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Subclasses of bi-univalent functions connected to the normalized error function

Aya Adel Al-romi, Basem Aref Frasin*

Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq, Jordan

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Abstract

In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions connected to the normalized error function. Also, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Furthermore, the Fekete-Szegö problem for these subclasses is solved. A number of new results are shown to follow upon specializing the parameters involved in our main results.

Keywords: Analytic and univalent functions, bi-univalent functions, starlike and convex functions, error function, coefficients bounds
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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}), \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}), \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are convex of order α in $\mathcal{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathcal{U}),$$

*Corresponding author
Email addresses: amalkawi599@gmail.com (Aya Adel Al-romi), bafrasin@yahoo.com (Basem Aref Frasin)

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and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.4}$$

A function is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). Example of functions in the class Σ are

$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{U}$ such as

$$\frac{2z-z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ. Lewin [16] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [8] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [18], on the other hand, showed that $\max_{f\in\Sigma}|a_2| = 4/3$.

The coefficient estimate problem for each of the Taylor–Maclaurin coefficients $|a_n|$ $(n \geq 3; n \in \mathbb{N})$ is presumably still an open problem.

Similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex function of order $\alpha(0 \leq \alpha < 1)$, respectively, Brannan and Taha [7] (see also [23]) introduced certain subclasses of the bi-univalent function class Σ, $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions and of bi-convex functions of order $\alpha(0 \leq \alpha < 1)$, respectively. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. For some intriguing examples of functions and characterization of the class Σ, see [2, 4, 5, 11, 12, 13, 17, 21, 22, 24, 26, 27, 28, 29].

The Fekete-Szegö problem introduced in 1933 [10], is the problem of maximizing the absolute value of $|a_3 - \eta a_2^2|$, $\eta \in \mathbb{R}$. Several researchers have found coefficient estimates for this problem, for elements of Σ (for example, see [9, 15, 14]). The error function defined by [1]

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)\,n!}$$

appears widely in mathematics and related disciplines. Especially, it has various applications in statistics, probability theory, partial differential equations, special functions and physics. It is important to mention here that the error function is also known as probability integral in the literature.

Ramachandran et al. [20] (see also, [3], [6]) introduced the normalized error function defined as follows:

$$E_rf(z) = \frac{\sqrt{\pi z}}{2}\operatorname{erf}\left(\sqrt{z}\right) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)\,(n-1)!}z^n, \qquad (z \in \mathcal{U}).$$

Using the Hadamard product, Ramachandran et al. [20] studied the function $Ef(z)$ defined as follows:

$$Ef(z) = E_rf(z) * f(z) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)\,(n-1)!}a_nz^n, \qquad (z \in \mathcal{U}). \tag{1.5}$$

The object of the present paper is to introduce two new subclasses of the function class Σ defined by the normalized error function $Ef(z)$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used by Srivastava et al. [21] (see also, [12] and [25]). Furthermore, the Fekete-Szegö problem for these subclasses is solved. In order to derive our main results, we have to recall here the following lemma [19].

Lemma 1.1. *If* $h \in \mathcal{P}$ *then* $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{U}$ for which $\operatorname{Re} h(z) > 0$, $h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$ for $z \in \mathcal{U}$.

2 Coefficient bounds for the function class $\mathcal{E}_\Sigma(\alpha, \lambda)$

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{E}_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left((1-\lambda)\frac{Ef(z)}{z} + \lambda E'f(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ \lambda \geq 1, z \in \mathcal{U}) \tag{2.1}$$

and

$$\left|\arg\left((1-\lambda)\frac{Eg(z)}{w} + \lambda Eg'(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ \lambda \geq 1, w \in \mathcal{U}), \tag{2.2}$$

where the function $g = f^{-1}$ is given by (1.4). For $\lambda = 1$, we have the class given by $\mathcal{E}_\Sigma(\alpha, 1) = \mathcal{E}_\Sigma(\alpha)$, whose functions satisfy the following conditions:

$$f \in \Sigma \text{ and } |\arg(E'f(z))| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, z \in \mathcal{U}) \tag{2.3}$$

and

$$|\arg(Eg'(z))| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, w \in \mathcal{U}), \tag{2.4}$$

where the function $g = f^{-1}$ is given by (1.4). We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{E}_\Sigma(\alpha, \lambda)$.

Theorem 2.2. Let the function $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then

$$|a_2| \leq \frac{6\sqrt{5}\alpha}{\sqrt{|9(2\lambda+1) - 5(\alpha-1)(\lambda+1)^2|}} \tag{2.5}$$

and

$$|a_3| \leq \frac{36\alpha^2}{(\lambda+1)^2} + \frac{20\alpha}{2\lambda+1}. \tag{2.6}$$

Proof . It follows from (2.1) and (2.2) that

$$(1-\lambda)\frac{Ef(z)}{z} + \lambda E'f(z) = [p(z)]^\alpha \tag{2.7}$$

and

$$(1-\lambda)\frac{Eg(z)}{w} + \lambda Eg'(z) = [q(w)]^\alpha \tag{2.8}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.9}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \tag{2.10}$$

Now, equating the coefficients in (2.7) and (2.8), we get

$$\frac{-(\lambda+1)}{3}a_2 = \alpha p_1, \tag{2.11}$$

$$\frac{(2\lambda+1)}{10}a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.12}$$

$$\frac{(\lambda+1)}{3}a_2 = \alpha q_1 \tag{2.13}$$

and

$$\frac{(2\lambda+1)}{10}(2a_2^2-a_3)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.14}$$

From (2.11) and (2.13), we get

$$p_1=-q_1 \tag{2.15}$$

and

$$\frac{2}{9}(\lambda+1)^2a_2^2=\alpha^2(p_1^2+q_1^2). \tag{2.16}$$

Now from (2.12), (2.14) and (2.16), we obtain

$$\begin{aligned}\frac{1}{5}(2\lambda+1)a_2^2 &= \alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)\\ &= \alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(\frac{2(\lambda+1)^2}{9\alpha^2}a_2^2\right).\end{aligned}$$

Therefore, we have

$$a_2^2=\frac{45\alpha^2(p_2+q_2)}{9(2\lambda+1)-5\,(\alpha-1)\,(\lambda+1)^2}. \tag{2.17}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we get the desired inequality (2.5)

$$|a_2|\le\frac{6\sqrt{5}\alpha}{\sqrt{|9(2\lambda+1)-5\,(\alpha-1)\,(\lambda+1)^2|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.14) from (2.12), we get

$$\frac{1}{5}(2\lambda+1)a_3-\frac{1}{5}(2\lambda+1)a_2^2=\alpha\,(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right). \tag{2.18}$$

Further, in view of (2.15), it follows from (2.18) that

$$a_3=a_2^2+\frac{5\alpha\,(p_2-q_2)}{2\lambda+1} \tag{2.19}$$

From (2.16) and (2.19), we get

$$a_3=\frac{9\alpha^2(p_1^2+q_1^2)}{2(\lambda+1)^2}+\frac{5\alpha(p_2-q_2)}{2\lambda+1}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get$|a_3|\le\frac{36\alpha^2}{(\lambda+1)^2}+\frac{20\alpha}{2\lambda+1}$. □

Putting $\lambda=1$ in Theorem 2.2, we have

Corollary 2.3. Let the function $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\alpha)$, $(0<\alpha\le1)$. Then

$$|a_2|\le\frac{6\sqrt{5}\alpha}{\sqrt{47-20\alpha}}\qquad\text{and}\qquad|a_3|\le\frac{\alpha(27\alpha+20)}{3}.$$

3 Coefficient bounds for the function class $\mathcal{E}_\Sigma(\beta,\lambda)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{E}_\Sigma(\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \operatorname{Re}\left((1-\lambda)\frac{Ef(z)}{z} + \lambda E'f(z)\right) > \beta \qquad (0 \le \beta < 1,\ \lambda \ge 1, z \in \mathcal{U}) \tag{3.1}$$

and

$$\operatorname{Re}\left((1-\lambda)\frac{Eg(z)}{w} + \lambda Eg'(z)\right) > \beta \qquad (0 \le \beta < 1,\ \lambda \ge 1, w \in \mathcal{U}), \tag{3.2}$$

where the function $g = f^{-1}$ is given by (1.4). For $\lambda = 1$, we have the class given by $\mathcal{E}_\Sigma(\beta, 1) = \mathcal{E}_\Sigma(\beta)$, whose functions satisfy the following conditions:

$$f \in \Sigma \text{ and } \operatorname{Re}(E'f(z)) > \beta \qquad (0 \le \beta < 1,\ z \in \mathcal{U}) \tag{3.3}$$

and

$$\operatorname{Re}(Eg'(z)) > \beta \qquad (0 \le \beta < 1,\ w \in \mathcal{U}), \tag{3.4}$$

where the function $g = f^{-1}$ is given by (1.4).

Theorem 3.2. Let $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\beta, \lambda)$, $0 \le \beta < 1$ and $\lambda \ge 1$. Then

$$|a_2| \le \sqrt{\frac{20(1-\beta)}{2\lambda+1}} \tag{3.5}$$

and

$$|a_3| \le \frac{36(1-\beta)^2}{(\lambda+1)^2} + \frac{20(1-\beta)}{2\lambda+1}. \tag{3.6}$$

Proof . It follows from (3.1) and (3.2) that there exist p and $q \in \mathcal{P}$ such that

$$(1-\lambda)\frac{Ef(z)}{z} + \lambda E'f(z) = \beta + (1-\beta)p(z) \tag{3.7}$$

and

$$(1-\lambda)\frac{Eg(z)}{w} + \lambda Eg'(z) = \beta + (1-\beta)q(w) \tag{3.8}$$

where $p(z)$ and $q(w)$ have the forms (2.9) and (2.10), respectively. Equating coefficients in (3.7) and (3.8) yields

$$\frac{-(\lambda+1)}{3}a_2 = (1-\beta)p_1, \tag{3.9}$$

$$\frac{(2\lambda+1)}{10}a_3 = (1-\beta)p_2, \tag{3.10}$$

$$\frac{(\lambda+1)}{3}a_2 = (1-\beta)q_1 \tag{3.11}$$

and

$$\frac{(2\lambda+1)}{10}(2a_2^2 - a_3) = (1-\beta)q_2. \tag{3.12}$$

From (3.9) and (3.11), we get $p_1 = -q_1$ and

$$\frac{2}{9}(\lambda+1)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.13}$$

Also, from (3.10) and (3.12), we find that

$$\frac{1}{5}(2\lambda+1)a_2^2 = (1-\beta)(p_2+q_2). \tag{3.14}$$

Thus, we have

$$|a_2^2| \leq \frac{5(1-\beta)}{(2\lambda+1)}(|p_2|+|q_2|) = \frac{20(1-\beta)}{2\lambda+1}$$

which is the bound on $|a_2|$ as given in (3.5). Next, in order to find the bound on $|a_3|$, by subtracting (3.13) form (3.12), we get

$$\frac{1}{5}(2\lambda+1)a_3 - \frac{1}{5}(2\lambda+1)a_2^2 = (1-\beta)(p_2-q_2)$$

or, equivalently,$a_3 = a_2^2 + \frac{5(1-\beta)(p_2-q_2)}{(2\lambda+1)}$. Upon substituting the value of a_2^2 from (3.13), we obtain

$$a_3 = \frac{9(1-\beta)^2(p_1^2+q_1^2)}{2(\lambda+1)^2} + \frac{5(1-\beta)(p_2-q_2)}{(2\lambda+1)}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{36(1-\beta)^2}{(\lambda+1)^2} + \frac{20(1-\beta)}{(2\lambda+1)}$$

which is the bound on $|a_3|$ as asserted in (3.6).□

Putting $\lambda = 1$ in Theorem 3.2, we have

Corollary 3.3. Let $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\beta)$, $(0 \leq \beta < 1)$. Then

$$|a_2| \leq \sqrt{\frac{20(1-\beta)}{3}} \qquad \text{and} \qquad |a_3| \leq 9(1-\beta)^2 + \frac{20}{3}(1-\beta).$$

4 Fekete-Szegö inequality

In this section, we will find the sharp bounds of Fekete-Szegö functional $|a_3 - \eta a_2^2|$, $\eta \in \mathbb{R}$, for $f \in \mathcal{E}_\Sigma(\alpha, \lambda)$.

Theorem 4.1. Let $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{20\alpha}{2\lambda+1}, & \text{for } 0 \leq |\varphi(\eta)| \leq \frac{1}{2\lambda+1}, \\ 20\alpha\,|\varphi(\eta)|, & \text{for } |\varphi(\eta)| \geq \frac{1}{2\lambda+1}, \end{cases}$$

where

$$\varphi(\eta) = (1-\eta)\,\frac{9\alpha}{9(2\lambda+1) - 5\,(\alpha-1)\,(\lambda+1)^2}.$$

Proof . Let $f \in \mathcal{E}_\Sigma(\alpha, \lambda)$. By using (2.17) and (2.19) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)\,\frac{45\alpha^2(p_2+q_2)}{9(2\lambda+1) - 5\,(\alpha-1)\,(\lambda+1)^2} + \frac{5\alpha\,(p_2-q_2)}{2\lambda+1} \\ &= 5\alpha\left[\left(\varphi(\eta) + \frac{1}{2\lambda+1}\right)p_2 + \left(\varphi(\eta) - \frac{1}{2\lambda+1}\right)q_2\right], \end{aligned}$$

where

$$\varphi(\eta) = (1-\eta)\,\frac{9\alpha}{9(2\lambda+1) - 5\,(\alpha-1)\,(\lambda+1)^2}.$$

Therefore, we have

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{20\alpha}{2\lambda+1}, & \text{for } 0 \leq |\varphi(\eta)| \leq \frac{1}{2\lambda+1}, \\ 20\alpha\,|\varphi(\eta)|\,, & \text{for } |\varphi(\eta)| \geq \frac{1}{2\lambda+1}, \end{cases}$$

The proof is completed. □

Putting $\lambda = 1$ in Theorem 4.1, we have

Corollary 4.2. Let the function $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\alpha)$, $(0<\alpha\leq 1)$. Then for some $\eta\in\mathbb{R}$,

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{20\alpha}{3}, & \text{for } 0 \leq |\varphi(\eta)| \leq \frac{1}{3}, \\ 20\alpha\,|\varphi(\eta)|\,, & \text{for } |\varphi(\eta)| \geq \frac{1}{3}, \end{cases}$$

where

$$\varphi(\eta) = (1-\eta)\,\frac{9\alpha}{27-45\,(\alpha-1)}.$$

Taking $\eta = 1$ in Corollary 4.2, we get the following corollary.

Corollary 4.3. Let the function $f(z)$ given by (1.1) be in the class $\mathcal{E}_\Sigma(\alpha)$, $(0<\alpha\leq 1)$. Then

$$\left|a_3-a_2^2\right| \leq \frac{20\alpha}{3}.$$

5 Conclusions

In this study, we introduce two new subclasses $\mathcal{E}_\Sigma(\alpha,\lambda)$ and $\mathcal{E}_\Sigma(\beta,\lambda)$ of the function class Σ of bi-univalent functions connected to the normalized error function $Ef(z)$ given in (1.5). We have derived estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for the function in the subclasses $\mathcal{E}_\Sigma(\alpha,\lambda)$ and $\mathcal{E}_\Sigma(\beta,\lambda)$. Furthermore, the Fekete-Szegö problem for these subclasses is solved. Making use of the normalized error function $Ef(z)$ could inspire researchers to derive the estimates of the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö functional problems for functions belonging to new subclasses of bi-univalent functions.

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