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Finite difference method for solving fractional cordial Volterra integro-differential equations with Caputo derivative

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Abstract

This study introduces, for the first time, a fractional formulation of cordial Volterra integro-differential equations and proposes an efficient numerical algorithm for their solution. The method combines a finite difference scheme of order $\mathscr{O}(\tau^{3-\alpha})$ for discretizing the Caputo fractional derivative with a composite trapezoidal rule $\mathscr{O}(\tau^2)$ to approximate the integral term. Theoretical analysis establishes that the unified scheme achieves an overall convergence order of $\mathscr{O}(\tau^2)$, a result rigorously confirmed through numerical experiments. The presented examples not only validate the algorithm's accuracy but also demonstrate its effectiveness in handling fractional-order systems, addressing a gap in the numerical analysis of such equations. This work provides a foundational framework for future research on fractional cordial integro-differential problems.

Keywords: Fractional cordial Volterra integro-differential equations, Finite difference, Caputo fractional derivative, Composite trapezoidal rule
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1 Introduction

Modeling of natural phenomena with differential equations and integral equations has long been the focus of researchers and authors. An important class of these equations, which is widely used in science and engineering, is intgro-differential equations. There are many numerical schemes to solve this type of equations. The analysis of solution existence for Ψ-Hilfer type fractional integro-differential equations with impulsive conditions and almost sectorial operators was recently advanced in [10]. A precise localized meshless collocation method for solving two-dimensional nonlinear integro-differential equations (2D-NIDE) with multi-term kernels was investigated in [5]. Ravichandran [19] dealt with integro-differential equations that incorporate the Atangana–Baleanu fractional derivatives, which feature generalized Mittag-Leffler functions in their kernels. The main aim of [3] was to explore the existence of solutions for a fractional hybrid integro-differential equation that comes with mixed hybrid integral boundary value conditions. A Volterra integro-differential equation is a type of mathematical equation that combines both differential and integral operators. Volterra integro-differential equations serve as fundamental mathematical tools across multiple scientific and engineering disciplines, demonstrating particular utility in modeling systems with memory effects, including physics, biology, economics, and control theory. Solving Volterra integro-differential equations can be a challenging task due to the combined presence of derivatives and integrals. Analytical solutions are often difficult to obtain, and numerical methods are commonly employed to approximate solutions. These equations have diverse applications,

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including in the modeling of dynamic systems, population dynamics, material science, and the analysis of various phenomena involving memory or hereditary effects. The study in [28] integrates the Laplace transform technique and the Adomian decomposition approach to investigate nonlinear Volterra integro-differential equations. Focusing on solution properties, Cemil Tunç and Osman Tunç [23] studied conducts a comparative analysis of nonlinear Volterra integro-differential systems, examining both time-delayed and instantaneous interaction cases. Stability of a system of Volterra integro-differential models was discussed in [25]. The analysis of specific nonlinear scalar Volterra integro-differential equations was obtained in [22]. The main objectives are to explore various qualitative properties of the solutions for both the scalar equations and the system. A spectral technique for parabolic Volterra integro-differential equations was studied in [8]. For further details, the reader is referred to [2].

Cordial Volterra integro-differential equations represent a significant subclass of such systems. Kangro seminal work [9] examined general cordial Volterra integral equations of the second kind and particular singular fractional integro-differential equations, with primary objectives being the characterization of operator properties including compactness and spectral analysis within relevant function spaces. The authors of [20] showed a numerical solution for cordial Volterra integro-differential equations with noncompact cordial operators using collocation techniques. Vainikko [24] examined the convergence and convergence rates of two different versions of spline collocation methods when applied to linear Volterra integral equations of the second kind. In [7] Diogo and Vainikko investigated the effectiveness of the standard spline collocation method to numerical solution of linear Volterra integral equations of the second kind. These integral equations involve cordial operators, which are known to be non compact. Toranj-Simin and Hadizadeh [21] indicated the exploring theoretical aspects related to the collocation method employed on a graded mesh. They focused on a specific class of non compact cordial Volterra integral equations. These equations were characterized by kernels that exhibit diagonal singularities, and the singularity properties depend on a real parameter s, which falls within the open interval $(0, 1)$. To see the other references about cordial Volterra integro-differential equation the interested readers could see [12, 15, 24, 27].

In this study, the cordial Volterra integro-differential equation is considered in fractional form with Caputo operator. So, we concentrate on this problem as follows

$$ {}_0^cD_t^\alpha u(t) + q(t)u(t) = \int_0^t \frac{1}{t}\Psi(\frac{s}{t})\mathcal{K}(t,s)u(s)ds + p(t), \qquad t \in [0, \mathcal{T}], \tag{1.1} $$

with initial condition $u(0) = u_0$ and the operator ${}_0^cD_t^\alpha$ denotes Caputo fractional derivative of order α, where α satisfies $0 < \alpha < 1$ which is defined as [18]

$$ {}_0^cD_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} u'(\xi)d\xi. \tag{1.2} $$

Also, q, p and $\mathcal{K}$ are continuous functions in $[0, \mathcal{T}]$ and the core function Ψ is an integrable function. The proposed algorithm primarily relies on finite difference methods and a Newton-Cotes integration scheme based on the compound trapezoidal rule to solve (1.1). For this purpose, a $3-\alpha$ order finite difference method is applied to discretize the Caputo fractional derivative of the fractional cordial Volterra integro-differential equation. The integral in the model is also approximated by compound trapezoidal technique. From the combination of these two approach, a numerical plan with a convergence order of $\mathscr{O}(\tau^2)$ results.

The finite difference method is a widely employed numerical technique that plays a crucial role in solving various types of differential equations, including fractional ordinary and partial, integro-differential equations, and more. Over the years, it has garnered significant attention from researchers and mathematicians due to its versatility and effectiveness in approximating solutions to complex mathematical problems.This introduction draws upon several notable references in the field of finite difference methods, particularly focusing on its applications in various types of differential equations. In [13], Li and Zeng delved into the application of finite difference method to fractional differential equations, an area that has gained increasing prominence in recent years. Furthermore, their work extends to the realm of fractional ordinary differential equations [14]. This research contributes to the understanding of how finite difference method can be effectively employed to tackle fractional ordinary differential equations, expanding the scope of its applications. In the realm of integro-differential equations, Zhao and Corless introduced the compact finite difference method in their work [29]. This approach offers a concise and efficient approach to solving equations involving both differential and integral operators, making it particularly valuable for a wide range of scientific and engineering problems. The versatility of finite difference extends even further to include Fredholm integro-differential equations, as explored by Dehghan and Saadatmandi [6]. Their work presents a variant of finite difference method tailored to address Fredholm integro-differential equations. In the domain of numerical analysis for perturbed systems, Kudu et al. [11] proposed a finite difference methodology specifically optimized for solving delay integro-differential equations with singular perturbation characteristics. Their study demonstrates the capability of finite difference to handle challenging equations involving singular perturbations and delays, further underlining its efficacy in solving complex problems.

The trapezoidal rule, a fundamental numerical integration technique, has found wide-ranging applications in diverse fields, from mathematics to engineering. This introduction draws upon several key references to provide insights into the versatility and significance of the trapezoidal rule in various contexts. Banjai and Ferrari introduced the concept of generalized convolution quadrature based on the trapezoidal rule [4]. Their work explores how the trapezoidal rule can be extended and adapted to handle more complex mathematical operations, revealing its adaptability and utility in advanced mathematical methodologies. Moreover, Wang contributed to the understanding of the trapezoidal rule's applicability in integrals featuring fractional order singularities [26]. Their research delves into the asymptotic expansion and extrapolation techniques, demonstrating the trapezoidal rule's effectiveness in addressing integrals with specialized characteristics. In another dimension of mathematical approximation, Pandiangan explored the utilization of the trapezoidal rule for approximating fractional integrals and Caputo derivatives [17]. This work is showcased how modifications to the trapezoidal rule can enhance its precision in approximating complex mathematical constructs. Additionally, Wang et al. [26] advanced the field by analyzing the asymptotic expansion and extrapolation of the trapezoidal rule for integrals containing fractional-order singularities, further establishing its efficacy as a robust numerical method for specialized integration problems. The trapezoidal rule has practical applications in engineering, as demonstrated by Pande and Nasikkar in their research related to wind power systems [16]. Their research demonstrates the application of the trapezoidal rule in Maximum Power Point Tracking methods for wind energy systems, emphasizing its critical role in enhancing power generation efficiency. In summary, the trapezoidal rule stands as a versatile and indispensable numerical integration method with a broad spectrum of applications.

The organization of this paper is outlined below:

- In Section 2, we explain and apply the devised numerical algorithm to solve the fractional cordial Volterra integro-differential model (1.1).
- To validate the proposed method's precision, Section 3 presents numerical implementations for selected test cases, with corresponding results documented.
- A general conclusion summarizing the findings is presented in Section 4.

2 Numerical implementation

In this section, two methods of Caputo derivative discretization of order of convergence $3-\alpha$ and compound trapezoidal rule are used to build a finite difference scheme for problem (1.1).

Let $u_i = u(t_i)$ and $0 = t_0 < t_1 < t_2 < \ldots < t_{\mathbf{N}-1} < t_{\mathbf{N}} = \mathcal{T}$ with $t_n = n\tau$ for $\tau = \mathcal{T}/\mathbf{N}$ and $n = 0, 1, \ldots, \mathbf{N}$. The discritization of Caputo derivative with $0 < \alpha < 1$ in t_{n+1} is as follows [1]:

$$ {}_0^cD_t^\alpha u_{n+1} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n+1} \mathscr{C}_{n+1,i} u_i + \mathscr{O}(\tau^{3-\alpha}), \tag{2.1} $$

in which the coefficients $\mathscr{C}_{n+1,i}$ can be calculated as follows

$$ \mathscr{C}_{1,i} = \begin{cases} -\mathfrak{p}_1, & i = 0, \\ \mathfrak{p}_1, & i = 1, \end{cases} \tag{2.2} $$

$$ \mathscr{C}_{2,i} = \begin{cases} -\mathfrak{p}_2 + \mathfrak{q}_{2,2}, & i = 0, \\ \mathfrak{p}_2 + \mathfrak{r}_{2,2}, & i = 1, \\ \mathfrak{s}_{2,2}, & i = 2, \end{cases} \tag{2.3} $$

and for $n \geq 3$, we have

$$\mathscr{C}_{n+1,i} = \begin{cases} -\mathfrak{p}_{n+1} + \mathfrak{q}_{n+1,2}, & i = 0, \\ \mathfrak{p}_{n+1} + \mathfrak{q}_{n+1,3} + \mathfrak{r}_{n+1,2}, & i = 1, \\ \mathfrak{q}_{n+1,i+2} + \mathfrak{r}_{n+1,i+1} + \mathfrak{s}_{n+1,i}, & 2 \leq i \leq n-1, \\ \mathfrak{r}_{n+1,n+1} + \mathfrak{s}_{n+1,n}, & i = n, \\ \mathfrak{s}_{n+1,n+1}, & i = n+1, \end{cases} \tag{2.4}$$

where

$$\mathfrak{p}_j = j^{1-\alpha} - (j-1)^{1-\alpha}, \tag{2.5}$$

$$\mathfrak{q}_{n+1,j} = \frac{1}{2-\alpha}\Big((n-j+2)^{1-\alpha}(n-j+\frac{\alpha}{2}+1) - (n-j+1)^{1-\alpha}(n-j-\frac{\alpha}{2}+2)\Big), \tag{2.6}$$

$$\mathfrak{r}_{n+1,j} = \frac{2}{2-\alpha}\Big((n-j+1)^{1-\alpha}(n-j-\alpha+3) - (n-j+2)^{2-\alpha}\Big), \tag{2.7}$$

and

$$\mathfrak{s}_{n+1,j} = \frac{1}{2-\alpha}\Big((n-j+2)^{1-\alpha}(n-j-\frac{\alpha}{2}+3) - (n-j+1)^{1-\alpha}(n-j-\frac{3\alpha}{2}+4)\Big). \tag{2.8}$$

Applying the finite difference scheme (2.1) in the problem (1.1), one obtain

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{n+1}\mathscr{C}_{n+1,i}u_i + q_{n+1}u_{n+1} = \int_0^{t_{n+1}} \frac{1}{t_{n+1}}\Psi(\frac{s}{t_{n+1}})\mathcal{K}(t_{n+1},s)u(s)ds + p_{n+1} + \mathscr{R}_\alpha, \tag{2.9}$$

where $\mathscr{R}_\alpha \leq \mathcal{C}_1\tau^{3-\alpha}$, $q_{n+1} = q(t_{n+1})$ and $p_{n+1} = p(t_{n+1})$. Using composite trapezoidal rule, we get

$$\begin{aligned} \int_0^{t_{n+1}} \frac{1}{t_{n+1}}\Psi(\frac{s}{t_{n+1}})\mathcal{K}(t_{n+1},s)u(s)ds = \frac{\tau}{2t_{n+1}}\sum_{k=0}^{n}\Big[\Psi(\frac{t_k}{t_{n+1}})\mathcal{K}(t_{n+1},t_k)u_k \\ + \Psi(\frac{t_{k+1}}{t_{n+1}})\mathcal{K}(t_{n+1},t_{k+1})u_{k+1}\Big] + \mathscr{O}(\tau^2). \end{aligned} \tag{2.10}$$

By (2.10) and (2.9), it can be written

$$\begin{aligned} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{n+1}\mathscr{C}_{n+1,i}u_i + q_{n+1}u_{n+1} = \frac{\tau}{2t_{n+1}}\sum_{k=0}^{n}\Big[\Psi(\frac{t_k}{t_{n+1}})\mathcal{K}(t_{n+1},t_k)u_k \\ + \Psi(\frac{t_{k+1}}{t_{n+1}})\mathcal{K}(t_{n+1},t_{k+1})u_{k+1}\Big] + p_{n+1} + \mathscr{R}_\alpha + \mathscr{R}_T, \end{aligned} \tag{2.11}$$

where $\mathscr{R}_T \leq \mathcal{C}_2\tau^2$. Equation (2.11) can be reformulated as shown below

$$\begin{aligned} &\Big(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathscr{C}_{n+1,n+1} + q_{n+1} - \frac{\tau}{2t_{n+1}}\Psi(1)\mathcal{K}(t_{n+1},t_{n+1})\Big)u_{n+1} \\ &= -\Big(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathscr{C}_{n+1,n} - \frac{\tau}{2t_{n+1}}\Psi(\frac{t_n}{t_{n+1}})\mathcal{K}(t_{n+1},t_n)\Big)u_n - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{n-1}\mathscr{C}_{n+1,i}u_i \\ &+ \frac{\tau}{2t_{n+1}}\sum_{k=0}^{n-1}\Big[\Psi(\frac{t_k}{t_{n+1}})\mathcal{K}(t_{n+1},t_k)u_k + \Psi(\frac{t_{k+1}}{t_{n+1}})\mathcal{K}(t_{n+1},t_{k+1})u_{k+1}\Big] + p_{n+1} + \mathscr{R}_\alpha + \mathscr{R}_T. \end{aligned} \tag{2.12}$$

So, we obtain a discrete scheme with removing the error terms $\mathscr{R}_\alpha$ and $\mathscr{R}_T$ as

$$\mathscr{U}_{n+1} = \mathscr{B}_n/\mathscr{A}_n, \qquad n = 0, 1, \quad , \mathbf{N}-1, \tag{2.13}$$

where $\mathscr{U}_{n+1} \simeq u(t_{n+1})$ and

$$\mathscr{A}_n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathscr{C}_{n+1,n+1} + q_{n+1} - \frac{\tau}{2t_{n+1}}\Psi(1)\mathcal{K}(t_{n+1},t_{n+1}), \tag{2.14}$$

$$\mathscr{B}_n = -\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathscr{C}_{n+1,n} - \frac{\tau}{2t_{n+1}}\Psi(\frac{t_n}{t_{n+1}})\mathcal{K}(t_{n+1},t_n)\right)u_n - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{n-1}\mathscr{C}_{n+1,i}u_i \tag{2.15}$$

$$+ \frac{\tau}{2t_{n+1}}\sum_{k=0}^{n-1}\left[\Psi(\frac{t_k}{t_{n+1}})\mathcal{K}(t_{n+1},t_k)u_k + \Psi(\frac{t_{k+1}}{t_{n+1}})\mathcal{K}(t_{n+1},t_{k+1})u_{k+1}\right] + p_{n+1}.$$

3 Numerical results

Numerical examples are provided in this section to evaluate the efficacy of the described algorithm, with the conventional ℓ_∞ error measure employed as follows.

$$\ell_\infty = \max_{0\le t\le \mathcal{T}} |u_E(t) - u_N(t)|, \tag{3.1}$$

where $u_E(t)$ and $u_N(t)$ are the exact and numerical solutions, respectively. Moreover, to validate the results, convergence rate is reported by the following formula

$$\mathcal{C} = \frac{\log\left(\dfrac{\mathscr{E}_1}{\mathscr{E}_2}\right)}{\log\left(\dfrac{\tau_1}{\tau_2}\right)}, \tag{3.2}$$

in which $\mathscr{E}_1$ and $\mathscr{E}_2$ are errors related to τ_1 and τ_2 time steps, respectively.

Example 3.1. At first, we examine (1.1) under the following representation

$$\begin{cases} {}_c^0D_t^\alpha u(t) + \dfrac{2}{9}tu(t) = t^{-1}\displaystyle\int_0^t \frac{s}{t}su(s)ds + p(t), \quad t\in[0,\mathcal{T}], \\ u(0) = 0, \end{cases} \tag{3.3}$$

with the source term

$$p(t) = \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\alpha)} + \frac{4}{99}t^{\frac{7}{2}}, \tag{3.4}$$

in which the true solution is $u(t) = t^{\frac{5}{2}}$, the core function $\Psi(x) = x$ and $\mathcal{K}(t,s) = s$. To investigate the numerical solution of the given problem, the proposed computational algorithm was implemented with varying step sizes τ and parameter values α over the intervals $[0,1]$, $[0,2]$, and $[0,3]$. The performance of the method was rigorously evaluated by analyzing the ℓ_∞ error norm and the corresponding order of convergence, which are systematically documented in Table 1. The structure of Table 1 is organized as follows: The first two columns present the discretization parameters $\mathcal{T}$ and τ, where τ undergoes successive refinement by halving at each step. This progressive reduction in τ allows for a thorough assessment of the algorithm's convergence behavior. The subsequent columns provide detailed error analysis, including the computed ℓ_∞ errors and the derived convergence rates for selected values of α. To further illustrate the accuracy of the method, Figure 1 graphically represents the absolute ℓ_∞ error distribution obtained using the proposed numerical scheme with $\tau = 0.01/16$ and $\alpha = 0.65$. The visualization highlights the spatial distribution of errors, offering additional insight into the algorithm's precision. The numerical results consistently demonstrate the robustness and efficiency of the proposed approach. Notably, the observed convergence rates align with theoretical expectations, confirming that the method maintains stability and accuracy across different parameter configurations. This comprehensive validation underscores the algorithm's suitability for solving the given class of problems, establishing its potential for broader applications in computational mathematics.

Example 3.2. We consider the mathematical problem defined by (1.1), represented in the following manner

$${}_0^cD_t^\alpha u(t) + \frac{1}{4}tu(t) = t^{-1}\int_0^t \frac{s}{t}su(s)ds + p(t), \tag{3.5}$$

Table 1: The outcome errors obtained and the convergence order for some values of α and τ in Example 3.1.

$\mathcal{T}$	τ	$\alpha = 0.25$	$\mathcal{C}$	$\alpha = 0.50$	$\mathcal{C}$	$\alpha = 0.75$	$\mathcal{C}$	$\alpha = 0.95$	$\mathcal{C}$
1	0.01	$3.1432e-05$	–	$2.9698e-05$	–	$4.7759e-05$	–	$1.1841e-04$	–
	0.01/2	$7.7888e-06$	2.0128	$7.0476e-06$	2.0752	$1.0709e-05$	2.1569	$2.8165e-05$	2.0718
	0.01/4	$1.9368e-06$	2.0077	$1.6962e-06$	2.0548	$2.4285e-06$	2.1407	$6.7365e-06$	2.0638
	0.01/8	$4.8258e-07$	2.0048	$4.1255e-07$	2.0397	$5.5631e-07$	2.1261	$1.6176e-06$	2.0581
	0.01/16	$1.2034e-07$	2.0037	$1.0117e-07$	2.0278	$1.2863e-07$	2.1127	$3.9026e-07$	2.0513
2	0.01	$4.1374e-04$	–	$4.1957e-04$	–	$5.3793e-04$	–	$9.6431e-04$	–
	0.01/2	$1.0298e-04$	2.0064	$1.0244e-04$	2.0341	$1.2669e-04$	2.0861	$2.3203e-04$	2.0552
	0.01/4	$2.5675e-05$	2.0039	$2.5185e-05$	2.0241	$3.0102e-05$	2.0734	$5.6102e-05$	2.0482
	0.01/8	$6.4082e-06$	2.0024	$6.2219e-06$	2.0171	$7.2056e-06$	2.0627	$1.3611e-05$	2.0433
	0.01/16	$1.6001e-06$	2.0018	$1.5427e-06$	2.0119	$1.7357e-06$	2.0536	$3.3147e-06$	2.0378
3	0.01	$1.8656e-03$	–	$2.0194e-03$	–	$2.5102e-03$	–	$3.8971e-03$	–
	0.01/2	$4.6491e-04$	2.0046	$4.9695e-04$	2.0228	$6.0281e-04$	2.0580	$9.4543e-04$	2.0434
	0.01/4	$1.1600e-04$	2.0028	$1.2287e-04$	2.0160	$1.4574e-04$	2.0483	$2.3032e-04$	2.0373
	0.01/8	$2.8967e-05$	2.0016	$3.0479e-05$	2.0112	$3.5425e-05$	2.0406	$5.6278e-05$	2.0330
	0.01/16	$7.2358e-06$	2.0012	$7.5785e-06$	2.0078	$8.6496e-06$	2.0341	$1.3792e-05$	2.0287

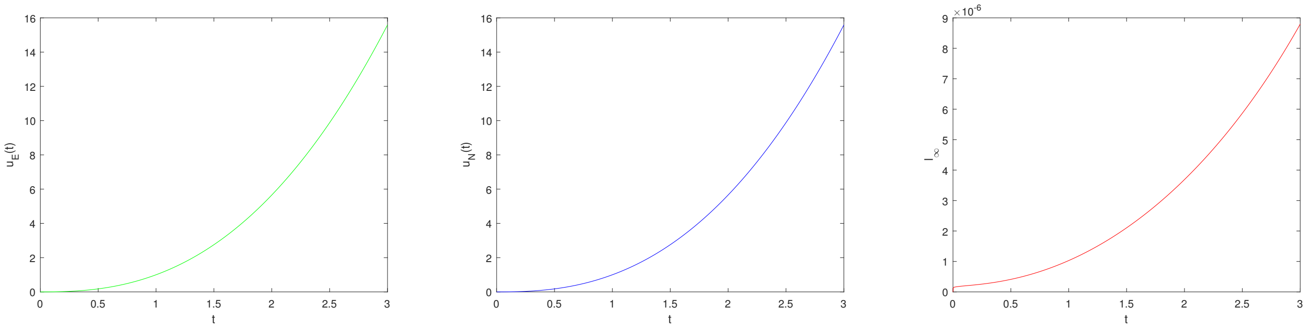

Figure 1: The exact solution (a), the numerical solution (b) and related absolute error (c) with $\alpha = 0.65$ and $\tau = 0.01/16$ in Example 3.1.

where

$$p(t) = \frac{\Gamma(\frac{9}{2})}{\Gamma(\frac{9}{2}-\alpha)} + \frac{5}{52} t^{\frac{9}{2}}. \tag{3.6}$$

Here, the homogeneous initial conditions and the non smooth exact solution is $u(t) = t^{\frac{7}{2}}$. The proposed numerical scheme was implemented to solve the given problem across multiple parameter configurations, with systematic variations in both the fractional order parameter α and the temporal step size τ over distinct computational intervals. A comprehensive error analysis was conducted, with particular emphasis on the ℓ_∞ norm of the absolute error and the corresponding empirical convergence rates, all of which are tabulated in detail in Table 2 for successive time intervals. To provide a more complete visualization of the method's performance, Figure 2 presents two key components: (i) the spatial distribution of the absolute error, and (ii) a comparative analysis between the exact solution and its numerical approximation. These results were generated using a refined temporal discretization with $\mathbf{N} = 1600$ time steps over the interval $\mathcal{T} = 3$, while maintaining $\alpha = 0.85$ as a representative case study. The numerical experiments yield several important observations:

- The ℓ_∞ error norms demonstrate consistent decay with temporal refinement, indicating proper convergence behavior.
- The computed convergence rates align well with theoretical expectations for problems of this class.
- The comparative solution plots in Figure 2 exhibit excellent agreement between exact and approximate solutions, with only minor deviations localized in expected regions.

These collective results not only validate the computational efficiency of the proposed method but also confirm its robustness in handling the mathematical challenges inherent to this category of integro-differential problems. The scheme's ability to maintain accuracy across different parameter regimes suggests its potential applicability to a broader range of related problems in applied mathematics.

Table 2: The outcome errors obtained and the convergence order for some values of α and τ in Example 3.2.

$\mathcal{T}$	τ	$\alpha = 0.25$	$\mathcal{C}$	$\alpha = 0.50$	$\mathcal{C}$	$\alpha = 0.75$	$\mathcal{C}$	$\alpha = 0.95$	$\mathcal{C}$
1	0.01	$3.5367e-05$	–	$3.9683e-05$	–	$8.7800e-05$	–	$2.3133e-04$	–
	0.01/2	$8.5901e-06$	2.0417	$8.8428e-06$	2.1659	$1.9272e-05$	2.1877	$5.6267e-05$	2.0396
	0.01/4	$2.1083e-06$	2.0266	$2.0153e-06$	2.1335	$4.2413e-06$	2.1839	$1.3653e-05$	2.0431
	0.01/8	$5.2100e-07$	2.0167	$4.6873e-07$	2.1042	$9.3750e-07$	2.1776	$3.3085e-06$	2.0450
	0.01/16	$1.2924e-07$	2.0112	$1.1092e-07$	2.0792	$2.0839e-07$	2.1695	$8.0095e-07$	2.0464
2	0.01	$8.5700e-04$	–	$9.0665e-04$	–	$1.4224e-03$	–	$2.9804e-03$	–
	0.01/2	$2.1137e-04$	2.0195	$2.1451e-04$	2.0795	$3.2581e-04$	2.1262	$7.2793e-04$	2.0336
	0.01/4	$5.2393e-05$	2.0123	$5.1421e-05$	2.0606	$7.5028e-05$	2.1185	$1.7742e-04$	2.0366
	0.01/8	$1.3029e-05$	2.0076	$1.2459e-05$	2.0452	$1.7387e-05$	2.1094	$4.3198e-05$	2.0381
	0.01/16	$3.2457e-06$	2.0051	$3.0439e-06$	2.0332	$4.0570e-06$	2.0995	$1.0510e-05$	2.0392
3	0.01	$5.4417e-03$	–	$5.8894e-03$	–	$8.1570e-03$	–	$1.4462e-02$	–
	0.01/2	$1.3483e-03$	2.0129	$1.4222e-03$	2.0500	$1.9186e-03$	2.0880	$3.5477e-03$	2.0273
	0.01/4	$3.3521e-04$	2.0080	$3.4648e-04$	2.0373	$4.5364e-04$	2.0804	$8.6878e-04$	2.0298
	0.01/8	$8.3510e-05$	2.0050	$8.4983e-05$	2.0275	$1.0786e-04$	2.0724	$2.1258e-04$	2.0310
	0.01/16	$2.0830e-05$	2.0033	$2.0953e-05$	2.0200	$2.5788e-05$	2.0644	$5.1988e-05$	2.0318

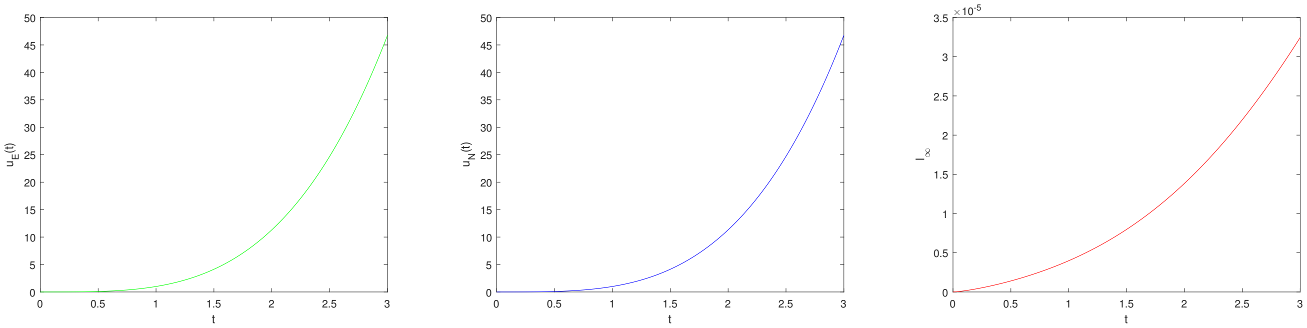

Figure 2: The exact solution (a), the numerical solution (b) and related absolute error (c) with $\alpha = 0.85$ and $\tau = 0.01/16$ in Example 3.2.

Example 3.3. Finally, the problem under consideration, as specified in (1.1), is formulated as follows

$$\begin{cases} {}_0^cD_t^\alpha u(t) + e^t u(t) = t^{-1}\displaystyle\int_0^t e^{t-s}u(s)ds + p(t), \quad t \in [0,1], \\ u(0) = 1, \end{cases} \tag{3.7}$$

and

$$p(t) = t^{1-\alpha}\mathbf{ML}_{1,2-\alpha}(t) + e^{2t} - e^t,$$

in which the true solution is $u(t) = e^t$ and **ML** is the generalized Mittag-Leffler function. The proposed methodology was implemented to resolve the given problem across varying parameters of α and τ. As illustrated in Table 3, the ℓ_∞ norm of the maximum error and the corresponding rate of convergence are documented for distinct α and τ values. The initial column of the table outlines the successively halved time step sizes τ for each computational run, while subsequent columns detail the associated ℓ_∞ error metrics and convergence rates for selected α values. Additionally, Figure 3 provides a visual representation of the absolute error distribution, alongside a comparative analysis between the exact solution and its numerical approximation, specifically for $\tau = 0.01/16$ and $\alpha = 0.65$. The obtained results demonstrate a high degree of precision, affirming the robustness and efficacy of the presented numerical scheme in addressing the problem with reliable performance.

Table 3: The outcome errors obtained and the convergence order for some values of α and τ in Example 3.3.

τ	$\alpha = 0.25$	$\mathcal{C}$	$\alpha = 0.50$	$\mathcal{C}$	$\alpha = 0.75$	$\mathcal{C}$	$\alpha = 0.95$	$\mathcal{C}$
0.01	$6.2567e-06$	–	$1.6042e-05$	–	$3.2840e-05$	–	$6.4390e-05$	–
0.01/2	$1.5941e-06$	1.9727	$4.0510e-06$	1.9855	$8.2372e-06$	1.9952	$1.5796e-05$	2.0273
0.01/4	$4.0522e-07$	1.9760	$1.0205e-06$	1.9890	$2.0642e-06$	1.9966	$3.8691e-06$	2.0295
0.01/8	$1.0278e-07$	1.9791	$2.5658e-07$	1.9918	$5.1691e-07$	1.9976	$9.6688e-07$	2.0006
0.01/16	$2.6019e-08$	1.9819	$6.4414e-08$	1.9940	$1.2937e-07$	1.9984	$2.7734e-07$	1.8017

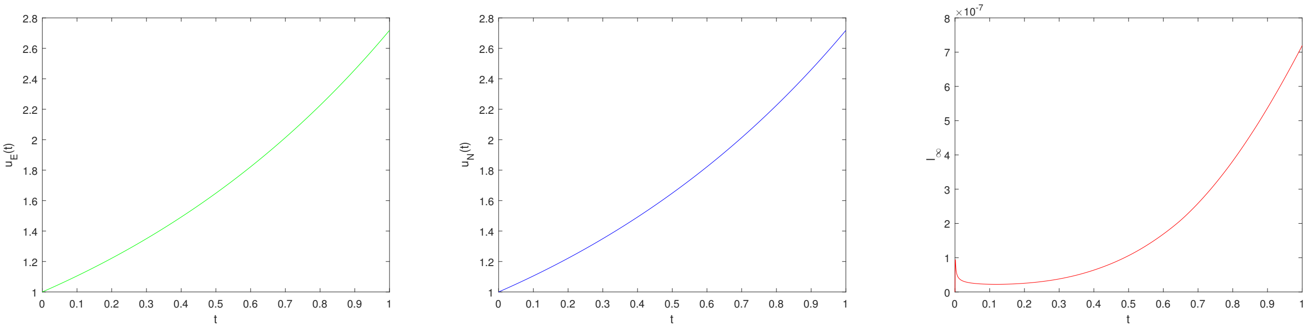

Figure 3: The exact solution (a), the numerical solution (b) and related absolute error (c) with $\alpha = 0.65$ and $\tau = 0.01/16$ in Example 3.3.

4 Conclusion

This study introduces a novel numerical algorithm designed for solving fractional cordial Volterra integro-differential equations. The proposed approach employs a hybrid discretization technique, integrating finite difference approximations with the composite trapezoidal rule. To evaluate the efficacy and robustness of the method, a series of illustrative test cases were examined. The numerical results confirm that the developed scheme is both accurate and computationally efficient, demonstrating its suitability for handling this class of problems.

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