Theorem 3.1 and Theorem 3.2, we can show the following theorems, respectively.

Theorem 3.8. Suppose that the following conditions are satisfied:

(f_0) there exist $R' > 0$ and a constant $\nu' > \frac{pM}{m}$ such that

$$0 < \nu' F(t) \leq t f(t), \;\; |t| > R'$$

(f_1) $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathèodory condition and

$$|f(t)| \leq c(1 + |t|^{q-1}) \;\;\; for \;\;\; t \in \mathbb{R},$$

where $q > p$,

(f_2) $f(t) = o(|t|^{p-1})$, $t \longrightarrow 0$, uniformly.

Then, if $f(t) \geq 0$ for all $t \in \mathbb{R}$, the problem (3.4) has at least two classical solutions.

Theorem 3.9. Assume that the assumption (f_0) holds. Then, if $f(t)$ is odd about t, the problem (3.4) has infinitely many classical solutions.

We obtain the following existence results as a consequence of Theorems 3.8 and 3.9, respectively .

Corollary 3.10. Suppose that the following conditions are satisfied:

(B_0) there exist $S, \tau > 0$ and a constant $\omega > \frac{pM}{m}$ such that

$$0 < \omega F(t) - tf(t) \leq |t|^{\tau}, \;\; |t| > S.$$

where $\tau \leq p$;

(B_1) there exist constant $L > 0$ and $\eta > p$, such that

$$f(t) \leq L|t|^{\eta}$$

when $\|t\| \to 0$;

(B_2) there exist constants $K_1, R_1 > 0$ and $\vartheta > p$, such that

$$f(t) \geq K_1|t|^{\vartheta}$$

when $\|t\| > R_1$.

If $f(t) > 0$, then the problem (3.4) has two classical solutions.

Corollary 3.11. Suppose that the assumption (B_0) and (B_2) hold. Then

If $f(t)$ about t is odd, the problem (3.4) has infinitely many classical solutions.

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