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MAXIMUM MODULUS OF THE DERIVATIVES OF A POLYNOMIAL

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Abstract. For an arbitrary entire function $f(z)$, let $M(f,R)=\max_{|z|=R}|f(z)|$ and $m(f,r)=\min_{|z|=r}|f(z)|$. If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$, it is proved by Aziz et al. that

$$M(P',\rho)\leq \frac{n}{\rho+k}\{(\frac{\rho+k}{k+r})^n[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}(\frac{\rho-r}{k+\rho})(\frac{k+r}{k+\rho})^{n-1}]M(P,r)$$
$$-[\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\times[((\frac{\rho+k}{r+k})^n-1)-n(\rho-r)]]m(P,k)\}.$$

In this paper, we obtain a refinement of the above inequality. Moreover, we obtain a generalization of above inequality for $M(P',R)$, where $R\geq k$.

1. Introduction and preliminaries

For an arbitrary entire function $f(z)$, let $M(f,R)=\max_{|z|=R}|f(z)|$ and $m(f,r)=\min_{|z|=r}|f(z)|$. Let $P(z)$ be a polynomial of degree n, then according to a famous result known as Bernstein's inequality on the derivative of a polynomial, we have

$$M(P',1)\leq nM(P,1). \tag{1.1}$$

The result is best possible and equality holds for the polynomials having all its zeros at the origin.
For polynomials having no zeros in $|z|<1$, Erdös conjectured and later Lax [6] proved that if $P(z)\neq 0$ in $|z|<1$, then (1.1) can be replaced by

$$M(P',1)\leq \frac{n}{2}M(P,1). \tag{1.2}$$

With equality for those polynomials, which have all their zeros on $|z|=1$.
As an extension of (1.2) Malik [7] proved that if $P(z)\neq 0$ in $|z|<k$, $k\geq 1$ then

$$M(P',1)\leq \frac{n}{1+k}M(P,1). \tag{1.3}$$

The result is best possible and equality holds for the polynomial $P(z)=(z+k)^n$.

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Dewan and Bidkham [2] obtained a generalization of inequality (1.3) for the class of polynomials $P(z)=\sum_{j=0}^{n}a_jz^j$ having no zeros in $|z|<k$, $k\geq 1$, by proving

$$M(P',\rho)\leq n\frac{(\rho+k)^{n-1}}{(1+k)^n}M(P,1), \tag{1.4}$$

where $1\leq\rho\leq k$. The result is best possible and equality holds for the polynomial $P(z)=(z+k)^n$.

Further, as a generalization of (1.4) Dewan and Mir [3] proved that if $P(z)=\sum_{j=0}^{n}a_jz^j$ having no zeros in $|z|<k$, $k\geq 1$ then for $0\leq r\leq\rho\leq k$,

$$M(P',\rho)\leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\frac{(\rho-r)(k+r)^{n-1}}{(k+\rho)^n}\}M(P,r). \tag{1.5}$$

The result is best possible and equality holds for the polynomial $P(z)=(z+k)^n$. Recently Aziz and Zargar [1] obtained a generalization of (1.5) and proved if $P(z)=\sum_{j=0}^{n}a_jz^j$ having no zeros in $|z|<k$, $k\geq 1$ then for $0\leq r\leq\rho\leq k$,

$$\begin{aligned}M(P',\rho)\leq&\frac{n}{\rho+k}\{(\frac{\rho+k}{k+r})^n[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}(\frac{\rho-r}{k+\rho})(\frac{k+r}{k+\rho})^{n-1}]M(P,r)\\&-[\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\times\{((\frac{\rho+k}{r+k})^n-1)-n(\rho-r)\}]m(P,k)\}.\end{aligned} \tag{1.6}$$

The result is best possible and equality holds for the polynomial $P(z)=(z+k)^n$. In this paper, first we obtain the following result which is a refinement of inequality (1.6).

Theorem 1.1. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, having no zeros in $|z|<k$, $k\geq 1$ then for $0\leq r\leq\rho\leq k$,*

$$\begin{aligned}M(P',\rho)\leq&\frac{n(n|a_0|\rho^2+k^2\rho|a_1|)}{\rho((\rho^2+k^2)n|a_0|+2k^2\rho|a_1|)}\times\\&\{(\frac{\rho+k}{k+r})^n[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}(\frac{\rho-r}{k+\rho})(\frac{k+r}{k+\rho})^{n-1}]M(P,r)\\&-[\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\times[((\frac{\rho+k}{r+k})^n-1)-n(\rho-r)]]m(P,k)\}.\end{aligned} \tag{1.7}$$

The result is best possible and equality holds for the polynomial $P(z)=(z+k)^n$.

Remark. Theorem 1.1 is, in general, an improvement of inequality (1.6). To see this, we note that for a polynomial $P(z)=\sum_{j=0}^{n}a_jz^j$ such that does not vanish in $|z|<k$, $k\geq 1$ and $0\leq r\leq\rho\leq k$, by using lemma 2.5 inequality $\frac{(n|a_0|\rho^2+k^2\rho|a_1|)}{\rho((\rho^2+k^2)n|a_0|+2k^2\rho|a_1|)}\leq\frac{1}{\rho+k}$ is true.

If we take $\rho=k$ in Theorem 1.1, then we have

Corollary 1.2. *If $P(z)$ be a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$ then for $0 \leq r \leq k$, we have*

$$M(P',k) \leq \frac{n}{2k}\{(\frac{2k}{k+r})^n M(P,r) - \frac{r+k}{2k}[(\frac{2k}{r+k})^n - 1 - n(k-r)]m(P,k)\}. \quad (1.8)$$

Next we prove the following interesting result which is a generalization of inequality (1.6) for radius greater than k.

Theorem 1.3. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$ then for $0 \leq r \leq k \leq R$*

$$\begin{aligned} M(P',R) \leq \frac{nR^{n-1}}{2k^n} &\times \{(\frac{2k}{k+r})^n M(P,r) - \\ &\frac{r+k}{2k}[(\frac{2k}{r+k})^n - 1 - n(k-r)]m(P,k)\}. \end{aligned} \quad (1.9)$$

If we take $R = k$ in Theorem 1.3, then we have inequality (1.8) again,

Corollary 1.4. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$ then for $0 \leq r \leq k$, we have*

$$M(P',k) \leq \frac{n}{2k}\{(\frac{2k}{k+r})^n M(P,r) - \frac{r+k}{2k}[(\frac{2k}{r+k})^n - 1 - n(k-r)]m(P,k)\}.$$

2. Lemmas

For the proof of theorems, we need the following lemmas. The first lemma is due to Govil, Rahman and Schmeisser [5].

Lemma 2.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, having no zeros in $|z| \leq k$, $k \geq 1$, then*

$$M(P',1) \leq n\{\frac{n|a_0| + k^2|a_1|}{n|a_0|(1+k^2) + 2k^2|a_1|}\}M(P,1). \quad (2.1)$$

Lemma 2.2. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq r \leq \rho \leq k$,*

$$\begin{aligned} M(P,\rho) \leq (\frac{\rho+k}{k+r})^n[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|}(\frac{\rho-r}{k+\rho})(\frac{k+r}{k+\rho})^{n-1}]M(P,r) \\ -[\frac{(n|a_0|\rho + k^2|a_1|)(r+k)}{(\rho^2+k^2)n|a_0| + 2k^2\rho|a_1|} \times [((\frac{\rho+k}{r+k})^n - 1) - n(\rho - r)]]m(P,k). \end{aligned} \quad (2.2)$$

The above lemma is due to Aziz and Zargar [1].

Lemma 2.3. *Let $F(z)$ be a polynomial of degree n, having all its zeros in the closed disk $|z| \leq 1$. Furthermore, let $f(z)$ be a polynomial of degree at most n such that $|f(z)| \leq |F(z)|$ for $|z| = 1$, then $|f'(z)| \leq |F'(z)|$ for $|z| \geq 1$.*

You can find the proof of Lemma 2.3 in [8].

Lemma 2.4. *If $P(z)$ is a polynomial of degree n and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for $|z| \geq 1$ we have*

$$|P'(z)| + |Q'(z)| \leq n|z|^{n-1}M(P,1). \quad (2.3)$$

Proof. Since $|P(z)| \leq M(P,1)$, where $|z| \leq 1$. Then by using Rouche's theorem it follows the polynomial

$$G(z) = P(z) - \lambda M(P,1),$$

does not vanish in $|z| \leq 1$, for λ with $|\lambda| > 1$. Now consider

$$H(z) = z^n\overline{G(1/\overline{z})} = Q(z) - \overline{\lambda} M(P,1)z^n.$$

Then the polynomial $H(z)$ has all its zeros in $|z| \leq 1$, and $|H(z)| = |G(z)|$, where $|z| = 1$.

Therefore on applying Lemma 2.3 to polynomials $G(z)$ and $H(z)$, we have for $|z| \geq 1$,

$$|P'(z)| \leq |Q'(z) - n\overline{\lambda} M(P,1)z^{n-1}|. \tag{2.4}$$

Since $M(Q,1) = M(P,1)$, then again we can apply Lemma 2.3 to polynomials $Q(z)$ and $M(P,1)z^n$, and we obtain

$$|Q'(z)| \leq nM(P,1)|z|^{n-1},$$

for $|z| \geq 1$.
Therefore for an appropriate choice of the argument of λ we have

$$|Q'(z) - n\overline{\lambda} M(P,1)z^{n-1}| = |\lambda| nM(P,1)|z|^{n-1} - |Q'(z)|.$$

Which helps us to rewrite inequality (2.4) as

$$|P'(z)| + |Q'(z)| \leq |\lambda| nM(P,1)|z|^{n-1}.$$

Make $|\lambda| \to 1$, we get inequality (2.3). □

Lemma 2.5. *If $P(z) = \sum_{j=0}^{n} a_j z^j$, is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then*

$$\frac{k|a_1|}{|a_0|} \leq n. \tag{2.5}$$

The above result is due to Gardner et al. [4].

3. Proof of the theorems

Proof of the Theorem 1.1. For ρ with $0 \leq \rho \leq k$, the polynomial $P(\rho z)$ has no zeros in $|z| \leq k/\rho$, $k/\rho \geq 1$. Now by applying Lemma 2.1, for $|z| = 1$, we have

$$\rho|P'(\rho z)| \leq n\{\frac{n|a_0| + \frac{k^2}{\rho^2}\rho|a_1|}{(1 + \frac{k^2}{\rho^2})n|a_0| + 2\frac{k^2}{\rho^2}\rho|a_1|}\}M(P,\rho). \tag{3.1}$$

Now, if $0 \leq r \leq \rho \leq k$, then by using Lemma 2.2, we have

$$\begin{aligned} M(P,\rho) \leq (\frac{\rho + k}{k + r})^n[1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(\rho^2 + k^2)n|a_0| + 2k^2\rho|a_1|}(\frac{\rho - r}{k + \rho})(\frac{k + r}{k + \rho})^{n-1}]M(P,r) \\ -[\frac{(n|a_0|\rho + k^2|a_1|)(r + k)}{(\rho^2 + k^2)n|a_0| + 2k^2\rho|a_1|} \times [((\frac{\rho + k}{r + k})^n - 1) - n(\rho - r)]]m(P,k). \end{aligned} \tag{3.2}$$

By combining (3.1) and (3.2), Theorem 1.1 follows. □

Proof of Theorem 1.3. Since $P(z)$ having no zero in $|z| < k$, therefore the polynomial $H(z) = P(kz)$ does not vanish in $|z| < 1$. Then the polynomial $G(z) =$

$z^n\overline{H(\frac{1}{\bar z})}$ has all its zeros in $|z| \le 1$, and $|H(z)| = |G(z)|$ for $|z| = 1$. By applying Lemma 2.3 we have

$$|H'(z)| \le |G'(z)| \text{ for } |z| \ge 1. \tag{3.3}$$

On the other hand by using Lemma 2.4, for $|z| \ge 1$ we have

$$|H'(z)| + |G'(z)| \le n|z|^{n-1}M(H,1). \tag{3.4}$$

Now combining (3.3) and (3.4) we have

$$|H'(te^{i\theta})| \le \frac{nt^{n-1}}{2}M(H,1) \quad t \ge 1.$$

Replacing $H(z)$ by $P(kz)$, we conclude that

$$k|P'(kte^{i\theta})| \le \frac{nt^{n-1}}{2}M(P,k) \quad t \ge 1. \tag{3.5}$$

Now if we take $\rho = k$ in Lemma 2.2 we have

$$M(P,k) \le (\frac{2k}{k+r})^n M(P,r) - \frac{r+k}{2k}[(\frac{2k}{r+k})^n - 1 - n(k-r)]m(P,k). \tag{3.6}$$

Hence for $R \ge k$, we take $t = R/k$ in (3.5), now combining (3.6) and (3.5), we have

$$|P'(Re^{i\theta})| \le \frac{nR^{n-1}}{2k^n}\{(\frac{2k}{k+r})^n M(P,r) - \frac{r+k}{2k}[(\frac{2k}{r+k})^n - 1 - n(k-r)]m(P,k)\}.$$

This completes the proof. □

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