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# Geodesic exponentially preinvex functions on Riemannian manifolds

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**Abstract**

In this paper, new concepts of geodesic exponentially preinvex and invex functions are introduced on Riemannian manifolds. In addition, some properties of aforesaid functions are investigated. Further, the classes of generalized geodesic exponentially preinvex and invex functions are introduced. Then, the optimality results are proved for an optimization problem in which the functions involved are geodesic exponentially invex.

Keywords: Riemannian manifold, geodesic invex set, geodesic preinvex function, geodesic invex function, geodesic generalized invex function, epigraph
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## 1 Introduction

Convex optimization has become increasingly influential across various domains, including mathematics, practical applications, and applied sciences. Within this field, invexity emerged as a generalization of convexity, initially proposed by Hanson [8]. Since then, numerous generalized concepts of convexity have been introduced. Jayakumar and Mond [11] addressed a novel class of multi-objective problems, providing substantial insights into optimality and duality in the scalar case. Additionally, Nie et al. [16] derived several inequalities pertaining to differentiable exponentially convex and exponentially quasiconvex mapping. Abdulaleem [1, 2, 3, 4, 5] focused on $E$-differentiable multiobjective programming problems, introducing innovative notions such as $E$-invexity, $V$-$E$-invexity, and $E$-$B$-invexity, and establishing the necessary and sufficient optimality conditions for such problems.

Rapcsák [19] and Udrişte [23] introduced the concept of geodesic convexity, including optimality conditions and its extension to complementarity problems, while Pini [18] investigated properties of invex functions on Riemannian manifolds and Mititelu [15] explored its generalization. Noor and Noor [17] proposed exponentially preinvex functions, and Ahmad et al. [6] introduced log-preinvex and log-invex functions on a Riemannian manifold. Iqbal and Ahmad [9] established the notion of strong geodesic convex function of order $m$ on Riemannian manifolds. Jost [12] expanded on the introduction of basic geometric concepts like geodesics and delved into important topics in analysis, connections and curvature, submanifolds, Kähler manifolds, and Floer homology. Kiliçman and Saleh [13] focused on geodesic strongly $E$-convexity and its associated properties in the realm of Riemannian geometry, whereas Kumari and Jayswal [14] investigated properties of geodesic $E$-preinvex functions. Iqbal et al. [10] generalized the notion of $\lambda$-radial

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contraction in a complete Riemannian manifold and developed the concept of $p^{\lambda}$-convex functions, Shaikh et al. [21], [22] explored properties of geodesic $\varphi$-convex functions, Saleh [20] presented a Hadamard-type inequality for a specific class of geodesic convex functions, and later, Upadhyay et al. [24, 25, 26, 27] conducted a comprehensive study on generalized geodesic convexity on Riemannian manifolds.

In this paper, we introduce new classes of geodesic exponentially preinvex functions and geodesic exponentially invex functions on Riemannian manifolds, extending the existing concepts in optimization theory. Additionally, we propose the concepts of geodesic exponentially quasi-preinvex functions and strictly geodesic exponentially quasi-preinvex functions, providing further insights into the properties of these functions. Moreover, we present a novel concept of generalized geodesic convexity on Riemannian manifolds, specifically focusing on the differentiable geodesic exponentially invexity notion. Also, we establish optimality conditions for mathematical optimization problems under appropriate geodesic exponentially invexity assumptions on Riemannian manifolds.

## 2 Preliminaries

Assume that $N$ is an $n$-dimensional Riemannian manifold with a Riemannian metric $\langle .,. \rangle$ on the tangent space $T_pN$ of $N$ at $p \in N$, with the corresponding norm denoted by $\|\cdot\|_p$. Assume that $TN = \bigcup_{p\in N} T_pN$ is a tangent bundle of $N$ and $p, q \in N$ are two points in Riemannian manifold, if $\gamma$: $[x_1, x_2] \to N$ is a piecewise smooth curve joining $\gamma(x_1) = p$ to $\gamma(x_2) = q$, then geodesic length $l(\gamma)$ is defined by

$$l(\gamma) = \int_{x_1}^{x_2} \|\gamma'(s)\| \, ds.$$

The following formula defines the Riemannian distance $d(p, q)$ between the points $p$ and $q$:

$$d(p, q) = \inf \{l(\gamma)\colon \gamma \text{ is a piecewise smooth curve joins the point } p \text{ to } q\}.$$

Let $\nabla$ be the Levi-Civita connection associated to $(N, \langle\cdot,\cdot\rangle)$. Let $\gamma$ be a smooth curve in $N$. A vector field $X$ is said to be parallel along $\gamma$ if $\nabla_{\gamma'} X = 0$. If $\gamma'$ itself is parallel along $\gamma$, we say that $\gamma$ is a geodesic. A geodesic joining $p$ to $q$ in $N$ is said to be minimal if its length equals $d(p, q)$. By the Hopf-Rinow Theorem, we know that, if $N$ is complete, then any points in $N$ can be joined by a minimal geodesic. Moreover, $(N, d)$ is a complete metric space and bounded closed subsets are compact.

Assuming that $N$ is complete. The exponential map $\exp_p : T_pN \to N$ at $p$ is defined by $\exp_p v = \gamma_v(1, p)$ for each $v \in T_pN$, where $\gamma(\cdot) = \gamma_v(\cdot, p)$ is the geodesic starting at $p$ with velocity $v$, that is $\gamma(0) = p$ and $\gamma'(0) = v$. It is easy to see that $\exp_p(sv) = \gamma_v(s, p)$ for each real number $s$. Note that the map $\exp_p$ is differentiable on $T_pN$ for any $p \in N$.

**Definition 2.1.** [23] A set $K \subset N$ is said to be geodesic convex, if it contains each geodesic $\gamma_{p,q}$ of $N$, whose end points $p$ and $q$ belong to $K$.

**Definition 2.2.** [23] Let a set $K \subset N$ be geodesic convex. A function $h$: $K \to \mathbb{R}$ is said to be (strictly) geodesic convex iff the following inequality

$$h(\gamma_{p,q}(s)) \leq sh(p) + (1 - s)h(q) \qquad (<) \tag{2.1}$$

holds for each $p, q \in K (p \neq q)$, $s \in [0, 1]$.

**Definition 2.3.** [7] A set $K \subset N$ is said to be geodesic invex with respect to (w.r.t.) $\eta$: $N \times N \to TN$ if, there is exactly one geodesic such that $\gamma_{p,q}$: $[0, 1] \to N$ such that the following relations

$$\gamma_{p,q}(0) = q,\ \gamma'_{p,q}(0) = \eta(p, q) \in TN,\ \gamma_{p,q}(s) \in K$$

hold for each $p, q \in K$, $s \in [0, 1]$.

**Definition 2.4.** [7] Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A function $h$: $K \to \mathbb{R}$ is said to be (strictly) geodesic preinvex w.r.t. $\eta$ iff the following inequality

$$h(\gamma_{p,q}(s)) \leq sh(p) + (1 - s)h(q) \qquad (<) \tag{2.2}$$

holds for each $p, q \in K (p \neq q)$, $s \in [0, 1]$ and $\gamma_{p,q}$ is the unique geodesic.

**Remark 2.5.** From Definition 2.4, if $K \subset N$ and $\eta(\cdot, \cdot)$: $N \times N \to TN$ is defined by $\eta(p, q) = p - q$, then we obtain the definition of (strictly) geodesic convex function.

## 3 Geodesic exponentially preinvex function

Now, we introduce a new concept of geodesic exponentially preinvex functions.

**Definition 3.1.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A function $h$: $K \to \mathbb{R}$ is said to be (strictly) geodesic exponentially preinvex w.r.t. $\eta$ iff the following inequality

$$e^{h(\gamma_{p,q}(s))} \leq se^{h(p)} + (1-s)e^{h(q)} \qquad (<) \tag{3.1}$$

holds for each $p, q \in K(p \neq q)$, $s \in [0,1]$ and $\gamma_{p,q}$ is the unique geodesic.

In other words, (3.1) is equivalent to the fact that the following inequality

$$h(\gamma_{p,q}(s)) \leq \log\left[se^{h(p)} + (1-s)e^{h(q)}\right] \qquad (<) \tag{3.2}$$

holds for each $p, q \in K(p \neq q)$, $s \in [0,1]$.

**Definition 3.2.** A function $h$: $K \to \mathbb{R}$ is said to be geodesic exponentially quasi-preinvex w.r.t. $\eta$: $N \times N \to TN$ iff the following inequality

$$e^{h(\gamma_{p,q}(s))} \leq \max\left\{e^{h(p)}, e^{h(q)}\right\} \tag{3.3}$$

holds for each $p, q \in K$, $s \in [0,1]$.

**Definition 3.3.** A function $h$: $K \to \mathbb{R}$ is said to be strictly geodesic exponentially quasi-preinvex w.r.t. $\eta$: $N \times N \to TN$ iff the following inequality

$$e^{h(\gamma_{p,q}(s))} < \max\left\{e^{h(p)}, e^{h(q)}\right\} \tag{3.4}$$

holds for each $p, q \in K$, $s \in [0,1]$.

Every geodesic exponentially preinvex function is geodesic exponentially quasi-preinvex, but the converse is not true.

**Condition (C).** Let $K \subset N$ and $\eta(\cdot,\cdot)$: $N \times N \to TN$ satisfy the following assumptions

$$\begin{aligned} \eta\left(q, \gamma_{p,q}(s)\right) &= -s\eta\left(p, q\right), \\ \eta\left(p, \gamma_{p,q}(s)\right) &= (1-s)\eta\left(p, q\right) \end{aligned} \tag{3.5}$$

for each $p, q \in K$, $s \in [0,1]$.

**Example 3.4.** Let $h$: $\mathbb{R} \to \mathbb{R}$ be defined by $h(p) = \begin{cases} -|p| & \text{if } |p| \leq 1, \\ -1 & \text{if } |p| \geq 1. \end{cases}$, the geodesic $\gamma$ and $\eta$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\gamma_{p,q}(s) = \begin{cases} sp + (1-s)q & \text{if } p \geq 0,\ q \geq 0, \\ sp + (1-s)q & \text{if } p \leq 0,\ q \leq 0, \\ q + s(1-q) & \text{if } p \leq 0,\ q \geq 0, \\ q + s(-1-q) & \text{if } p \geq 0,\ q \leq 0. \end{cases},$$

$$\eta(p,q) = \begin{cases} p - q & \text{if } p \geq 0,\ q \geq 0, \\ p - q & \text{if } p \leq 0,\ q \leq 0, \\ 1 - q & \text{if } p \leq 0,\ q \geq 0, \\ -1 - q & \text{if } p \geq 0,\ q \leq 0. \end{cases}.$$

Note that this example shows that a geodesic exponentially quasi-preinvex function $h$ w.r.t. $\eta$ given above satisfying Condition $C$ is not necessarily a geodesic quasi-convex function.

Now, we present an example of a geodesic exponentially quasi-preinvex function that does not satisfy the conditions of being a geodesic exponentially preinvex function.

**Example 3.5.** Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(p) = \begin{cases} -p & \text{if} \quad p > 0, \\ 0 & \text{if} \quad p \leq 0. \end{cases}$, the geodesic $\gamma$ and $\eta\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\gamma_{p,q}(s) = \begin{cases} sp + (1-s)q & \text{if} \quad p \geq 0,\ q \geq 0, \\ sp + (1-s)q & \text{if} \quad p \leq 0,\ q \leq 0, \\ (1+s)q - sp & \text{if} \quad p \leq 0,\ q \geq 0, \\ (1+s)q - sp & \text{if} \quad p \geq 0,\ q \leq 0. \end{cases},$$

$$\eta(p,q) = \begin{cases} p - q & \text{if} \quad p \geq 0,\ q \geq 0, \\ p - q & \text{if} \quad p \leq 0,\ q \leq 0, \\ q - p & \text{if} \quad p \leq 0,\ q \geq 0, \\ q - p & \text{if} \quad p \geq 0,\ q \leq 0. \end{cases}.$$

Then, $h$ is geodesic exponentially quasi-preinvex function, but $h$ is not geodesic exponentially preinvex w.r.t. $\eta$ defined above as can be seen by taking $p = 1,\ q = -1,\ s = \frac{1}{2}$, since the inequality

$$e^{h(\gamma_{p,q}(s))} = 1 > se^{h(p)} + (1-s)e^{h(q)} \quad \approx 0.8678 \tag{3.6}$$

holds. Hence, by the definition of a geodesic exponentially preinvex function, it follows that $h$ is not geodesic exponentially preinvex function w.r.t. $\eta$ given above.

Now, we present an example of such a geodesic strictly exponentially quasi-preinvex function which is not geodesic strictly quasi-convex.

**Example 3.6.** Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(p) = -|p|$, the geodesic $\gamma$ and $\eta\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\gamma_{p,q}(s) = \begin{cases} sp + (1-s)q & \text{if} \quad p \geq 0,\ q \geq 0, \\ sp + (1-s)q & \text{if} \quad p \leq 0,\ q \leq 0, \\ (1+s)q - sp & \text{if} \quad p \leq 0,\ q \geq 0, \\ (1+s)q - sp & \text{if} \quad p \geq 0,\ q \leq 0. \end{cases},$$

$$\eta(p,q) = \begin{cases} p - q & \text{if} \quad p \geq 0,\ q \geq 0, \\ p - q & \text{if} \quad p \leq 0,\ q \leq 0, \\ q - p & \text{if} \quad p \leq 0,\ q \geq 0, \\ q - p & \text{if} \quad p \geq 0,\ q \leq 0. \end{cases}.$$

Then, $h$ is geodesic strictly exponentially quasi-preinvex function, but $h$ is not strictly geodesic quasi-convex on $R$.

**Theorem 3.7.** Let $K \subset N$ be a geodesic invex set w.r.t. $\eta\colon N \times N \to TN$. If the function $h\colon K \to \mathbb{R}$ is geodesic exponentially preinvex on $K$, then the level set $L(\alpha) = \{q \in K :\ e^{h(q)} \leq \alpha\}$ is geodesic invex for all $\alpha \in R$.

**Proof .** Let $\alpha \in R$, $p, q \in L(\alpha)$ and $s \in [0,1]$, such that $e^{h(q)} \leq \alpha$ and $e^{h(p)} \leq \alpha$. Since a set $K$ is geodesic invex, there is exactly one geodesic $\gamma_{p,q}\colon [0,1] \to N$ such that $\gamma_{p,q}(0) = q,\ \gamma'_{p,q}(0) = \eta(p,q) \in TN,\ \gamma_{p,q}(s) \in K$. By geodesic exponentially preinvex of $h$, we have

$$\begin{aligned} e^{h(\gamma_{p,q}(s))} &\leq se^{h(p)} + (1-s)\, e^{h(q)} \\ &\leq s\alpha + (1-s)\, \alpha = \alpha \end{aligned} \tag{3.7}$$

it follows that $\gamma_{p,q}(s)) \in L(\alpha)$. Hence $L(\alpha)$ is a geodesic invex set. □

**Theorem 3.8.** Let $K \subset N$ be a geodesic invex set w.r.t. $\eta\colon N \times N \to TN$. A function $h\colon K \to \mathbb{R}$ is geodesic exponentially preinvex on $K$, iff $\operatorname{epi}(h) = \{(q,\alpha) \in K \times \mathbb{R} :\ e^{h(q)} \leq \alpha\}$ is a geodesic invex set.

**Proof .** Let $(p,\beta), (q,\alpha) \in \operatorname{epi}(h)$ and a function $h\colon K \to \mathbb{R}$ be geodesic exponentially preinvex on $K$. Since a set $K$ is geodesic invex, there is exactly one geodesic $\gamma_{p,q}\colon [0,1] \to N$, such that $\gamma_{p,q}(0) = q,\ \gamma'_{p,q}(0) = \eta(p,q) \in TN,\ \gamma_{p,q}(s) \in K$, by the definition of geodesic exponentially preinvex function, we have

$$\begin{aligned} e^{h(\gamma_{p,q}(s))} &\leq se^{h(p)} + (1-s)\, e^{h(q)} \\ &\leq s\beta + (1-s)\, \alpha, \end{aligned} \tag{3.8}$$

implies that $(\gamma_{p,q}(s), \alpha + s(\beta - \alpha)) \in \mathrm{epi}(h)$. Thus, $\mathrm{epi}(h)$ is a geodesic invex set. Conversely, let $\mathrm{epi}(h)$ be a geodesic invex set. Let $p, q \in K$, and $(p, \beta), (q, \alpha) \in \mathrm{epi}(h)$, since $\mathrm{epi}(h)$ is a geodesic invex set, then we have

$$(\gamma_{p,q}(s), se^{h(p)} + (1-s)e^{h(q)}) \in \mathrm{epi}(h), \tag{3.9}$$

which implies that

$$e^{h(\gamma_{p,q}(s))} \leq se^{h(p)} + (1-s)\, e^{h(q)}. \tag{3.10}$$

This shows that $h$ is geodesic exponentially preinvex on $K$. □

**Theorem 3.9.** Let $K \subset N$ be a geodesic invex set w.r.t. $\eta$: $N \times N \to TN$. A function $h$: $K \to \mathbb{R}$ is said to be geodesic exponentially quasi-preinvex on $K$, if and only if the level set $L(\alpha) = \{q \in K : \ e^{h(q)} \leq \alpha\}$ is a geodesic invex set for each real number $\alpha$.

**Proof .** Assume that $h$ is geodesic exponentially quasi-preinvex on $K$, and let $p, q \in L(\alpha)$. Therefore, $p, q \in K$ and $\max\{e^{h(p)}, e^{h(q)}\} \leq \alpha$. Let $\widetilde{p} = \gamma_{p,q}(s)$. By the geodesic invexity of set $K$, $\widetilde{p} \in K$. Furthermore, by the geodesic exponentially quasi-preinvex on $K$,

$$e^{h(\widetilde{p})} \leq \max\{e^{h(p)}, e^{h(q)}\} \leq \alpha. \tag{3.11}$$

Hence, $\widetilde{p} \in L(\alpha)$ and thus $L(\alpha)$ is geodesic invex. Conversely, suppose that $L(\alpha)$, is geodesic invex for each real number $\alpha$. Let $p, q \in K$. Furthermore, let $\widetilde{p} = \gamma_{p,q}(s)$, $s \in (0, 1)$. Note that $p, q \in L(\alpha)$ for $\max\{e^{h(p)}, e^{h(q)}\} = \alpha$. By assumption, $L(\alpha)$ is geodesic invex, so that $\widetilde{p} \in L(\alpha)$. Therefore,

$$e^{h(\widetilde{p})} \leq \alpha = \max\{e^{h(p)}, e^{h(q)}\}. \tag{3.12}$$

Hence, $h$ is geodesic exponentially quasi-preinvex on $K$, and the proof of this theorem is complete. □

**Theorem 3.10.** Let $K \subset N$ be a geodesic invex set w.r.t. $\eta$: $N \times N \to TN$. If a function $h$: $K \to \mathbb{R}$ is geodesic exponentially preinvex on $K$ such that $e^{h(p)} < e^{h(q)}$ for each $p, q \in K$. Then, $h$ is strictly geodesic exponentially preinvex on $K$.

**Proof .** Let $h$ be a geodesic exponentially preinvex function, $p, q \in K$, $s \in [0, 1]$. Then, we have

$$e^{h(\gamma_{p,q}(s))} \leq se^{h(p)} + (1-s)\, e^{h(q)} < e^{h(q)} \tag{3.13}$$

since $e^{h(p)} < e^{h(q)}$ which shows that a function $h$ is strictly geodesic exponentially preinvex. □

**Theorem 3.11.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$ and $\sup_{j\in J} h_j(p)$, $p \in K$ exists in $\mathbb{R}$. If each function $h_j$: $K \to \mathbb{R}$, $j \in J$ is geodesic exponentially preinvex on $K$, then $h$: $K \to \mathbb{R}$ defined by $e^{h(p)} = \sup_{j\in J} e^{h_j(p)}$, for each $p \in K$ is a geodesic exponentially preinvex function on $K$.

**Proof .** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$, then for each $p, q \in K$ there is exactly one geodesic $\gamma_{p,q}(0) = q$, $\gamma'_{p,q}(0) = \eta(p, q)$, $\gamma_{p,q}(s) \in K$, for any $s \in [0, 1]$. By Definition 3.1, the geodesic exponentially preinvex function of $h_j, j \in J$, we obtain

$$e^{h_j(\gamma_{p,q}(s))} \leq se^{h_j(p)} + (1-s)e^{h_j(q)}. \tag{3.14}$$

Then

$$\begin{aligned}
\sup_{j\in J} e^{h_j(\gamma_{p,q}(s))} &\leq \sup_{j\in J}\left(se^{h_j(p)} + (1-s)e^{h_j(q)}\right) \\
&\leq s\left(\sup_{j\in J} e^{h_j(p)}\right) + (1-s)\left(\sup_{j\in J} e^{h_j(q)}\right) \\
&= se^{h(p)} + (1-s)e^{h(q)}.
\end{aligned}$$

Thus,

$$e^{h(\gamma_{p,q}(s))} \leq se^{h(p)} + (1-s)e^{h(q)}. \tag{3.15}$$

Hence, a function $h$ is geodesic exponentially preinvex on $K$. □

Now, we introduce a new concept of generalized geodesic convexity on Riemannian manifolds.

**Definition 3.12.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A differentiable function $h$: $K \to \mathbb{R}$ is said to be geodesic exponentially invex on $K$, if the following inequality

$$e^{h(p)} - e^{h(q)} \geq e^{h(q)} dh_q \eta(p,q) \tag{3.16}$$

holds for each $p$, $q \in N$.

**Definition 3.13.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A differentiable function $h$: $K \to \mathbb{R}$ is said to be strictly geodesic exponentially invex on $K$, if the following inequality

$$e^{h(p)} - e^{h(q)} > e^{h(q)} dh_q \eta(p,q) \tag{3.17}$$

holds for each $p$, $q \in N, p \neq q$.

Now, we introduce various classes of generalized geodesic exponentially invex functions.

**Definition 3.14.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A differentiable function $h$: $K \to \mathbb{R}$ is said to be geodesic exponentially quasi-invex at $q$ on $K$ if the following relation

$$e^{h(p)} \leq e^{h(q)} \Rightarrow e^{h(q)} dh_q \eta(p,q) \leq 0 \tag{3.18}$$

holds for each $p \in K$. If (3.18) is satisfied for each $q \in K$, then $h$ is said to be a geodesic exponentially quasi-invex function on $K$.

**Definition 3.15.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A differentiable function $h$: $K \to \mathbb{R}$ is said to be geodesic exponentially pseudo-invex at $q$ on $K$ if the following relation

$$e^{h(p)} < e^{h(q)} \Rightarrow e^{h(q)} dh_q \eta(p,q) < 0 \tag{3.19}$$

holds for each $p \in K$. If (3.19) is satisfied for each $q \in K$, then $h$ is said to be a geodesic exponentially pseudo-invex function on $K$.

**Definition 3.16.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A differentiable function $h$: $K \to \mathbb{R}$ is said to be strictly geodesic exponentially pseudo-invex at $q$ on $K$ if the following relation

$$e^{h(p)} \leq e^{h(q)} \Rightarrow e^{h(q)} dh_q \eta(p,q) < 0 \tag{3.20}$$

holds for each $p \in K$, $p \neq q$. If (3.20) is satisfied for each $q \in K$, $p \neq q$, then $h$ is said to be a strictly geodesic exponentially pseudo-invex function on $K$.

**Theorem 3.17.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. If a differentiable function $h$: $K \to \mathbb{R}$ is geodesic exponentially preinvex on $K$, then a function $h$ is geodesic exponentially invex on $K$.

**Proof .** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$, there is exactly one geodesic $\gamma_{p,q}$: $[0,1] \to N$ such that the following relations

$$\gamma_{p,q}(0) = q,\ \gamma'_{p,q}(0) = \eta(p,q),\ \gamma_{p,q}(s) \in K \tag{3.21}$$

hold for each $p, q \in K$, $s \in [0,1]$. Since $h$ is a geodesic exponentially preinvex function, by Definition 3.1, the following inequalities

$$e^{h(\gamma_{p,q}(s))} \leq s e^{h(p)} + (1-s) e^{h(q)} \tag{3.22}$$

holds for each $p, q \in K$ and $s \in [0,1]$. Thus, the above inequality yields,

$$\frac{e^{h(\gamma_{p,q}(s))} - e^{h(q)}}{s} \leq e^{h(p)} - e^{h(q)}. \tag{3.23}$$

Letting $s \to 0$, we get

$$e^{h(\gamma_{p,q}(0))} dh_{\gamma_{p,q}(0)} \gamma'_{p,q}(0) \leq e^{h(p)} - e^{h(q)} \tag{3.24}$$

that is,

$$e^{h(p)} - e^{h(q)} \geq e^{h(q)} dh_q \eta(p,q). \tag{3.25}$$

Thus, a function $h$ is geodesic exponentially invex on $K$. □

**Theorem 3.18.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$. A differentiable function $h$: $K \to \mathbb{R}$ satisfies the Condition $(C)$. Then a function $h$ is geodesic exponentially preinvex on $K$ if $h$ is a geodesic exponentially invex function on $K$.

**Proof .** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$, there is exactly one geodesic such that the following relations

$$\gamma_{p,q}(0) = q,\ \gamma'_{p,q}(0) = \eta(p,q) \in TN,\ \gamma_{p,q}(s) \in K \tag{3.26}$$

hold for each $p, q \in K$, $s \in [0,1]$. Now, let $\widehat{p} = \gamma_{p,q}(s)$, $s \in (0,1)$, then by Definition 3.12, the following inequalities

$$e^{h(p)} - e^{h(\widehat{p})} \geq e^{h(\widehat{p})} dh_{\widehat{p}}\eta(p,\widehat{p}) \tag{3.27}$$

$$e^{h(q)} - e^{h(\widehat{p})} \geq e^{h(\widehat{p})} dh_{\widehat{p}}\eta(q,\widehat{p}) \tag{3.28}$$

hold. Multiplying (3.27) by $s$ and (3.28) by $(1-s)$, and then adding them together, we get

$$se^{h(p)} + (1-s)e^{h(q)} - e^{h(\widehat{p})} \geq e^{h(\widehat{p})} dh_{\widehat{p}}\left(s\eta(p,\widehat{p}) + (1-s)\eta(q,\widehat{p})\right). \tag{3.29}$$

By using the Condition (C), $\widehat{p} = \gamma_{p,q}(s)$, we get

$$s\eta(p,\widehat{p}) + (1-s)\eta(q,\widehat{p}) = s(1-s)\eta(p,q) - s(1-s)\eta(p,q) = 0. \tag{3.30}$$

By (3.30) and (3.29), we get

$$e^{h(\widehat{p})} \leq se^{h(p)} + (1-s)e^{h(q)}. \tag{3.31}$$

Thus,

$$e^{h(\gamma_{p,q}(s))} \leq se^{h(p)} + (1-s)e^{h(q)}, \tag{3.32}$$

which show $h$ is geodesic exponentially preinvex function on $K$. □

## 4 Applications

Now, we consider the following mathematical optimization problem (MOP):

$$\begin{aligned} &\text{minimize } h(p) \\ &\text{subject to } g_j(p) \leq 0,\ \ j \in J = \{1, ..., m\}, \end{aligned} \qquad \text{(MOP)}$$

where $h : K \to R$ and $g_j : K \to R$, $j \in J$ are differentiable functions. Let

$$\Omega := \{p \in K : g_j(p) \leq 0,\ j \in J\}$$

be the set of all feasible solutions of (MOP). For a given $\overline{p} \in \Omega$, denote $J(\overline{p}) = \{j \in J : g_j(\overline{p}) = 0\}$ the index set of all active constraints at $\overline{p}$.

**Definition 4.1.** A feasible point $\overline{p} \in \Omega$ is called an optimal solution of (MOP) iff there is no other feasible solution $p \in \Omega$ such that

$$h(p) \leq h(\overline{p}).$$

**Theorem 4.2.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$: $N \times N \to TN$ and $\overline{p} \in \Omega$ be a feasible solution of the problem (MOP). If $h$ and $\overline{\mu}g_j, j \in J$ are geodesic exponentially invex functions on $K$ and there exists multiplier $\overline{\mu} \in R^m$ such that $(\overline{p}, \overline{\mu})$ satisfies the following conditions:

$$dh_{\overline{p}}\eta(p,\overline{p}) + \sum_{j=1}^{m} \overline{\mu}_j dg_{j\overline{p}}\eta(p,\overline{p}) = 0, \tag{4.1}$$

$$g_j(p) \leq g_j(\overline{p}),\ \ j \in J,\ \forall p \in \Omega, \tag{4.2}$$

$$\overline{\mu}_j g_j(\overline{p}) = 0,\ \ j \in J, \tag{4.3}$$

$$\overline{\mu} \geq 0. \tag{4.4}$$

then $\overline{p}$ is an optimal solution of the problem (MOP).

**Proof .** Suppose, contrary to the result, that $\overline{p}$ is not an optimal solution of the problem (MOP). Hence, by Definition 4.1, there exists another $p \in \Omega$ such that

$$h(p) \leq h\left(\overline{p}\right). \tag{4.5}$$

Since a function $h$ is differentiable and geodesic exponentially invex, by Definition 3.12, the following inequality

$$e^{h(p)} - e^{h(\overline{p})} \geq e^{h(\overline{p})} dh_{\overline{p}}(\eta(p,\overline{p})) \tag{4.6}$$

holds for each $p, \overline{p} \in K$. Combining (4.5)-(4.6), we get

$$dh_{\overline{p}}(\eta(p,\overline{p})) \leq 0. \tag{4.7}$$

Since each function $g_j,\ j \in J$ is differentiable and geodesic exponentially invex, by Definition 3.12, the following inequality

$$e^{g_j(p)} - e^{g_j(\overline{p})} \geq e^{g_j(\overline{p})} dg_{j\overline{p}}(\eta(p,\overline{p})), \quad j \in J(\overline{p}) \tag{4.8}$$

holds. Multiplying above inequality by $\overline{\mu}_j \geq 0,\ j \in J(\overline{p})$, we obtain

$$\overline{\mu}_j e^{g_j(p)} - \overline{\mu}_j e^{g_j(\overline{p})} \geq \overline{\mu}_j e^{g_j(\overline{p})} dg_{j\overline{p}}(\eta(p,\overline{p})), \quad j \in J(\overline{p}). \tag{4.9}$$

Using conditions (4.2)-(4.4), $p, \overline{p} \in \Omega$, we get

$$\overline{\mu}_j dg_{j\overline{p}}(\eta(p,\overline{p})) \leq 0, \quad j \in J(\overline{p}). \tag{4.10}$$

Combining (4.7) and (4.10), we obtain

$$dh_{\overline{p}}(\eta(p,\overline{p})) + \sum_{j=1}^{m} \overline{\mu}_j dg_{j_{\overline{p}}}(\eta(p,\overline{p})) \leq 0 \tag{4.11}$$

which contradicts (4.1). Hence, the proof of this theorem is completed. □

**Theorem 4.3.** Let a set $K \subset N$ be geodesic invex w.r.t. $\eta$ and $\overline{p} \in \Omega$ be a feasible solution of the problem (MOP). If a function $h$ is strictly geodesic exponentially pseudo-invex on $K$, each function $\overline{\mu}_j g_j, j \in J$ is geodesic exponentially quasi-invex w.r.t. $\eta$ on $K$ and there exists multiplier $\overline{\mu} \in R^m$ such that $(\overline{p}, \overline{\mu})$ satisfies the conditions (4.1)-(4.4), then $\overline{p}$ is an optimal solution of the problem (MOP).

**Proof .** Suppose, contrary to the result, that $\overline{p}$ is not an optimal solution of the problem (MOP). Hence, by Definition 4.1, there exists another $p \in \Omega$ such that

$$h(p) \leq h\left(\overline{p}\right). \tag{4.12}$$

Thus,

$$e^{h(p)} \leq e^{h(\overline{p})}. \tag{4.13}$$

Since a function $h$ is strictly geodesic exponentially pseudo-invex w.r.t. $\eta$, by Definition 3.16 and $\overline{p} \in \Omega$, the inequality

$$dh_{\overline{p}}\eta(p,\overline{p}) < 0 \tag{4.14}$$

holds. Since $p \in \Omega$, therefore, the conditions (4.2) and (4.4) imply

$$\sum_{j=1}^{m} \overline{\mu}_j e^{g_j(p)} \leq \sum_{j=1}^{m} \overline{\mu}_j e^{g_j(\overline{p})}. \tag{4.15}$$

Since $\overline{\mu}_j g_j, j \in J$ is geodesic exponentially quasi-invex w.r.t. $\eta$, by Definition 3.14, $\overline{p} \in \Omega$ and $\overline{\mu}_j \geq 0$, we get

$$\sum_{j=1}^{m} \overline{\mu}_j dg_{j_{\overline{p}}}\eta(p,\overline{p}) \leq 0. \tag{4.16}$$

Combining (4.14) and (4.16), we obtain that the inequality

$$dh_{\overline{p}}(\eta(p,\overline{p})) + \sum_{j=1}^{m} \overline{\mu}_j dg_{j_{\overline{p}}}(\eta(p,\overline{p})) < 0 \tag{4.17}$$

which contradicts (4.1). Hence, the proof of this theorem is completed. □

## 5 Concluding remarks

This paper has introduced new classes of geodesic exponentially preinvex functions and geodesic exponentially invex functions on Riemannian manifolds. Additionally, the concepts of geodesic exponentially quasi-preinvex and strictly geodesic exponentially quasi-preinvex functions have been presented. Furthermore, a novel concept of generalized geodesic convexity has been introduced, specifically focusing on the differentiable geodesic exponentially invexity notion on Riemannian manifolds. Moreover, the paper has derived optimality conditions for mathematical optimization problems under appropriate geodesic exponentially invexity hypotheses. These findings contribute to a deeper understanding and analysis of optimization problems in the context of Riemannian manifolds. In addition, the presented results were enriched by providing appropriate examples that illustrate the defined concepts of preinvex (generalized preinvex) exponential geodesic functions on Riemannian manifolds.

## Data Availability

No data were used to support this study.

## Competing interests

The author declares to have no competing interests.

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