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A three-step iterative process for G-quasinonexpansive multivalued mappings in Banach spaces endowed with graphs

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Abstract

In this paper, we introduce a three-step iterative process for G-quasinonexpansive multivalued mappings that satisfy the condition (G-E) in Banach spaces endowed with directed graphs. We establish both the weak and strong convergence results for the proposed iteration scheme. Finally, we present a numerical example to illustrate the theoretical results.

Keywords: G-quasinonexpansive mapping, Condition (G-E), Common fixed point.
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1. Introduction

Let X be a real Banach space and C a nonempty subset of X. The family of nonempty closed and bounded subsets of C is denoted by $CB(C)$, while the family of nonempty compact subsets of C is denoted by $K(C)$. We define

$$\operatorname{dist}(x, C) = \inf\{\|x - y\| : y \in C\}, \forall x \in X.$$

The Hausdorff metric H on $CB(C)$ is defined by

$$H(E, F) := \max\{\sup_{e \in E} \operatorname{dist}(e, F), \sup_{f \in F} \operatorname{dist}(f, E)\}, \text{ for all } E, F \in CB(C).$$

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A multivalued mapping $T : C \to CB(C)$ is called:
i) nonexpansive if $H(T(x), T(y)) \leq \|x - y\|$ for all $x, y \in C$.
ii) quasinonexpansive if $F(T) \neq \emptyset$ and $H(T(x), T(p)) \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$.
It is clear that every nonexpansive multivalued mapping T with $F(T) \neq \emptyset$ is quasinonexpansive.
In 2011, Garcia-Falset et al. [7] introduced a condition weaker than nonexpansiveness, called condition (E), for single-valued mappings. Later, Abkar and Eslamian [1] proposed a modified version of this condition for multivalued mappings, defined as follows:

Definition 1.1. *A multivalued mapping $T : C \to CB(C)$ is said to satisfy condition (E_μ) for some $\mu \geq 1$ if*

$$\mathrm{dist}(x, Ty) \leq \mu \mathrm{dist}(x, Tx) + \|x - y\|, \ \forall x, y \in C.$$

We say that T satisfies condition (E) if it satisfies condition (E_μ) for some $\mu \geq 1$.

It is known that the class of quasinonexpansive multivalued mappings differs from the class of mappings satisfying condition (E) [7].
In 2008, Jachymski [8] combined fixed point theory with graph theory to generalize the Banach fixed point theorem in a complete metric space endowed with a graph. Later, Beg et al. [4] extended this result to multivalued G-contractions. Aleomraninejad et al. [3] introduced several iterative schemes for G-contraction and G-nonexpansive mappings in real Banach space endowed with a graph.
Tripak [20] studied the Ishikawa iteration method to approximate common fixed points of two G-nonexpansive mappings in a Banach space endowed with a graph. Agarwal et al. [2] studied the S-iteration process for nearly asymptotically nonexpansive mappings and proved that it converges faster than the Ishikawa iteration for a class of contractions in metric spaces. Suparatulatorn et al. [17] proposed a modified S-iteration process for two G-nonexpansive mappings and established weak and strong convergence results in uniformly convex Banach space endowed with a graph. In 2011, Phuengrattana and Suantai [12] introduced the SP-iteration process and provided necessary and sufficient conditions for its convergence. Motivated by these studies, Sridarat et al. [15] modified the SP-iteration process for three G-nonexpansive mappings and proved several convergence theorems.
Recently, A. Kaewkhao et al. [9] studied the Ishikawa iteration process for G-quasinonexpansive mappings in a metric space endowed with a graph and established several convergence results.
Motivated by above mentioned studies, in this paper we introduce the condition (G-E) for multivalued mappings and propose a new three-step iteration process for three G-quasinonexpansive multivalued mappings satisfying the condition (G-E), which can be described as follows:
Let T_1, T_2, $T_3 : C \to CB(C)$ be three multivalued mappings with $\mathcal{F} = \cap_{i=1}^{3} F(T_i)$ nonempty. We define the sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ x'_n \in T_3 x_n, \\ z_n = (1 - \gamma_n) x_n + \gamma_n x'_n, \\ z'_n \in T_2 z_n, \\ y_n = (1 - \beta_n) x'_n + \beta_n z'_n, \\ y'_n \in T_1 y_n, \\ x_{n+1} = (1 - \alpha_n) y'_n + \alpha_n z'_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Additionally, we establish weak and strong convergence theorems for the proposed algorithm when the mappings are defined on uniformly convex Banach space endowed with a graph. Our results extend and improve the finding in [9, 10, 12, 15], and several other related works.

2. Preliminaries

Let X be a real Banach space. We use the notation $x_n \to x$ to denote strong convergence of the sequence $\{x_n\}$ to a point x, and $x_n \rightharpoonup x$ to denote the weak convergence of the sequence $\{x_n\}$ to a point $x \in X$. Let C be a nonempty subset of X. A point $p \in C$ is called a fixed point of a mapping $T : C \to CB(C)$ if $p \in T(p)$. The set of all fixed points of T is denoted by $F(T)$.

Definition 2.1. *Let C be a nonempty subset of a Banach space X. A sequence $\{x_n\}$ in X is called Fejér monotone with respect to C if*

$$\|x_{n+1} - q\| \leq \|x_n - q\|, \ \textit{for each} \ q \in C \quad \textit{and} \ n \in \mathbb{N}.$$

Definition 2.2. *[5] A Banach space X is called uniformly convex if for each $\epsilon \in (0, 2]$ there is a $\sigma > 0$ such that for every $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| > \epsilon$, it follows that $\frac{1}{2}\|x + y\| \leq 1 - \sigma$.*

Definition 2.3. *[11] A Banach space X is said to have Opial's property if, for every sequence $\{x_n\}$ in X that converges weakly to $x \in X$ and for every $y \neq x \in X$, the following inequality holds:*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

It is worth mentioning that Hilbert spaces and all l^p spaces $(1 < p < \infty)$ are examples of Banach spaces that satisfy Opial's property.

Recall that the mappings $T_i : C \to CB(C) \quad$ for $i = 1, 2, 3$ are said to satisfy condition (JC) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$ such that, for each $x \in C$,

$$\max\{\operatorname{dist}(x, T_1(x)), \operatorname{dist}(x, T_2(x)), \operatorname{dist}(x, T_3(x))\} \geq f(\operatorname{dist}(x, F)).$$

where, $\mathcal{F} = \cap_{i=1}^{3} F(T_i)$.

Lemma 2.4. *[14] Let X be a uniformly convex Banach space, and let $\{\alpha_n\}$ be a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Suppose that $\{x_n\}$ and $\{y_n\}$ in X satisfy $\limsup_{n\to\infty} \|x_n\| \leq c, \limsup_{n\to\infty} \|y_n\| \leq c$, and $\limsup_{n\to\infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = c$ for some $c \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5. *[16] Let X be a Banach space with Opial's property, and let $\{x_n\}$ be a sequence in X. Suppose $x, y \in X$ are such that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exist. If $\{x_{n_k}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ that converge weakly to x and y, respectively, then $x = y$.*

Lemma 2.6. *[13] Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If for any weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$, both $\{x_{n_j}\}$ and $\{x_{n_j+1}\}$ converge weakly to the same point in X, then the sequence $\{x_n\}$ converges weakly.*

Consider a directed graph G with a vertices set $V(G)$ and edges set $E(G)$. We assume that G contains all loops and has no parallel edges. For $u, v \in V(G)$, we say that u dominates v if $(u, v) \in E(G)$. Let C and D be nonempty subsets of $V(G)$, we say that C dominates D if $(u, v) \in E(G)$ for all $u \in C$ and $v \in D$.

Definition 2.7. *[19] Let X be a normed space, C a nonempty subset of X, and $G = (V(G), E(G))$ a directed graph with $V(G) = C$. Then C is said to have property (G) if, for any sequence $\{x_n\}$ in C satisfying $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, and $\{x_n\}$ converges weakly to $x \in C$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Definition 2.8. *Let X be a Banach space with a nonempty subset $C \subseteq X$, and $G = (V(G), E(G))$ be a directed graph with $V(G) \subseteq C$. A multivalued mapping $T : C \to CB(C)$ is called*
i) edge-preserving if for each $(x, y) \in E(G)$, with $u \in T(x)$ and $v \in T(y)$, it follows that $(u, v) \in E(G)$.
ii) G-nonexpansive if it is edge-preserving and

$$H(T(x), T(y)) \leq \|x - y\|, \text{ for all } (x, y) \in E(G).$$

iii) G-quasinonexpansive if it is edge-preserving and $F(T) \neq \emptyset$ and

$$H(T(x), T(p)) \leq \|x - p\|, \text{ for all } x \in C \text{ and } p \in F(T) \text{ with } (x, p) \in E(G).$$

Now, the given example demonstrates that the classes of quasinonexpansive mappings, G-quasinonexpansive mappings, and G-nonexpansive mappings are not identical.

Example 2.9. *Let $X = \mathbb{R}$, C be the closed interval $[0, 1]$, and G be a graph represented by $G = (V(G), E(G))$ such that $V(G) = (0, 1]$ and $E(G) = \{(x, y) : x, y \in V(G)\}$. Let $T : C \to CB(C)$ defined as follows:*

$$T(x) = \begin{cases} [\sqrt{x}, 1] & \text{if } 0 < x \leq 1 \\ \{0\} & \text{if } x = 0. \end{cases}$$

Clearly, the mapping T preserves edges and $F(T) = \{0, 1\}$. Since $(x, 0) \notin E(G)$, it follows that if $(x, y) \in E(G)$ with $y \in F(T)$, then $y = 1$. Therefore, we have

$$H(T(x), T(y)) = H([\sqrt{x}, 1], \{1\}) = 1 - \sqrt{x} \leq 1 - x = |x - y|.$$

This indicates that T is a G-quasinonexpansive mapping. However, if $x = \frac{1}{4}$ and $y = \frac{1}{16}$, then $(x, y) \in E(G)$, and

$$H(T(x), T(y)) = H(T(\frac{1}{4}), T(\frac{1}{16})) = H([\frac{1}{2}, 1], [\frac{1}{4}, 1]) = \frac{1}{4} > |x - y|.$$

Thus, it follows that the mapping T fails to be G-nonexpansive. Moreover, if $x = 0$ and $y = \frac{1}{4}$, then we obtain

$$H(T(x), T(y)) = H(\{0\}, [\frac{1}{2}, 1]) = 1 > \frac{1}{4} = |x - y|.$$

Consequently, T is not quasinonexpansive.

Definition 2.10. *Suppose X is a Banach space and C is a nonempty subset of X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G)$ is a subset of C. A mapping $T : C \to CB(C)$ is said to satisfy the condition (G-E_μ) for $\mu \geq 1$, if for all $x, y \in C$, with $(x, y) \in E(G)$ we have*

$$\operatorname{dist}(x, T(y)) \leq \mu \operatorname{dist}(x, T(x)) + \|x - y\|. \tag{2.1}$$

The mapping T satisfies condition (G-E) whenever it satisfies condition (G-E_μ) for some $\mu \geq 1$.

It is clear that every G-nonexpansive mapping satisfies condition (G-E_1). Moreover, for single-valued mappings, condition (G-E) implies G-quasinonexpansiveness.
Notice that the following example demonstrates that the class of multivalued mappings satisfies condition (E), while condition (G-E) differs from it.

Example 2.11. *Let $X = \mathbb{R}$, C be the closed interval $[0, 1]$, and G be a graph represented by $G = (V(G), E(G))$ such that $V(G) = [0, \frac{1}{2}]$ and $E(G) = V(G) \times V(G)$. Let $T : C \to CB(C)$ defined as follows:*

$$T(x) = \begin{cases} [0, x^2] & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \{1\} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

We show that T satisfies condition (G-E). Observe that the mapping T preserves edges. Let $(x, y) \in E(G)$. Then $0 \leq x, y \leq \frac{1}{2}$. Now, we are taking into account the following pair of cases:
Case 1: If $x \leq y^2$ then

$$\text{dist}(x, T(y)) = dist(x, [0, y^2]) = 0 \leq \mu\text{dist}(x, T(x)) + |x - y|, \ \text{for all } \mu \geq 1.$$

Case 2: If $y^2 < x$, then

$$\begin{aligned} \text{dist}(x, T(y)) = \text{dist}(x, [0, y^2]) &= |x - y^2| \leq |x - x^2| + |x^2 - y^2| \\ &\leq |x - x^2| + |x - y||x + y| \leq \text{dist}(x, T(x)) + |x - y|. \end{aligned}$$

This demonstrates that T satisfies condition (G-E). On the other hand, if $x = 1$, $y = \frac{1}{2}$, then

$$\text{dist}(x, T(y)) = \text{dist}(1, [0, \frac{1}{4}]) = \frac{3}{4} > |x - y| = \mu\text{dist}(x, T(x)) + |x - y|, \ \text{for all } \mu \geq 1.$$

This indicates that T fails to satisfy condition (E).

Additionally, it should be noted that the classes of G-quasinonexpansive multivalued mappings and the classes of mappings satisfying condition (G-E) are distinct.

Example 2.12. *Let $X = \mathbb{R}$, C be the closed interval $[0, 1]$, and G be a graph represented by $G = (V(G), E(G))$ such that $V(G) = (0, 1]$ and $E(G) = \{(x, y) : x, y \in V(G)\}$. Let $T : C \to CB(C)$ defined as follows:*

$$T(x) = \begin{cases} [\sqrt{x}, 1] & \text{if } 0 < x \leq 1, \\ \{0\} & \text{if } x = 0. \end{cases}$$

It can be seen that T preserves edges and $F(T) = \{0, 1\}$. Since $(x, 0) \notin E(G)$, Therefore, if $(x, y) \in E(G)$ with $y \in F(T)$, then $y = 1$. Hence

$$H(T(x), T(1)) = H([\sqrt{x}, 1], 1) = 1 - \sqrt{x} \leq |x - 1|.$$

Thus, T is G-quasinonexpansive. Now, for all $n \in \mathbb{N}$ let $x_n := \frac{1}{4n^2}$ and $y_n := \frac{1}{4}$, then

$$\frac{\text{dist}(x_n, T(y_n)) - |x_n - y_n|}{\text{dist}(x_n, T(x_n))} = \frac{\frac{1}{2} - \frac{1}{4n^2} - \frac{1}{4} + \frac{1}{4n^2}}{\frac{1}{2n} - \frac{1}{4n^2}} = \frac{n^2}{2n - 1} \to \infty.$$

As a result, the mapping T fails to satisfy condition (G-E).

Example 2.13. *Let $X = \mathbb{R}$, C be the closed interval $[0,\infty)$, and G be a graph represented by $G = (V(G), E(G))$ such that $V(G) = C$ and $E(G) = V(G) \times V(G)$. Let $T : C \to CB(C)$ defined as follows:*

$$T(x) = [x, 2x], \quad \textit{for all } x \in C.$$

Then T satisfies the condition of (G-E) while not being G-quasinonexpansive.

Definition 2.14. *Let $T : C \to CB(C)$ be a multivalued mapping. We say that $I - T$ is G-demiclosed at 0 if, for any sequence $\{x_n\}$ in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup x$, and $\lim_{n\to\infty} dist(x_n, T(x_n)) = 0$, it follows that $x \in F(T)$.*

Lemma 2.15. *Let C be a nonempty, closed, and convex subset of Banach space X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Let T_1, T_2, and $T_3 : C \to CB(C)$ be three multivalued mappings, and $\mathcal{F} = \cap_{i=1}^{3} F(T_i)$. If $\mathcal{F} \times \mathcal{F} \subseteq E(G)$ and $I - T_i$ is G-demiclosed at 0, for $i = 1, 2, 3$, then $\mathcal{F}$ is closed in X.*

Proof . Let $\{x_n\}$ be a sequence in $\mathcal{F}$ such that $\lim_{n\to\infty} x_n = x$. Then $(x_n, x_{n+1}) \in E(G)$ and $dist(x_n, T_i x_n) = 0$ for $i = 1, 2, 3$. Since $I - T_i$ is G-demiclosed for each i, we conclude that $x \in \mathcal{F}$. Hence, $\mathcal{F}$ is closed. $\square$

Proposition 2.16. *Let C be a nonempty, closed, and convex subset of Banach space X that fulfills Opial's condition. Suppose $G = (V(G), E(G))$ be a directed graph, with $V(G) = C$, and C possesses property G. Let $T : C \to K(C)$ satisfy condition (G-E). Then $I - T$ is G-demiclosed at 0.*

Proof . Assume that $\{x_n\}$ be a sequence in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup x$ and $\lim_{n\to\infty} dist(x_n, T(x_n)) = 0$. Our objective is to demonstrate that $x \in T(x)$. By definition of T, $T(x)$ is compact, so we can choose $z_n \in T(x)$ such that $\|x_n - z_n\| = dist(x_n, T(x))$. Moreover, by compactness of $T(x)$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\to\infty} z_{n_k} = z \in T(x)$. Since C has property G, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Thus, by the definition of condition (G-E), we have for some $\mu \geq 1$,

$$\mathrm{dist}(x_{n_k}, T(x)) \leq \mu \mathrm{dist}(x_{n_k}, T(x_{n_k})) + \|x_{n_k} - x\|$$

Therefore, we obtain

$$\begin{aligned} \|x_{n_k} - z\| &\leq \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - z\| \\ &= \mathrm{dist}(x_{n_k}, T(x)) + \|z_{n_k} - z\| \\ &\leq \mu \mathrm{dist}(x_{n_k}, T(x_{n_k})) + \|x_{n_k} - x\| + \|z_{n_k} - z\| \end{aligned}$$

This implies that

$$\limsup_{k\to\infty} \|x_{n_k} - z\| \leq \limsup_{k\to\infty} \|x_{n_k} - x\|.$$

In accordance with the Opial's condition, we obtain $x = z \in T(x)$. $\square$

Definition 2.17. *[6] Let X be a vector space, C be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then $E(G)$ is said to be coordinate-convex if for all $(x, y), (x, z), (y, x), (z, x) \in E(G)$ and for all $\gamma \in [0, 1]$, we have*

$$\gamma(x, y) + (1 - \gamma)(x, z) \in E(G) \ \textit{ and } \ \gamma(y, x) + (1 - \gamma)(z, x) \in E(G).$$

Also, $E(G)$ is convex if for any $(x, z), (y, w) \in E(G)$ and $\gamma \in [0, 1]$, we have $((1 - \gamma)x + \gamma y, (1 - \gamma)z + \gamma w) \in E(G)$.

It is important to note that when $E(G)$ is convex, it implies that $E(G)$ is also coordinate-convex, but the converse is not necessarily true [18].

Definition 2.18. *A directed graph $G = (V(G), E(G))$ is said to be transitive if, for all $u, v, w \in V(G)$, the conditions $(u, v) \in E(G)$ and $(v, w) \in E(G)$ imply that $(u, w) \in E(G)$.*

3. Main results

In this section, we assume that X is a uniformly convex Banach space endowed with a directed graph G. Let C be a nonempty, closed, and convex subset of X with $V(G) = C$. We further assume that $E(G)$ is coordinate-convex and G is transitive. The sequence $\{x_n\}$ is defined by (1.1). We start this section with a useful lemma.

Lemma 3.1. *Let $T_1, T_2, T_3 : C \to CB(C)$ be three edge-preserving mappings. Let $p \in \mathcal{F}$ be such that $(x_1, p), (p, x_1) \in E(G)$. Then $(x_n, p), (z_n, p), (y_n, p), (p, x_n)$, $(p, z_n), (p, y_n), (x_n, y_n), (x_n, z_n), (y_n, x_n), (z_n, x_n)$, and (x_n, x_{n+1}) are in $E(G)$.*

Proof . Because $(x_1, p) \in E(G)$, with $x_1' \in T_3(x_1)$ and $p \in T_3(p)$, the edge-preserving property of T_3 implies $(x_1', p) \in E(G)$. Moreover, using $(x_1, p), (x_1', p) \in E(G)$ together with the coordinate-convexity of $E(G)$, we obtain

$$\begin{aligned}(z_1, p) &= ((1-\gamma_1)x_1 + \gamma_1 x_1', p)\\ &= (1-\gamma_1)(x_1, p) + \gamma_1(x_1', p) \in E(G).\end{aligned}$$

Again, since $(z_1, p) \in E(G)$, $z_1' \in T_2(z_1)$, and $p \in T_2(p)$, by the edge-preserving property of T_2 we have $(z_1', p) \in E(G)$. Because $(x_1', p), (z_1', p) \in E(G)$ using the coordinate-convexity of $E(G)$, it follows that

$$\begin{aligned}(y_1, p) &= ((1-\beta_1)x_1' + \beta_1 z_1', p)\\ &= (1-\beta_1)(x_1', p) + \beta_1(z_1', p) \in E(G).\end{aligned}$$

Similarly, given that $(y_1, p) \in E(G)$, $y_1' \in T_1(y_1)$, and $p \in T_1(p)$, the edge-preserving property of T_1 that $(y_1', p) \in E(G)$. By the coordinate-convexity of $E(G)$ and $(z_1', p), (y_1', p) \in E(G)$, hence

$$\begin{aligned}(x_2, p) &= ((1-\alpha_1)y_1' + \alpha_1 z_1', p)\\ &= (1-\alpha_1)(y_1', p) + \alpha_1(z_1', p) \in E(G).\end{aligned}$$

In the same way, we can show that $(z_2, p), (y_2, p), (x_3, p) \in E(G)$. Hence, for each $n \geq 1$ we have $(x_n, p), (y_n, p), (z_n, p) \in E(G)$.
By a similar argument, since T_3 preserves edges and $(p, x_1) \in E(G)$, we can conclude that $(p, x_1') \in E(G)$. Using $(p, x_1), (p, x_1') \in E(G)$ and the coordinate-convexity of $E(G)$, we have $(p, z_1) \in E(G)$. Again, since T_2 preserves edges, we have $(p, z_1') \in E(G)$. By the coordinate-convexity of $E(G)$ and $(p, x_1'), (p, z_1') \in E(G)$, so $(p, y_1) \in E(G)$. Since T_1 preserves edges, we get $(p, y_1') \in E(G)$. By the coordinate-convexity of $E(G)$ and $(p, y_1'), (p, z_1') \in E(G)$, so $(p, x_2) \in E(G)$. Using an analogous reasoning, we deduce that $(p, z_2), (p, y_2), (p, x_3) \in E(G)$. Therefore for each $n \geq 1$ we conclude that $(p, x_n), (p, y_n), (p, z_n) \in E(G)$. By transitivity of G, we obtain that $(x_n, y_n), (x_n, z_n), (y_n, x_n)$, (z_n, x_n) and (x_n, x_{n+1}) are in $E(G)$. This completes the proof.
$\square$

Lemma 3.2. *Let $T_1, T_2, T_3 : C \to CB(C)$ be mappings satisfying $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, 3$. Suppose that $(x_1, p), (p, x_1) \in E(G)$ for arbitrary $x_1 \in C$ and all $p \in \mathcal{F}$. Then the following statements hold:*
(i) If the mappings T_1, T_2, and T_3 are G-quasinonexpansive then $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is Fejér monotone with respect to $\mathcal{F}$.
(ii) If the mappings T_1, T_2, and T_3 are G-quasinonexpansive and T_1, T_2 satisfy the condition (G-E) then

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_1(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_2(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_3(x_n)) = 0.$$

Proof . (i) Assume that $p \in \mathcal{F}$. Since $(x_1, p), (p, x_1) \in E(G)$ for arbitrary $x_1 \in C$, and by definition of G-quasinonexpansive mapping, T_1, T_2, and T_3 are edge-preserving, Lemma 3.1 implies that (x_n, p),(y_n, p),$(z_n, p) \in E(G)$.
Since $(x_n, p) \in E(G)$, $T_3(p) = \{p\}$, and T_3 is G-quasinonexpansive, we obtain that

$$\begin{aligned} \|z_n - p\| &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|x'_n - p\| \\ &= (1-\gamma_n)\|x_n - p\| + \gamma_n \operatorname{dist}(x'_n, T_3(p)) \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n H(T_3(x_n), T_3(p)) \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| = \|x_n - p\|, \end{aligned} \tag{3.1}$$

From $(x_n, P), (z_n, p) \in E(G)$, T_2 and T_3 are G-quasinonexpansive, and $T_2(p) = T_3(p) = \{p\}$, it follows from (3.1) that

$$\begin{aligned} \|y_n - p\| &\le (1-\beta_n)\|x'_n - p\| + \beta_n\|z'_n - p\| \\ &= (1-\beta_n)\operatorname{dist}(x'_n, T_3(p)) + \beta_n \operatorname{dist}(z'_n, T_2(p)) \\ &\le (1-\beta_n)H(T_3(x_n), T_3(p)) + \beta_n H(T_2(z_n), T_2(p)) \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \le \|x_n - p\|. \end{aligned} \tag{3.2}$$

In view of $(y_n, p), (z_n, p) \in E(G)$, T_1 and T_2 are G-quasinonexpansive and $T_1(p) = T_2(p) = \{p\}$, and as a result of (3.1) and (3.2), we conclude that

$$\begin{aligned} \|x_{n+1} - p\| &\le (1-\alpha_n)\|y'_n - p\| + \alpha_n\|z'_n - p\| \\ &= (1-\alpha_n)\operatorname{dist}(y'_n, T_1(p)) + \alpha_n \operatorname{dist}(z'_n, T_2(p)) \\ &\le (1-\alpha_n)H(T_1(y_n), T_1(p)) + \alpha_n H(T_2(z_n), T_2(p)) \\ &\le (1-\alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \le \|x_n - p\|. \end{aligned} \tag{3.3}$$

Thus, $\{\|x_n - p\|\}$ is decreasing and bounded below, so $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence, $\{x_n\}$ is Fejér monotone with respect to $\mathcal{F}$. Consequently, the sequence $\{x_n\}$ is bounded.

(ii) Suppose that $\lim_{n\to\infty} \|x_n - p\| = c$, then for all $w^i_n \in T_i(x_n)$ we conclude that

$$\operatorname{dist}(x_n, T_i(x_n)) \le \|x_n - w^i_n\| \le \|x_n - p\| + \|p - w^i_n\|.$$

Therefore, we have

$$\operatorname{dist}(x_n, T_i(x_n)) \le \|x_n - p\| + \operatorname{dist}(p, T_i(x_n)).$$

Since $(x_n, p) \in E(G)$, and by the G-quasinonexpansiveness of T_i, we can get

$$\begin{aligned}\operatorname{dist}(x_n, T_i(x_n)) &\leq \|x_n - p\| + \operatorname{dist}(p, T_i(x_n))\\ &\leq \|x_n - p\| + H(T_i(x_n), T_i(p))\\ &\leq \|x_n - p\| + \|x_n - p\|.\end{aligned}$$

If $c = 0$, then the result holds. Assume that $c > 0$, By taking lim sup of both sides of (3.2), we can get

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c. \tag{3.4}$$

Since $(y_n, p) \in E(G)$, by utilizing the G-quasinonexpansiveness of T_1, we conclude that

$$\|y_n' - p\| = \operatorname{dist}(y_n', T_1(p)) \leq H(T_1(y_n), T_1(p)) \leq \|y_n - p\|. \tag{3.5}$$

Taking the lim sup on both sides of the inequality and using (3.4), we obtain

$$\limsup_{n\to\infty} \|y_n' - p\| \leq c. \tag{3.6}$$

By applying the lim sup to both sides of (3.1), we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c. \tag{3.7}$$

From $(z_n, p) \in E(G)$ and the G-quasinonexpansiveness of T_2, it follows that

$$\|z_n' - p\| = \operatorname{dist}(z_n', T_2(p)) \leq H(T_2(z_n), T_2(p)) \leq \|z_n - p\|. \tag{3.8}$$

Using the lim sup on both sides of (3.7), we deduces

$$\limsup_{n\to\infty} \|z_n' - p\| \leq c. \tag{3.9}$$

Furthermore, by (1.1), we obtain

$$c = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1 - \alpha_n)(y_n' - p) + \alpha_n(z_n' - p)\|. \tag{3.10}$$

By (3.6), (3.9), (3.10), and Lemma 2.4, it follows that

$$\lim_{n\to\infty} \|y_n' - z_n'\| = 0. \tag{3.11}$$

Additionally, we can observe that

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - \alpha_n)\|y_n' - p\| + \alpha_n\|z_n' - p\|\\ &\leq (1 - \alpha_n)(\|y_n' - z_n'\| + \|z_n' - p\|) + \alpha_n\|z_n' - p\|\\ &= (1 - \alpha_n)\|y_n' - z_n'\| + \|z_n' - p\|.\end{aligned} \tag{3.12}$$

combining (3.11) and (3.12), we obtain that

$$\liminf_{n\to\infty} \|z_n' - p\| \geq c. \tag{3.13}$$

From (3.9) and (3.13), we obtain

$$\lim_{n\to\infty} \|z'_n - p\| = c. \tag{3.14}$$

Furthermore, applying the triangle inequality and (3.5), we obtain that

$$\begin{aligned}\|z'_n - p\| &\leq \|z'_n - y'_n\| + \|y'_n - p\| \\ &\leq \|z'_n - y'_n\| + \|y_n - p\|.\end{aligned}$$

In view of (3.11), we conclude that

$$\liminf_{n\to\infty} \|y_n - p\| \geq c. \tag{3.15}$$

And so, by (3.4) and (3.15), we conclude

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.16}$$

Since $(x_n, p) \in E(G)$, due to the G-quasinonexpansiveness of T_3, we conclude

$$\|x'_n - p\| = \operatorname{dist}(x'_n, T_3(p)) \leq H(T_3(x_n), T_3(p)) \leq \|x_n - p\|.$$

After applying the $\limsup$ to both sides of this inequality, we conclude that

$$\limsup_{n\to\infty} \|x'_n - p\| \leq c. \tag{3.17}$$

Using (1.1) and (3.16), we have

$$\lim_{n\to\infty} \|(1-\beta_n)(x'_n - p) + \beta_n(z'_n - p)\| = c. \tag{3.18}$$

Utilizing (3.9), (3.17), and (3.18), along with Lemma 2.4, the following result can be derived

$$\lim_{n\to\infty} \|x'_n - z'_n\| = 0. \tag{3.19}$$

Furthermore, it can be established that

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|y'_n - p\| + \alpha_n\|z'_n - p\| \\ &\leq (1-\alpha_n)\|y'_n - p\| + \alpha_n(\|z'_n - y'_n\| + \|y'_n - p\|) \\ &= \|y'_n - p\| + \alpha_n\|z'_n - y'_n\|.\end{aligned} \tag{3.20}$$

By (3.11) and (3.20), we obtain

$$\liminf_{n\to\infty} \|y'_n - p\| \geq c. \tag{3.21}$$

And so, by (3.6) and (3.21), we obtain

$$\lim_{n\to\infty} \|y'_n - p\| = c. \tag{3.22}$$

Moreover, by using the triangle inequality and (3.8), we have

$$\begin{aligned}\|y'_n - p\| &\leq \|y'_n - z'_n\| + \|z'_n - p\| \\ &\leq \|y'_n - z'_n\| + \|z_n - p\|.\end{aligned}$$

Thus, by (3.11), we have

$$\liminf_{n\to\infty} \|z_n - p\| \geq c. \tag{3.23}$$

Therefore, using (3.7) and (3.23), we have

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{3.24}$$

Using (1.1) and (3.24), we have

$$\lim_{n\to\infty} \|(1-\gamma_n)(x_n - p) + \gamma_n(x'_n - p)\| = c. \tag{3.25}$$

Utilizing (3.17), (3.25), and Lemma 2.4, we obtain the following result

$$\lim_{n\to\infty} \|x_n - x'_n\| = 0. \tag{3.26}$$

Since $\operatorname{dist}(x_n, T_3(x_n)) \leq \|x_n - x'_n\|$ then, by (3.26) we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_3(x_n)) = 0. \tag{3.27}$$

Through the utilization of equations (1.1) and (3.26), it follows that

$$\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \gamma_n \|x_n - x'_n\| = 0. \tag{3.28}$$

By (1.1), we obtain that

$$\begin{aligned}
\|y_n - x_n\| &= \|(1-\beta_n)(x'_n - x_n) + \beta_n(z'_n - x_n)\| \\
&\leq (1-\beta_n)\|x'_n - x_n\| + \beta_n\|z'_n - x_n\| \\
&\leq (1-\beta_n)\|x'_n - x_n\| + \beta_n(\|z'_n - x'_n\| + \|x'_n - x_n\|) \\
&= \|x'_n - x_n\| + \beta_n\|z'_n - x'_n\|.
\end{aligned} \tag{3.29}$$

By using (3.19), (3.26) and (3.29), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.30}$$

By our assumptions and Lemma 3.1, we have $(y_n, x_n) \in E(G)$. Therefore, since T_1 satisfies condition (G-E), there exist $\mu \geq 1$ such that

$$\operatorname{dist}(y_n, T_1(x_n)) \leq \mu \operatorname{dist}(y_n, T_1(y_n)) + \|x_n - y_n\|.$$

Thus, from triangle inequality, it follows that

$$\begin{aligned}
\operatorname{dist}(x_n, T_1(x_n)) &\leq \|x_n - y_n\| + \operatorname{dist}(y_n, T_1(x_n)) \\
&\leq \|x_n - y_n\| + \mu \operatorname{dist}(y_n, T_1(y_n)) + \|x_n - y_n\| \\
&\leq 2\|x_n - y_n\| + \mu(\|y_n - x_n\| + \operatorname{dist}(x_n, T_1(y_n))) \\
&\leq (2+\mu)\|x_n - y_n\| + \mu\|x_n - y'_n\| \\
&\leq (2+\mu)\|x_n - y_n\| + \mu(\|x_n - x'_n\| + \|x'_n - z'_n\| + \|z'_n - y'_n\|).
\end{aligned} \tag{3.31}$$

By using (3.11), (3.19), (3.26), (3.30), and (3.31) , we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_1(x_n)) = 0. \tag{3.32}$$

Similarly, by Lemma 3.1, we have $(z_n, x_n) \in E(G)$. Since T_2 satisfies condition (G-E), there exists $\mu \geq 1$ such that

$$\operatorname{dist}(z_n, T_2(x_n)) \leq \mu \operatorname{dist}(z_n, T_2(z_n)) + \|x_n - z_n\|.$$

Hence, by triangle inequality, it follows that

$$\begin{aligned}
\operatorname{dist}(x_n, T_2(x_n)) &\leq \|x_n - z_n\| + \operatorname{dist}(z_n, T_2(x_n)) \\
&\leq \|x_n - z_n\| + \mu \operatorname{dist}(z_n, T_2(z_n)) + \|x_n - z_n\| \\
&\leq 2\|x_n - z_n\| + \mu(\|z_n - x_n\| + \operatorname{dist}(x_n, T_2(z_n))) \\
&\leq (2+\mu)\|x_n - z_n\| + \mu\|x_n - z'_n\| \\
&\leq (2+\mu)\|x_n - z_n\| + \mu(\|x_n - x'_n\| + \|x'_n - z'_n\|).
\end{aligned} \tag{3.33}$$

By employing equation (3.19), (3.26), (3.28), and (3.33), we deduce the following

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_2(x_n)) = 0. \tag{3.34}$$

Thus, by (3.27), (3.32) and (3.34), we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_1(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_2(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_3(x_n)) = 0.$$

This completes the proof. □

Theorem 3.3. *Suppose that X satisfies Opial's condition and C have property G. Let the mappings $T_1, T_2, T_3 : C \to CB(C)$ be G-quasinonexpansive mappings with $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, 3$. Assume that T_1 and T_2 satisfying the condition (G-E), $I - T_i$ is G-demiclosed at 0 for $i = 1, 2, 3$, and $(x_1, p), (p, x_1) \in E(G)$ for arbitrary $x_1 \in C$ and all $p \in \mathcal{F}$. Then the sequence $\{x_n\}$ defined by (1.1) converges weakly to an element of $\mathcal{F}$.*

Proof . Lemma 3.2 states that $\lim_{n\to\infty} \|x_n - p\|$ exists, so $\{x_n\}$ is bounded. Moreover, we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_1(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_2(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_3(x_n)) = 0.$$

Since X is uniform convex and $\{x_n\}$ is bounded, we may, without loss of generality, assume that $x_n \rightharpoonup x$. Since $(x_1, p), (p, x_1) \in E(G)$, it follows from Lemma 3.1 that $(x_n, x_{n+1}) \in E(G)$. Hence, by the G-demiclosedness of $I - T_i$ at 0 for $i = 1, 2, 3$, we obtain that $x \in \mathcal{F}$. To demonstrate that $\{x_n\}$ converges weakly to an element in $\mathcal{F}$, it is sufficient to show that $\{x_n\}$ has only one weak subsequential limit in $\mathcal{F}$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_m}\}$ of $\{x_n\}$ converge weakly to x and y, respectively. Using Lemma 3.2, we conclude that $\lim_{k\to\infty} \operatorname{dist}(x_{n_k}, T_i(x_{n_k})) = \lim_{m\to\infty} \operatorname{dist}(x_{n_m}, T_i(x_{n_m})) = 0$, for $i = 1, 2, 3$. Since $(x_n, x_{n+1}) \in E(G)$ and $I - T_i$ is G-demiclosed at 0, thus $x, y \in \mathcal{F}$. By applying Lemma 3.2, we can deduce that both $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exists, and utilizing Lemma 2.5 we can conclude that $x = y$. This completes the proof. □

Next, we establish the weak convergence of the sequence $\{x_n\}$ generated by (1.1) for G-quasinonexpansive multivalued mappings under condition (G-E), without relying on the assumption of Opial's condition.

Theorem 3.4. *Suppose that C possesses property G. Let $T_1, T_2, T_3 : C \to CB(C)$ be G-quasinonexpansive mappings with $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, 3$, where T_1 and T_2 satisfying the condition (G-E). If $(I - T_i)$ $(i = 1, 2, 3)$ is G-demiclosed at 0, and $(x_1, p), (p, x_1) \in E(G)$ for arbitrary $x_1 \in C$ and all $p \in F$. Then, $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.*

Proof . According to Lemma 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$, and, moreover, $\{x_n\}$ is bounded. Since C is a nonempty, closed, and convex subset of a uniformly convex Banach space X, it is weakly compact. Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to a point $q \in C$. From Lemma 3.2 (ii), we have

$$\lim_{j\to\infty} \operatorname{dist}(x_{n_j}, T_1(x_{n_j})) = \lim_{j\to\infty} \operatorname{dist}(x_{n_j}, T_2(x_{n_j})) = \lim_{j\to\infty} \operatorname{dist}(x_{n_j}, T_3(x_{n_j})) = 0. \tag{3.35}$$

Furthermore, by the triangle inequality, we obtain

$$\|y_n - z_n\| \leq \|y_n - x_n\| + \|x_n - z_n\|.$$

From equations (3.28) and (3.30), it follows that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.36}$$

Furthermore

$$\|z_n - z'_n\| \leq \|z_n - x_n\| + \|x_n - x'_n\| + \|x'_n - z'_n\|.$$

From (3.28), (3.26), and (3.19), we deduce

$$\lim_{n\to\infty} \|z_n - z'_n\| = 0. \tag{3.37}$$

Moreover

$$\|y_n - y'_n\| \leq \|y_n - x_n\| + \|x_n - x'_n\| + \|x'_n - z'_n\| + \|z'_n - y'_n\|.$$

By considering (3.30), (3.26), (3.19), and (3.11), we derive

$$\lim_{n\to\infty} \|y_n - y'_n\| = 0. \tag{3.38}$$

Due to the G-demiclosedness of $I - T_i$ at 0 for $i = 1, 2, 3$, it follows that $q \in \mathcal{F}$. Finally, we will establish that $\{x_n\}$ converges weakly to q. To do this, we need to show that $\{x_n\}$ satisfies the assumptions of Lemma 2.6. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ that converges weakly to some $p_0 \in C$. Using analogous reasoning, we can establish that $p_0 \in \mathcal{F}$.
Now, using (3.26) and the fact that $x_{n_j} \rightharpoonup p_0$, we have $x'_{n_j} \rightharpoonup p_0$. Now, according to (1.1), we have

$$z_{n_j} = (1 - \gamma_{n_j})x_{n_j} + \gamma_{n_j} x'_{n_j} \rightharpoonup p_0. \tag{3.39}$$

From equations (3.37) and (3.39), we have $z'_{n_j} \rightharpoonup p_0$. Then, by considering (1.1), it follows that

$$y_{n_j} = (1 - \beta_{n_j})x'_{n_j} + \beta_{n_j} z'_{n_j} \rightharpoonup p_0. \tag{3.40}$$

From (3.38) and (3.40), we have $y'_{n_j} \rightharpoonup p_0$. Now, by using (1.1), we have

$$x_{n_j+1} = (1 - \alpha_{n_j})y'_{n_j} + \alpha_{n_j} z'_{n_j} \rightharpoonup p_0.$$

Therefore, since the sequence $\{x_n\}$ satisfies the assumptions of Lemma 2.6, we can conclude that $\{x_n\}$ converges weakly to p_0, which implies that $p_0 = q$. This completes the proof.
□

Theorem 3.5. *Suppose that X satisfies Opial's condition and C has property G. Let $T_1, T_2, T_3 : C \to K(C)$ be G-quasinonexpansive mappings with $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, 3$, where T_1 and T_2 satisfying the condition (G-E). If $(x_1, p), (p, x_1) \in E(G)$ for arbitrary $x_1 \in C$ and all $p \in \mathcal{F}$. Then, $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.*

Proof . According to Proposition 2.16 we have $I - T_1, I - T_2$, and $I - T_3$ are G-demiclosed at 0. Hence, by Theorem 3.3, $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. This completes the proof. □

Theorem 3.6. *Assume that the set C has Property G. Let $T_1, T_2, T_3 : C \to CB(C)$ be G-quasinonexpansive mappings with $T_i(p) = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, 3$ and satisfying condition (JC). Suppose that T_1 and T_2 satisfy the condition (G-E). If $\mathcal{F} \times \mathcal{F} \subseteq E(G)$, and $(x_1, p), (p, x_1) \in E(G)$ for arbitrary $x_1 \in C$ and all $p \in \mathcal{F}$. Then $\{x_n\}$ converges strongly to an element of the set $\mathcal{F}$.*

Proof . From Lemma 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$. Consequently, $\lim_{n\to\infty} dist(x_n, \mathcal{F})$ exists, and

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_1(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_2(x_n)) = \lim_{n\to\infty} \operatorname{dist}(x_n, T_3(x_n)) = 0.$$

By condition (JC), it follows that $\lim_{n\to\infty} f(\operatorname{dist}(x_n, \mathcal{F})) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $f(0) = 0, f(r) > 0$ for each $r \in (0, 1)$, we conclude that $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$. Consequently, we obtain a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{q_j\} \subset \mathcal{F}$ such that $\|x_{n_j} - q_j\| \leq \frac{1}{2^j}$. Set $n_{j+1} = n_j + k$ for some $k \in \mathbb{N}$. From (3.3), we obtain

$$\|x_{n_{j+1}} - q_j\| \leq \|x_{n_{j+k-1}} - q_j\| \leq \|x_{n_j} - q_j\| \leq \frac{1}{2^j}.$$

Hence,

$$\|q_{j+1} - q_j\| \leq \|q_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - q_j\| \leq \frac{1}{2^{j+1}} + \frac{1}{2^j} = \frac{3}{2^{j+1}}.$$

Thus, $\{q_j\}$ is a Cauchy sequence. Suppose that $\lim_{j\to\infty} q_j = q_0 \in C$. By utilizing Lemma 2.15, we conclude that $\mathcal{F}$ is closed. So, $q_0 \in \mathcal{F}$. Thus, we have $\lim_{j\to\infty} x_{n_j} = q_0 \in C$. Due to the fact that $\lim_{n\to\infty} \|x_n - q_0\|$ exists, we conclude $\lim_{n\to\infty} x_n = q_0$. This completes the proof. □

4. Examples and numerical results

In this section, we demonstrate the effectiveness of our iterative method through the following example and numerical results.

Example 4.1. *Let $X = \mathbb{R}$, C be the closed interval $[0, 2]$, and G be a directed graph represented $G = (V(G), E(G))$ such that $V(G) = C$ and $(x, y) \in E(G)$ if and only if $\frac{1}{4} \leq x \neq y \leq 2$ or $x = y \in C$. In this case $E(G)$ is coordinate-convex and $\{(x, x) : x \in V(G)\} \subset E(G)$. Let $T_1, T_2, T_3 : C \to CB(C)$ are mappings defined as follows:*

$$T_1(x) = \begin{cases} \{\frac{1}{16}\} & if \;\; 0 \leq x < \frac{1}{4} \\ [x^{\frac{1}{2}}, 1] & if \;\; \frac{1}{4} \leq x \leq 1 \\ [1, x^{\frac{1}{2}}] & if \;\; 1 \leq x \leq 2, \end{cases}$$

$$T_2(x) = \begin{cases} \{\frac{1}{16}\} & if \;\; 0 \leq x < \frac{1}{4} \\ [x^{\frac{1}{3}}, 1] & if \;\; \frac{1}{4} \leq x \leq 1 \\ [1, x^{\frac{1}{3}}], & if \;\; 1 \leq x \leq 2, \end{cases}$$

and

$$T_3(x) = \begin{cases} \{\frac{1}{16}\} & if \;\; 0 \leq x < \frac{1}{4} \\ [2^{\frac{3}{4}(x-1)}, 1] & if \;\; \frac{1}{4} \leq x \leq 1 \\ [1, 2^{\frac{3}{4}(x-1)}] & if \;\; 1 \leq x \leq 2. \end{cases}$$

It is evident that T_1, T_2, and T_3 are edge-preserving, and $\mathcal{F} = \cap_{i=1}^{3} F(T_i) = \{1\}$. We show that T_1 is G-quasinonexpansive. Let $(x, p) \in E(G)$ with $p \in F(T_1)$. It follows that $p = 1$ and $\frac{1}{4} \leq x \leq 2$. Therefore, we distinguish the following two cases:
Case 1: if $\frac{1}{4} \leq x \leq 1$, then we have

$$H(T_1(x), T_1(1)) = H([x^{\frac{1}{2}}, 1], \{1\}) = 1 - x^{\frac{1}{2}} \leq |x - 1|,$$

Case 2: if $1 \leq x \leq 2$, then we have

$$H(T_1(x), T_1(1)) = H([1, x^{\frac{1}{2}}], \{1\}) = x^{\frac{1}{2}} - 1 \leq |x - 1|,$$

Thus, T_1 is G-quasinonexpansive. Similarly, we can show that T_2 and T_3 are G-quasinonexpansive. We note that $T_1(1) = T_2(1) = T_3(1) = \{1\}$. Next, we show that T_1 satisfies condition (G-E). To this end, let $(x, y) \in E(G)$. Then $\frac{1}{4} \leq x \neq y \leq 2$ or $x = y \in C$. If $x = y \in C$, the condition obviously holds. Thus, we only need to consider the case $\frac{1}{4} \leq x \neq y \leq 2$. In this case, we consider the following four subcases:
Subcase 1: Let $\frac{1}{4} \leq x \neq y \leq 1$. If $x \geq y^{\frac{1}{2}}$ then we have $\mathrm{dist}(x, T_1(y)) = 0$ and the condition (G-E) clearly holds. Otherwise, if $x < y^{\frac{1}{2}}$, we obtain

$$\begin{aligned} \mathrm{dist}(x, T_1(y)) &= \mathrm{dist}(x, [y^{\frac{1}{2}}, 1]) = y^{\frac{1}{2}} - x \leq (y^{\frac{1}{2}} - x^{\frac{1}{2}}) + (x^{\frac{1}{2}} - x) \\ &\leq \mathrm{dist}(x, T_1(x)) + (y - x) \leq \mathrm{dist}(x, T_1(x)) + |y - x| \end{aligned}$$

Subcase 2: Let $1 \leq x \neq y \leq 2$. If $x \leq y^{\frac{1}{2}}$ then we have $\mathrm{dist}(x, T_1(y)) = 0$ and the condition (G-E) clearly holds. Otherwise, if $x > y^{\frac{1}{2}}$, we obtain

$$\begin{aligned} \mathrm{dist}(x, T_1(y)) &= \mathrm{dist}(x, [1, y^{\frac{1}{2}}]) = x - y^{\frac{1}{2}} \leq (x - x^{\frac{1}{2}}) + (x^{\frac{1}{2}} - y^{\frac{1}{2}}) \\ &\leq \mathrm{dist}(x, T_1(x)) + (x - y) \leq \mathrm{dist}(x, T_1(x)) + |x - y| \end{aligned}$$

Subcase 3: Let $\frac{1}{4} \leq x \leq 1$ and $1 \leq y \leq 2$, then we have

$$\begin{aligned} \mathrm{dist}(x, T_1(y)) &= \mathrm{dist}(x, [1, y^{\frac{1}{2}}]) = 1 - x \leq y - x \\ &\leq \mathrm{dist}(x, T_1(x)) + |x - y| \end{aligned}$$

Subcase 4: Let $\frac{1}{4} \leq y \leq 1$ and $1 \leq x \leq 2$, then we have

$$\begin{aligned} \mathrm{dist}(x, T_1(y)) &= \mathrm{dist}(x, [y^{\frac{1}{2}}, 1]) = x - 1 \leq x - 1 \leq x - y \\ &\leq \mathrm{dist}(x, T_1(x)) + |x - y| \end{aligned}$$

Therefore, T_1 satisfies the condition (G-E) in all four cases. Similarly, we can prove that T_2 satisfies the condition (G-E). However, we note that T_1 and T_2 are neither quasinonexpansive nor G-nonexpansive. Let

$$\alpha_n = \frac{n+2}{3n+1}, \quad \beta_n = \frac{n+3}{7n+4}, \quad \gamma_n = \frac{n+1}{2n+3} \quad \textit{and.}$$

Take

$$x_n' = \frac{1}{2}(x_n^{\frac{1}{2}} + 1) \in T_3 x_n, z_n' = \frac{1}{2}(x_n^{\frac{1}{3}} + 1) \in T_2 z_n$$

and

$$y_n' = \frac{1}{2}(2^{(\frac{3}{4}(x_n - 1))} + 1) \in T_1 y_n.$$

Table 1 and Figure 1 show the convergence of $\{x_n\}$ generated by Algorithm 1.1 for two different initial values, $x_1 = 1.7500$ and $x_1 = 0.2500$. In both cases, the sequence converges to the common fixed point $x = 1$.

Table 1: Convergence of Algorithm (1.1) for two different initial values.

n	$x_1 = 1.75$	$x_1 = 0.2500$
1	1.75000	0.2500
2	1.0989	1.0842
3	1.0112	1.0110
4	1.0012	1.0012
5	1.0001	1.0001
6	1.0000	1.0000

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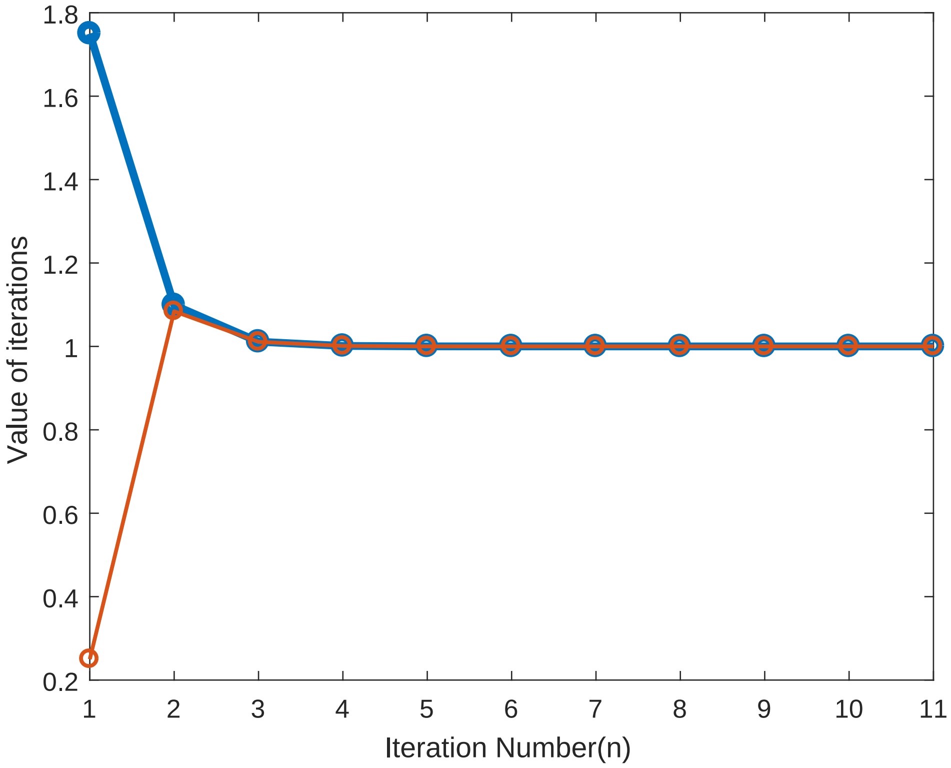

Figure 1: Convergence of Algorithm (1.1) for two different initial values.

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