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Properly efficiency for multiobjective mathematical programming problems with vanishing constraints with non-differentiable data

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Abstract

In this article, we investigate a class of nonsmooth multiobjective mathematical optimization problems with vanishing constraints (abbreviated as MMPVC) in the framework of Euclidean space. Corresponding to MMPVC, some generalized Abadie and Guignard constraint qualifications are introduced. Additionally, some Karush–Kuhn–Tucker type necessary conditions, known as stationarity conditions, are derived at the properly efficient solutions of MMPVC. The results of the paper are established by employing the tools of Clarke subdifferential.

Keywords: qualification condition, stationarity condition, multiobjective optimization, vanishing constraint, properly efficient solution
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1 Introduction

This paper focuses on the following *multiobjective mathematical programming with vanishing constraints* (MMPVC, in brief):

$$(\mathbb{MP}): \qquad \min\ (f_1(x), \ldots, f_p(x)) \quad \text{subject to} \quad x \in S,$$

where the objective functions f_j, for $j \in J := \{1, \ldots, p\}$, are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, and the feasible set S defined by locally Lipschitz functions $H_i, G_i : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$S := \Big\{ x \in \mathbb{R}^n \mid H_i(x) \geq 0, \quad H_i(x)\, G_i(x) \leq 0, \qquad i \in I := \{1, \ldots, m\} \Big\}.$$

Note that MMPVC is an extension of well-known *mathematical programming with vanishing constraints*, denoted by MPVC, which is a generalization of *mathematical programming with equilibrium constraints* [2, 3, 19, 20]. Considering the applications of MPVC in several branches of engineering ([1]), finding optimality conditions and introducing different constraint qualifications for MPVCs are studied by many researchers; see, e.g., [1, 10, 11, 12, 17] in smooth case and [4, 5, 13, 14, 15, 16, 18] in nonsmooth case. Very recently, these results are extended to MPSVs on Hadamard manifolds in [24].

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If $p > 1$ and the appearing functions f_j, H_i, G_i as $(j, i) \in J \times I$ are smooth, some constraint qualifications and optimality conditions, named stationarity conditions, are stated at the optimal solutions of smooth MMPVC in [8, 9]. These results are extended to nondifferentiable convex MMPVC in [23] and to nondifferentiable locally Lipschitz case in [22]. Recently, the nonsmooth MMPVCs on Hadamard manifolds are studied in [24, 25, 26, 27, 28, 29, 30].

It is worth mentioning that [22] focused on efficient and weakly efficient solutions of nonsmooth MMPVCs, and [23] analyzed the properly efficient solutions of convex MMPVCs. To the best of our knowledge, there is no research on properly efficient solutions of nonsmooth MMPVCs. The first goal of this article is to fill this gap. Indeed, we present a parametric equation (3.1) and then, we conclude several well-known stationarity conditions from this equation.

2 Preliminaries

In this section, we mention some definitions, notations, and theorems in convex analysis and nonsmooth analysis widely used in the sequel. Since the contents of this section are indicative, we suggest the references [6, 7, 21] to study examples, proofs, and more details.

The zero vector of $\mathbb{R}^n$ and the set of non-negative real numbers are denoted by 0_n and $\mathbb{R}_+$, respectively. Let W_1 and W_2 be subsets of $\mathbb{R}^n$ and $r \in \mathbb{R}$. The sum of W_1 and W_2, and the scalar product of r and W_1 are respectively defined by

$$W_1 + W_2 := \{w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2\}, \quad \text{and} \quad rW_1 := \{rw_1 \mid w_1 \in W_1\}.$$

Remark 2.1. It is worth mentioning that $(r_1 + r_2)W_1$ is not necessarily equal to $r_1W_1 + r_2W_1$. However, in some special cases, such as when the set W_1 is single-membered, this equality holds.

Let $W \subseteq \mathbb{R}^n$ be given. The closure of W, the convex hull of W, and the convex cone of W are respectively denoted by $\overline{W}$, $conv(W)$, and $cone(W)$.

Theorem 2.2. [7] Suppose that $W \subseteq \mathbb{R}^n$ is compact.

- Then, $conv(W)$ is closed.
- If W is a finite set, then $cone(W)$ is closed.
- If $0_n \notin conv(W)$, then $cone(W)$ is closed.

Let $\{A_\ell \mid \ell \in L\}$ be a collection of sets. The union of all A_ℓ, $\ell \in L$, is denoted by $\bigcup_{\ell \in L} A_\ell$. In many contexts, one may wish to restrict the union to those indices satisfying a certain condition $P(\ell)$, which may be a long or multi-line statement. Since such conditions cannot be conveniently placed directly under the union symbol, we adopt the following notation:

$$\bigcup_{\ell \in L} \{A_\ell \mid P(\ell)\} := \bigcup_{\ell \in L\ \&\ P(\ell)} A_\ell.$$

This allows us to succinctly denote the union over all A_ℓ for which $P(\ell)$ holds. As this notation will be employed extensively in the sequel, we present the following theorem using it, thereby illustrating its practical application.

Theorem 2.3. [7] Suppose that $W_1, \ldots, W_q$ are convex subsets of $\mathbb{R}^n$, then

$$cone\Big(\bigcup_{s=1}^{q} W_s\Big) = \bigcup_{(\lambda_1,\ldots,\lambda_q)\in\mathbb{R}^q} \left\{\sum_{s=1}^{q} \lambda_s W_s \mid \lambda_s \geq 0,\ s = 1, \ldots, q\right\}.$$

$$conv\Big(\bigcup_{s=1}^{q} W_s\Big) = \bigcup_{\lambda_1,\ldots,\lambda_q\geq 0} \left\{\sum_{s=1}^{q} \lambda_s W_s \mid \sum_{s=1}^{q} \lambda_s = 1\right\}.$$

For a given $W \subseteq \mathbb{R}^n$, the polar cone of W is defined by

$$W^- := \{x \in \mathbb{R}^n \mid \langle x, w\rangle \leq 0, \quad \forall w \in W\},$$

where $\langle x, w\rangle$ refers to standard inner product of x and w in $\mathbb{R}^n$. Also, the contingent cone and the normal cone of $\emptyset \neq W \subseteq \mathbb{R}^n$ at $w^* \in \overline{W}$ are respectively defined as

$$\Gamma(W, w^*) := \Big\{ y \in \mathbb{R}^n \mid \exists t_s \downarrow 0,\ \exists y_s \to y \text{ such that } w^* + t_s y_s \in W \quad \forall s \in \mathbb{N} \Big\},$$
$$N(W, w^*) := \Big(\Gamma(W, w^*) \Big)^-.$$

It is shown in [7] that if $\emptyset \neq W \subseteq \mathbb{R}^n$ is a convex set, then the following equalities are true:

$$N(W, w^*) = \{ d \in \mathbb{R}^n \mid \langle d, w - w^* \rangle \leq 0, \quad \forall w \in W \},$$
$$N\big(W^-, 0_n\big) = W^{--} := (W^-)^-. \tag{2.1}$$

It is easy to show that

$$W_1 \subseteq W_2 \implies W_2^- \subseteq W_1^-. \tag{2.2}$$

The bipolar theorem ([7]) states that

$$W^{--} = \overline{cone}(W), \tag{2.3}$$

where $\overline{cone}(W)$ denotes the closed convex cone of W, i.e.,

$$\overline{cone}(W) := \overline{cone(W)}.$$

The real-valued function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for each $x_0 \in \mathbb{R}^n$ there exist a neighborhood U_0 for x_0 and a positive number $L_0 > 0$ such that

$$|\psi(x) - \psi(y)| \leq L_0 \|x - y\|, \quad \text{for all } x, y \in U_0.$$

For a given locally Lipschitz function $\psi : \mathbb{R}^n \to \mathbb{R}$, the Clarke subdifferential of ψ at x_0 is defined by ([6])

$$\partial_c \psi(x_0) := \big\{ \xi \in \mathbb{R}^n \mid \langle \xi, v \rangle \leq \psi^0(x_0; v) \qquad \forall\, v \in \mathbb{R}^n \big\},$$

in which

$$\psi^0(x_0; v) := \limsup_{y \to x_0,\ t \downarrow 0} \frac{\psi(y + tv) - \psi(y)}{t}.$$

It is worth mentioning that if $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable (smooth) function, we have $\partial_c g(x_0) = \{\nabla g(x_0)\}$ for all $x_0 \in \mathbb{R}^n$, where $\nabla g(x_0)$ denotes the classic gradient of g at x_0. Also, if $h : \mathbb{R}^n \to \mathbb{R}$ is a convex function, for all $x_0 \in \mathbb{R}^n$ we have $\partial_c h(x_0) = \partial h(x_0)$, in which $\partial h(x_0)$ denotes the classic subdifferential of h at x_0 in convex analysis sense, defined as

$$\partial h(x_0) := \{ \xi \in \mathbb{R}^n \mid h(x) - h(x_0) \geq \langle \xi, x - x_0 \rangle, \quad \forall x \in \mathbb{R}^n \}.$$

Let us recall some important properties of Clarke subdifferentials from [6] which are widely used in what follows.

Theorem 2.4. Suppose that φ and ψ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

- $\partial_c \psi(x_0)$ is a nonempty, convex and compact set in $\mathbb{R}^n$.
- $\varphi^0(x_0; \cdot)$ is a convex function with respect to direction, and we have
$$\partial \varphi^0(x_0; \cdot)(0_n) = \partial_c \varphi(x_0). \tag{2.4}$$
- For all real numbers λ_1, λ_2, one has
$$\partial_c(\lambda_1 \varphi + \lambda_2 \psi)(x_0) \subseteq \lambda_1 \partial_c \varphi(x_0) + \lambda_2 \partial_c \psi(x_0). \tag{2.5}$$

Theorem 2.5. [7] Let $w^* \in W \subseteq \mathbb{R}^n$ be a minimizer of a locally Lipschitz function $\psi : \mathbb{R}^n \to \mathbb{R}$ on W.

(i) One has
$$\psi^0(w^*, y) \geq 0, \qquad \forall y \in \Gamma(W, w^*).$$

(ii) If W is a convex set, we have
$$0_n \in \partial_c \psi(w^*) + N(W, w^*).$$

(iii) If ψ is smooth at w^*, we get
$$\psi^0(w^*, y) \geq 0, \qquad \forall y \in \overline{cone}\big(\Gamma(W, w^*)\big).$$

As the final point of this section, we recall the following well-known definition.

Definition 2.6. We say that the feasible point x_0 is a properly efficient solution to the problem ($\mathbb{MP}$) if there exist some positive scalars $\lambda_j > 0$ as $j \in J$ such that for each feasible point x one has
$$\sum_{j \in J} \lambda_j f_j(x_0) \leq \sum_{j \in J} \lambda_j f_j(x).$$

3 Main Results

As the beginning of this section, we assign some symbols for the whole of this paper.

Suppose that the feasible set of ($\mathbb{MP}$) is nonempty, i.e., $S \neq \emptyset$. Also, let $\hat{x} \in S$ be a feasible point of ($\mathbb{MP}$) which is assumed to be fixed throughout this article. The index set I can be divided into $I = I_+ \cup I_0$, where
$$I_+ := \{i \in I \mid H_i(\hat{x}) > 0\},$$
$$I_0 := \{i \in I \mid H_i(\hat{x}) = 0\},$$

and I_+ and I_0 divide as
$$I_{+0} := \{i \in I_+ \mid G_i(\hat{x}) = 0\},$$
$$I_{+-} := \{i \in I_+ \mid G_i(\hat{x}) < 0\},$$
$$I_{0+} := \{i \in I_0 \mid G_i(\hat{x}) > 0\},$$
$$I_{00} := \{i \in I_0 \mid G_i(\hat{x}) = 0\},$$
$$I_{0-} := \{i \in I_0 \mid G_i(\hat{x}) < 0\}.$$

Motivated by [13, 14], for each $K \subseteq I$ we set
$$\mathcal{H}_K := \bigcup_{i \in K} \partial_c H_i(\hat{x}) \quad \text{and} \quad \mathcal{G}_K := \bigcup_{i \in K} \partial_c G_i(\hat{x}).$$

Following [13], we define the following constraint qualifications.

Definition 3.1. We say that ($\mathbb{MP}$) satisfies

- the Abadie constraint qualification, denoted by ACQ, at $\hat{x}$ if $\mathcal{L}^- \subseteq \Gamma(S, \hat{x})$, where
$$\mathcal{L} := \mathcal{G}_{I_{+0}} \cup (-\mathcal{H}_{I_0}) \cup \mathcal{H}_{I_{0+}}.$$

- the Guignard constraint qualification, denoted by GCQ, at $\hat{x}$ if
$$\mathcal{L}^- \subseteq \overline{cone}\,(\Gamma(S, \hat{x})).$$

It is worth mentioning that GCQ is an extension of Guignard constraint qualification, defined in [1, 10, 11, 12] for smooth MPVCs.

The following theorem will have a key role in the rest of this paper.

Theorem 3.2. If K_1, K_2 and K_3 are three subsets of I, then

$$cone\Big(\mathcal{G}_{K_1}\cup\big(-\mathcal{H}_{K_2}\big)\cup\mathcal{H}_{K_3}\Big)=\bigcup_{\mu_i,\hat{\eta}_i,\tilde{\eta}_i\geq 0}\left\{\sum_{i\in I}\Big[\mu_i\partial_c G_i(\hat{x})-\big(\hat{\eta}_i\partial_c H_i(\hat{x})-\tilde{\eta}_i\partial_c H_i(\hat{x})\big)\Big]\;\middle|\;\begin{array}{ll}\mu_i=0, & i\in I\setminus K_1\\ \hat{\eta}_i=0, & i\in I\setminus K_2\\ \tilde{\eta}_i=0, & i\in I\setminus K_3\end{array}\right\}. \tag{3.1}$$

If, in addition, the functions G_i and H_i as $i\in I$ are smooth at $\hat{x}$ and $K_3\subseteq K_2$, then

$$cone\Big(\mathcal{G}_{K_1}\cup\big(-\mathcal{H}_{K_2}\big)\cup\mathcal{H}_{K_3}\Big)=\left\{\sum_{i\in I}\Big(\mu_i\nabla G_i(\hat{x})-\eta_i\nabla H_i(\hat{x})\Big)\;\middle|\;\begin{array}{ll}\mu_i\geq 0, & i\in I\\ \mu_i=0, & i\in I\setminus K_1\\ \eta_i=0, & i\in I\setminus K_2\\ \eta_i\geq 0, & i\in K_2\setminus K_3\\ \eta_i \text{ is free}, & i\in K_3\end{array}\right\}.$$

Proof . According to the convexity of $\partial_c G_i(\hat{x})$ and $\partial_c H_i(\hat{x})$ as $i\in I$ and Theorem 2.3, we have

$$\begin{aligned}cone\Big(\mathcal{G}_{K_1}\cup\big(-\mathcal{H}_{K_2}\big)\cup\mathcal{H}_{K_3}\Big)&=\bigcup_{(\mu_i)\in\mathbb{R}_+^{|K_1|}}\ \bigcup_{(\hat{\eta}_i)\in\mathbb{R}_+^{|K_2|}}\ \bigcup_{(\tilde{\eta}_i)\in\mathbb{R}_+^{|K_3|}}\Big\{\sum_{i\in K_1}\mu_i\partial_c G_i(\hat{x})+\sum_{i\in K_2}-\hat{\eta}_i\partial_c H_i(\hat{x})+\sum_{i\in K_3}\tilde{\eta}_i\partial_c H_i(\hat{x})\Big\}\\&=\bigcup_{(\mu_i)\in\mathbb{R}_+^{|I|}}\ \bigcup_{(\hat{\eta}_i)\in\mathbb{R}_+^{|I|}}\ \bigcup_{(\tilde{\eta}_i)\in\mathbb{R}_+^{|I|}}\Big\{\sum_{i\in I}\mu_i\partial_c G_i(\hat{x})+\sum_{i\in I}-\hat{\eta}_i\partial_c H_i(\hat{x})+\sum_{i\in I}\tilde{\eta}_i\partial_c H_i(\hat{x})\Big\},\end{aligned}$$

where $\mu_i=0$ for $i\notin K_1$, $\hat{\eta}_i=0$ for $i\notin K_2$, and $\tilde{\eta}_i=0$ for $i\notin K_3$. The last equality deduces that

$$\begin{aligned}&cone\Big(\mathcal{G}_{K_1}\cup\big(-\mathcal{H}_{K_2}\big)\cup\mathcal{H}_{K_3}\Big)\\&=\bigcup_{(\mu_i)\in\mathbb{R}_+^{|I|}}\ \bigcup_{(\hat{\eta}_i)\in\mathbb{R}_+^{|I|}}\ \bigcup_{(\tilde{\eta}_i)\in\mathbb{R}_+^{|I|}}\left\{\sum_{i\in I}\Big[\mu_i\partial_c G_i(\hat{x})-\big(\hat{\eta}_i\partial_c H_i(\hat{x})-\tilde{\eta}_i\partial_c H_i(\hat{x})\big)\Big]\;\middle|\;\begin{array}{ll}\mu_i=0, & i\in I\setminus K_1\\ \hat{\eta}_i=0, & i\in I\setminus K_2\\ \tilde{\eta}_i=0, & i\in I\setminus K_3\end{array}\right\},\end{aligned}$$

which concludes (3.1). Now, if the functions G_i and H_i as $i\in I$ are smooth at $\hat{x}$, we get from (3.1) and Remark 2.1 that

$$\begin{aligned}&cone\Big(\mathcal{G}_{K_1}\cup\big(-\mathcal{H}_{K_2}\big)\cup\mathcal{H}_{K_3}\Big)\\&=\bigcup_{\mu_i,\hat{\eta}_i,\tilde{\eta}_i\geq 0}\left\{\sum_{i\in I}\Big[\mu_i\{\nabla G_i(\hat{x})\}-\big(\hat{\eta}_i\{\nabla H_i(\hat{x})\}-\tilde{\eta}_i\{\nabla H_i(\hat{x})\}\big)\Big]\;\middle|\;\begin{array}{ll}\mu_i=0, & i\in I\setminus K_1\\ \hat{\eta}_i=0, & i\in I\setminus K_2\\ \tilde{\eta}_i=0, & i\in I\setminus K_3\end{array}\right\}\\&=\bigcup_{\mu_i,\hat{\eta}_i,\tilde{\eta}_i\geq 0}\left\{\sum_{i\in I}\Big[\mu_i\{\nabla G_i(\hat{x})\}-(\hat{\eta}_i-\tilde{\eta}_i)\{\nabla H_i(\hat{x})\}\Big]\;\middle|\;\begin{array}{ll}\mu_i=0, & i\in I\setminus K_1\\ \hat{\eta}_i=0, & i\in I\setminus K_2\\ \tilde{\eta}_i=0, & i\in I\setminus K_3\end{array}\right\}.\end{aligned}$$

Put $\eta_i:=\hat{\eta}_i-\tilde{\eta}_i$ for all $i\in I$. Thus, $\eta_i=0$ for $i\in I\setminus K_2$ (since $\eta_i=0-0$ in this case), $\eta_i\geq 0$ for $i\in K_2\setminus K_3$ (since $\eta_i=\hat{\eta}_i-0$ in this case), and η_i has free sign for $i\in K_3$ (since η_i is equal to two non-negative numbers in this case). Hence,

$$cone\Big(\mathcal{G}_{K_1}\cup\big(-\mathcal{H}_{K_2}\big)\cup\mathcal{H}_{K_3}\Big)=\left\{\sum_{i\in I}\Big(\mu_i\nabla G_i(\hat{x})-\eta_i\nabla H_i(\hat{x})\Big)\;\middle|\;\begin{array}{ll}\mu_i\geq 0, & i\in I\\ \mu_i=0, & i\in I\setminus K_1\\ \eta_i=0, & i\in I\setminus K_2\\ \eta_i\geq 0, & i\in K_2\setminus K_3\\ \eta_i \text{ is free}, & i\in K_3\end{array}\right\},$$

and the proof is complete. □

Now, we present the strongly stationarity condition, denoted by SSC, for MMPVCs.

Theorem 3.3. *(SSC Under ACQ; Nonsmooth Objective and Constraints Functions):* Let $\hat{x}$ be a properly efficient solution to the problem ($\mathbb{MP}$), and ACQ holds at $\hat{x}$. Then, there exist some positive scalars λ_j as $j\in J$, such that

$$0_n\in\sum_{j\in J}\lambda_j\partial_c f_j(\hat{x})+\overline{cone}\big(\mathcal{L}\big). \tag{3.2}$$

If, in addition, $cone(\mathcal{L})$ is closed, we can find some coefficients $\mu_i,\ \hat{\eta}_i$ and $\tilde{\eta}_i$ as $i \in I$, such that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \sum_{i\in I} \Big[\mu_i \partial_c G_i(\hat{x}) - \big(\hat{\eta}_i \partial_c H_i(\hat{x}) - \tilde{\eta}_i \partial_c H_i(\hat{x})\big)\Big], \tag{3.3}$$

$$\mu_i = 0 \quad (i \in I_{+-} \cup I_0)\,, \qquad \mu_i \geq 0 \quad (i \in I_{+0})\,, \tag{3.4}$$

$$\hat{\eta}_i = 0 \quad (i \in I_{+})\,, \qquad \hat{\eta}_i \geq 0 \quad (i \in I_0)\,, \tag{3.5}$$

$$\tilde{\eta}_i = 0 \quad (i \in I_{+} \cup I_{0-} \cup I_{00})\,, \qquad \tilde{\eta}_i \geq 0 \quad (i \in I_{0+})\,. \tag{3.6}$$

Proof . The definition of properly efficiency leads us to find some positive scalars $\lambda_j > 0$ as $j \in J$ such that $\hat{x}$ is a minimizer of $\sum_{j\in J} \lambda_j f_j(x)$ on S. Thus, owing to Theorem 2.5(i), we get

$$\Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; d) \geq 0, \qquad \forall d \in \Gamma(S, \hat{x}).$$

This inequality and the fact that $\Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; 0_n) = 0$ conclude that

$$\Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; d) \geq \Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; 0_n), \qquad \forall d \in \Gamma(S, \hat{x}).$$

From the last relation and ACQ at $\hat{x}$, we have

$$\Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; d) \geq \Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; 0_n), \qquad \forall d \in \mathcal{L}^-. \tag{3.7}$$

Thus, the convex function $d \to \Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; d) = 0$ on the convex set $\mathcal{L}^-$ obtains its minimum at $\hat{d} := 0_n$. As a result, Theorem 2.5(ii) and relations (2.1)-(2.4) imply that

$$\begin{aligned} 0_n &\in \partial \left(\Big(\sum_{j\in J} \lambda_j f_j\Big)^0 (\hat{x}; \cdot) \right) (\hat{d}) + N\big(\mathcal{L}^-, \hat{d}\,\big) \\ &= \partial_c \Big(\sum_{j\in J} \lambda_j f_j\Big)(\hat{x}) + \mathcal{L}^{--} \\ &\subseteq \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \overline{cone}\big(\mathcal{L}\big), \end{aligned}$$

where the last inclusion holds by (2.5) and the bipolar theorem (2.3). So, (3.2) is proved. Now, if $cone(\mathcal{L})$ is closed, (3.2) concludes that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + cone\big(\mathcal{L}\big).$$

Taking $K_1 = I_{+0}$, $K_2 = I_0$, and $K_3 = I_{0+}$ in Theorem 3.2, the last inclusion implies that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \bigcup_{\mu_i, \hat{\eta}_i, \tilde{\eta}_i \geq 0} \left\{ \sum_{i\in I} \Big[\mu_i \partial_c G_i(\hat{x}) - \big(\hat{\eta}_i \partial_c H_i(\hat{x}) - \tilde{\eta}_i \partial_c H_i(\hat{x})\big)\Big] \;\middle|\; \begin{array}{ll} \mu_i = 0, & i \in I \setminus I_{+0} \\ \hat{\eta}_i = 0, & i \in I \setminus I_0 \\ \tilde{\eta}_i = 0, & i \in I \setminus I_{0+} \end{array} \right\}.$$

Since $I \setminus I_{+0} = I_{+-} \cup I_0$ and $I \setminus I_0 = I_+$ and $I \setminus I_{0+} = I_+ \cup I_{0-} \cup I_{00}$, the last relation implies that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \sum_{i\in I} \Big[\mu_i \partial_c G_i(\hat{x}) - \big(\hat{\eta}_i \partial_c H_i(\hat{x}) - \tilde{\eta}_i \partial_c H_i(\hat{x})\big)\Big],$$

for some non-negative scalars μ_i, $\hat{\eta}_i$, and $\tilde{\eta}_i$ as $i \in I$ with

$$\mu_i = 0, \quad i \in I_{+-} \cup I_0; \qquad \hat{\eta}_i = 0, \quad i \in I_+; \qquad \tilde{\eta}_i = 0, \quad i \in I_+ \cup I_{0-} \cup I_{00},$$

and the proof is complete. □

The following example, motivated by [13], shows that we can not replace ACQ with GCQ in Theorem 3.3.

Example 3.4. Consider the optimization problem as

$$\min_{x\in S} \ \big(-x_2 + |x_1 - x_2| \ , \ -x_2 + |x_1 - x_2|\big),$$

such that $S = (\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \{0\})$. We can formalize this problem as (MP) with following data,

$$\begin{aligned} &H_1(x_1,x_2) = x_2, \quad H_2(x_1,x_2) = x_1, \quad G_1(x_1,x_2) = -x_1, \quad G_2(x_1,x_2) = -x_2, \\ &f_1(x_1,x_2) = f_2(x_1,x_2) = -x_2 + |x_1 - x_2|. \end{aligned}$$

We observe that $\hat{x} = 0_2$ is a properly efficient solution of the problem, and $I_{00} = \{1,2\}$. Since

$$\mathcal{L}^- = (-\mathcal{H}_{I_{00}})^- = (\{(-1,0),(0,-1)\})^- = \mathbb{R}_+ \times \mathbb{R}_+ = conv(\Gamma(S,\hat{x})),$$

the GCQ is satisfied at $\hat{x}$. It is easy to show that the $cone(\mathcal{L})$ is closed, and also, the below relations do not hold for any non-negative scalars λ_1, λ_2, μ_1, μ_2, $\hat{\eta}_1$, $\hat{\eta}_2$, $\tilde{\eta}_1$, and $\tilde{\eta}_2$:

$$\begin{aligned} &0_2 = \lambda_1(\gamma, 1-\gamma) + \lambda_2(\gamma, 1-\gamma) + \mu_1(-1,0) + \mu_2(0,1) - \Big(\hat{\eta}_1(0,1) + \hat{\eta}_2(1,0) - \tilde{\eta}_1(0,1) - \tilde{\eta}_2(1,0)\Big), \\ &\quad \lambda_1, \lambda_2 > 0, \quad \gamma \in [-1,1], \quad \mu_1 = \mu_2 = 0, \quad \tilde{\eta}_1 = \tilde{\eta}_2 = 0. \end{aligned}$$

The following theorem shows that when the objective functions f_j as $j \in J$ are smooth at $\hat{x}$, we can replace ACQ with GCQ in Theorem 3.3.

Theorem 3.5. *(USC Under GCQ; Smooth Objective and Nonsmooth Constraints Functions):* Assume that $\hat{x}$ is a properly efficient solution to the problem (MP) and the objective functions f_j as $j \in J$ are smooth at $\hat{x}$. Also, suppose that GCQ holds at $\hat{x}$. There exist some positive scalars λ_j as $j \in J$ such that

$$-\sum_{j\in J} \lambda_j \nabla f_j(\hat{x}) \in \overline{cone}\big(\mathcal{L}\big).$$

If, in addition, $cone\big(\mathcal{L}\big)$ is closed, we can find some coefficients μ_i, $\hat{\eta}_i$ and $\tilde{\eta}_i$ as $i \in I$, such that the following inclusion and (3.4)-(3.6) hold:

$$-\sum_{j\in J} \lambda_j \nabla f_j(\hat{x}) \in \sum_{i\in I} \Big[\mu_i \partial_c G_i(\hat{x}) - \big(\hat{\eta}_i \partial_c H_i(\hat{x}) - \tilde{\eta}_i \partial_c H_i(\hat{x})\big)\Big]. \tag{3.8}$$

Proof . Following the proof of Theorem 3.3, the definition of properly efficiency, and Theorem 2.5(iii) conclude that there exist some positive scalars $\lambda_j > 0$, for $j \in J$, such that

$$\left(\sum_{j\in J} \lambda_j f_j\right)^0 (\hat{x}; d) \geq 0, \qquad \forall d \in \overline{cone}\big(\Gamma(S,\hat{x})\big).$$

Since GCQ holds at $\hat{x}$, the last inequality implies that (3.7) is true, and hence, (3.2) holds. Thus,

$$0_n \in \sum_{j\in J} \lambda_j \{\nabla f_j(\hat{x})\} + \overline{cone}\big(\mathcal{L}\big),$$

which concludes that

$$\sum_{j\in J}\lambda_j\nabla f_j(\hat{x})\in\overline{cone}\big(\mathcal{L}\big).$$

If $cone\big(\mathcal{L}\big)$ is closed, the proof of Theorem 3.3 shows that (3.4)-(3.6) and (3.8) hold. □

It should be noted that the closedness of $cone\big(\mathcal{L}\big)$ is a restrictive assumption in Theorem 3.5. In [13, Example 4.1] it is shown that this assumption can not be ignored in this theorem. The following theorem shows that this restrictive condition can be removed when the constraint functions G_i and H_i as $i\in I$ are smooth at $\hat{x}$.

Theorem 3.6. *(USC Under GCQ; Smooth Objective and Constraint Functions):* Let $\hat{x}$ be a properly efficient solution to the problem $(\mathbb{MP})$ with smooth objective and constraint functions. If GCQ holds at $\hat{x}$, we can find some positive scalars λ_j as $j\in J$, and some coefficients μ_i and η_i as $i\in I$, such that the following relations and (3.4) hold:

$$\sum_{j\in J}\lambda_j\nabla f_j(\hat{x})+\sum_{i\in I}\Big(\mu_i\nabla G_i(\hat{x})-\eta_i\nabla H_i(\hat{x})\Big)=0_n, \tag{3.9}$$

$$\eta_i=0\quad(i\in I_{+}),\qquad \eta_i\geq 0\quad(i\in I_{0-}\cup I_{00}),\qquad \eta_i\ is\ free\quad(i\in I_{0+}). \tag{3.10}$$

Proof . Owing to

$$\mathcal{L}=\big\{\nabla G_i(\hat{x})\mid i\in I_{+0}\big\}\cup\big\{-\nabla H_i(\hat{x})\mid i\in I_0\big\}\cup\big\{\nabla H_i(\hat{x})\mid i\in I_{0+}\big\},$$

we can see $\mathcal{L}$ is a finite set, and Theorem 2.2 implies that $cone\big(\mathcal{L}\big)$ is closed. Thus, Theorem 3.5 implies that

$$-\sum_{j\in J}\lambda_j\nabla f_j(\hat{x})\in cone\big(\mathcal{L}\big).$$

Tacking $K_1=I_{+0}$ and $K_2=I_0$ and $K_3=I_{0+}$ in Theorem 3.2, and regarding to $I\setminus I_{+0}=I_{+-}\cup I_0$ and $I\setminus I_0=I_+$ and $I_0\setminus I_{0+}=I_{0-}\cup I_{00}$, we obtain that the relations (3.4) and (3.9)-(3.10) hold. □

Remark 3.7. We should note that if all appearing functions $G_i,H_i:\mathbb{R}^n\to\mathbb{R}$ are piecewise affine, their Clarke subdifferentials are polyhedrons, and hence, $cone\big(\mathcal{L}\big)$ is finitely generated and naturally closed. So, the limiting condition of closedness of $cone\big(\mathcal{L}\big)$ in Theorem 3.5 is automatically fulfilled in this case.

Motivated by [13], we consider the following tightened problems which don't have product constraints (a tightened problem is an optimization problem such that whose feasible set is included in S):

$$\begin{array}{llll}(\mathbb{MP}_W): & \min & (f_1(x),\ldots,f_p(x)) & \\ & \text{s.t.} & H_i(x)=0, & i\in I_{0+}\cup I_{00},\\ & & H_i(x)\geq 0, & i\in I_{0-}\cup I_{+},\\ & & G_i(x)\leq 0, & i\in I_{0-}\cup I_{00}\cup I_{+},\end{array}$$

and

$$\begin{array}{llll}(\mathbb{MP}_V): & \min & (f_1(x),\ldots,f_p(x)) & \\ & \text{s.t.} & H_i(x)=0, & i\in I_{0+},\\ & & H_i(x)\geq 0, & i\in I_{0-}\cup I_{00}\cup I_{+},\\ & & G_i(x)\leq 0, & i\in I_{0-}\cup I_{00}\cup I_{+}.\end{array}$$

According to the above problems, and following [13], we define some constraint qualifications.

Definition 3.8. We say that $(\mathbb{MP})$ satisfies

- ACQ_W at $\hat{x}$ if $\mathcal{L}_W^-\subseteq\Gamma\,(S,\hat{x})$, where

$$\mathcal{L}_W:=\mathcal{G}_{I_{+0}\cup I_{00}}\cup(-\mathcal{H}_{I_0})\cup\mathcal{H}_{I_{0+}\cup I_{00}}.$$

- GCQ_W at $\hat{x}$ if $\mathcal{L}_W^-\subseteq\overline{cone}\,(\Gamma\,(S,\hat{x}))$.

- ACQ_V at $\hat{x}$ if $\mathcal{L}_V^- \subseteq \Gamma(S, \hat{x})$, where

$$\mathcal{L}_V := \mathcal{G}_{I_{+0} \cup I_{00}} \cup (-\mathcal{H}_{I_0}) \cup \mathcal{H}_{I_{0+}}.$$

- GCQ_V at $\hat{x}$ if $\mathcal{L}_V^- \subseteq \overline{cone}(\Gamma(S, \hat{x}))$.

It is worth mentioning that GCQ_V (resp. ACQ_V) is a generalization of the well-known VC-Guignard constraint qualification (resp. VC-Abadie constraint qualification) defined in [1, 10, 11, 12] (resp. [13, 18]) for smooth (resp. nonsmooth) MPVCs. Also, GCQ_W and ACQ_W are extensions of weakly Guignard and weakly Abadie constraint qualifications, defined in [13, 14, 18] for nonsmooth MPVCs.

It should be noted that the feasible set of $(\mathbb{MP}_W)$ (resp. $(\mathbb{MP}_V)$) is not necessarily equal to S, and hence ACQ_W (resp. ACQ_V) and GCQ_W (resp. GCQ_V) are not equivalent to Abadie and Guignard constraint qualifications for $(\mathbb{MP}_W)$ (resp. $(\mathbb{MP}_V)$).

Note that the relation of

$$\mathcal{L} \subseteq \mathcal{L}_V \subseteq \mathcal{L}_W,$$

is trivially true by the definition, and so $\mathcal{L}_W^- \subseteq \mathcal{L}_V^- \subseteq \mathcal{L}^-$ by (2.2). Thus,

$$\begin{array}{ccccc} \text{ACQ} & \Longrightarrow & \text{ACQ}_V & \Longrightarrow & \text{ACQ}_W \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{GCQ} & \Longrightarrow & \text{GCQ}_V & \Longrightarrow & \text{GCQ}_W \end{array}$$

Note that, it is shown in [1] that the converse of implication $[\text{ACQ}_V \Longrightarrow \text{ACQ}_W]$ is not necessarily true, even in the smooth case. The following example shows that the converse of the implications $[\text{ACQ} \Longrightarrow \text{GCQ}]$ and $[\text{ACQ} \Longrightarrow \text{ACQ}_V]$ may be false.

Example 3.9. Consider problem $(\mathbb{MP})$, with $n = 2$ and

$$f_1(x) = x_1x_2 + x_2^2, \qquad f_2(x) = x_1x_2 + |x_2|,$$
$$G_1(x) = x_2 - |x_1|, \qquad H_1(x) = x_2.$$

It is easy to check that $\hat{x} = 0_2$ is a properly efficient solution of the considered problem. Since

$$I_{00} = \{1\}, \qquad \partial_c H_1(\hat{x}) = \{(0,1)\}, \qquad \partial_c G_1(\hat{x}) = [-1,1] \times \{1\},$$

we have

$$\begin{aligned} \mathcal{L}^- &= \{-(0,1)\}^- = \mathbb{R} \times \mathbb{R}_+, \\ \mathcal{L}_V^- &= \{-(0,1)\}^- \cap \Big([-1,1] \times \{1\}\Big)^- = \{0_2\}, \\ \mathcal{L}_W^- &= \{(0,1)\}^- \cap \{-(0,1)\}^- \cap \Big([-1,1] \times \{1\}\Big)^- = \{0_2\}. \end{aligned}$$

Above equalities show that ACQ fails whereas GCQ (and so, GCQ_V and GCQ_W) and ACQ_V (and so, ACQ_W) hold at $\hat{x}$. Thus, the inverse of the following implications do not necessarily hold:

$$[\text{ACQ} \Longrightarrow \text{GCQ}] \qquad \text{and} \qquad [\text{ACQ} \Longrightarrow \text{ACQ}_V].$$

Now, we present some tightened stationarity conditions for MMPVCs, named weakly and VC-stationarity condition, denoted by WSC and VSC, respectively.

Theorem 3.10. *(WSC Under ACQ_W and GCQ_W):* Let $\hat{x}$ be a properly efficient solution to the problem $(\mathbb{MP})$.

(i): Assume that ACQ_W holds at $\hat{x}$. Then, there exist some positive scalars λ_j as $j \in J$, such that

$$0_n \in \sum_{j \in J} \lambda_j \partial_c f_j(\hat{x}) + \overline{cone}\big(\mathcal{L}_W\big). \tag{3.11}$$

If, in addition, $cone\big(\mathcal{L}_W\big)$ is closed, we can find some coefficients μ_i, $\hat{\eta}_i$ and $\tilde{\eta}_i$ as $i \in I$, such that (3.3), (3.5), and the following relations hold:

$$\mu_i = 0 \quad (i \in I_{+-} \cup I_{0+} \cup I_{0-}), \qquad \mu_i \geq 0 \quad (i \in I_{+0} \cup I_{00}), \tag{3.12}$$
$$\tilde{\eta}_i = 0 \quad (i \in I_+ \cup I_{0-}), \qquad \tilde{\eta}_i \geq 0 \quad (i \in I_{0+} \cup I_{00}). \tag{3.13}$$

(ii): Assume that the objective functions f_j as $j \in J$ are smooth at $\hat{x}$ and GCQ$_W$ holds at $\hat{x}$. Then, there exist some positive scalars λ_j as $j \in J$ such that

$$-\sum_{j\in J} \lambda_j \nabla f_j(\hat{x}) \in \overline{cone}\big(\mathcal{L}_W\big).$$

If, in addition, $cone\big(\mathcal{L}_W\big)$ is closed, we can find some coefficients μ_i, $\hat{\eta}_i$ and $\tilde{\eta}_i$ as $i \in I$, such that (3.8) and (3.12)-(3.13) hold.

(iii): Assume that the objective and the constraint functions of ($\mathbb{MP}$) are smooth at $\hat{x}$. If GCQ$_W$ holds at $\hat{x}$, we can find some positive scalars λ_j as $j \in J$, and some coefficients μ_i and η_i as $i \in I$, such that the following relation, (3.9), and (3.12) hold:

$$\eta_i = 0 \quad (i \in I_+)\,, \qquad \eta_i \geq 0 \quad (i \in I_{0-})\,, \qquad \eta_i \ is \ free \quad (i \in I_{0+} \cup I_{00})\,. \tag{3.14}$$

Proof .

(i): Because the proof of (3.11) is exactly the same as the proof of (3.2), we refrain from repeating it. Now, let $cone\big(\mathcal{L}_W\big)$ be closed. Taking $K_1 = I_{+0} \cup I_{00}$, $K_2 = I_0$, and $K_3 = I_{0+} \cup I_{00}$ in Theorem 3.2, and regarding to (3.11), we deduce that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \bigcup_{\mu_i,\hat{\eta}_i,\tilde{\eta}_i \geq 0} \left\{ \sum_{i\in I} \Big[\mu_i \partial_c G_i(\hat{x}) - \big(\hat{\eta}_i \partial_c H_i(\hat{x}) - \tilde{\eta}_i \partial_c H_i(\hat{x})\big)\Big] \;\middle|\; \begin{array}{ll} \mu_i = 0, & i \in I \setminus (I_{+0} \cup I_{00}) \\ \hat{\eta}_i = 0, & i \in I \setminus I_0 \\ \tilde{\eta}_i = 0, & i \in I \setminus (I_{0+} \cup I_{00}) \end{array} \right\}.$$

Since $I \setminus (I_{+0} \cup I_{00}) = I_{+-} \cup I_{0+} \cup I_{0-}$ and $I \setminus I_0 = I_+$ and $I \setminus (I_{0+} \cup I_{00}) = I_+ \cup I_{0-}$, the last relation implies that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \sum_{i\in I} \Big[\mu_i \partial_c G_i(\hat{x}) - \big(\hat{\eta}_i \partial_c H_i(\hat{x}) - \tilde{\eta}_i \partial_c H_i(\hat{x})\big)\Big],$$

for some non-negative scalars μ_i, $\hat{\eta}_i$, and $\tilde{\eta}_i$ as $i \in I$ with

$$\mu_i = 0, \quad i \in I_{+-} \cup I_{0+} \cup I_{0-}; \qquad \hat{\eta}_i = 0, \quad i \in I_+; \qquad \tilde{\eta}_i = 0, \quad i \in I_+ \cup I_{0-},$$

and the proof is complete.

(ii): The proof is exactly the same as the proof of Theorem 3.5.

(iii): Since $\mathcal{L}_W$ is a finite set, Theorem 2.2 implies that $cone\big(\mathcal{L}_W\big)$ is closed. So, part *(ii)* of this theorem concludes that

$$-\sum_{j\in J} \lambda_j \nabla f_j(\hat{x}) \in cone\big(\mathcal{L}_W\big).$$

Tacking $K_1 = I_{+0} \cup I_{00}$ and $K_2 = I_0$ and $K_3 = I_{0+} \cup I_{00}$ in Theorem 3.2, we obtain the result.

□

Theorem 3.11. *(VSC Under ACQ$_V$ and GCQ$_V$):* Let $\hat{x}$ be a properly efficient solution to the problem ($\mathbb{MP}$).

(i): Assume that ACQ$_V$ holds at $\hat{x}$. Then, there exist some positive scalars λ_j as $j \in J$, such that

$$0_n \in \sum_{j\in J} \lambda_j \partial_c f_j(\hat{x}) + \overline{cone}\big(\mathcal{L}_V\big).$$

If, in addition, $cone\big(\mathcal{L}_V\big)$ is closed, we can find some coefficients μ_i, $\hat{\eta}_i$ and $\tilde{\eta}_i$ as $i \in I$, such that the following relations, (3.3), (3.5)-(3.6), and (3.12) hold:

(ii): Assume that the objective functions f_j as $j \in J$ are smooth at $\hat{x}$ and GCQ$_V$ holds at $\hat{x}$. Then, there exist some positive scalars λ_j as $j \in J$ such that

$$-\sum_{j\in J} \lambda_j \nabla f_j(\hat{x}) \in \overline{cone}\big(\mathcal{L}_W\big).$$

If, in addition, $cone\big(\mathcal{L}_V\big)$ is closed, we can find some coefficients μ_i, $\hat{\eta}_i$ and $\tilde{\eta}_i$ as $i \in I$, such that (3.5)-(3.8) and (3.12) hold.

(iii): Assume that the objective and the constraint functions of ($\mathbb{MP}$) are smooth at $\hat{x}$. If GCQ$_V$ holds at $\hat{x}$, we can find some positive scalars λ_j as $j \in J$, and some coefficients μ_i and η_i as $i \in I$, such that (3.9)-(3.10) and (3.12) hold.

Proof . The proof is exactly the same as the proof of Theorem 3.10 with $K_1 = I_{+0} \cup I_{00}$, $K_2 = I_0$, and $K_3 = I_{0+}$.□

As the final point of this section, we mention that the following implications are straightforward consequence of the definitions of stationarity conditions:

$$USC \Rightarrow VSC \Rightarrow WSC.$$

Note that these three stationarity conditions coincide when $I_{00} = \emptyset$.

4 Conclusions and Comparison with Previous Works

In this paper, we have studied a class of MMPVCs which is defined by locally Lipschitz functions. ACQ and GCQ are introduced for MMPVC, using the Clarke subdifferential, and these constraint qualifications are employed to establish the well-known Karush-Kuhn-Tucker type necessary criteria, named stationarity condition, of properly efficiency for MMPVCs. Moreover, we have introduced several other constraint qualifications for our considered problem, such as ACQ$_W$, GCQ$_W$, ACQ$_V$, and GCQ$_V$, and established some new stationarity conditions.

Now, we compare the results established in this article with the results presented in previous researches.

The results derived in the article extend the corresponding results derived in [13, 14, 18] from single-objective to the setting of multi-objective, and generalize the corresponding results stated in [1, 8, 9, 10, 11, 12, 17] from smooth to the nonsmooth case. Also, this article extends [23] from convex to locally Lipschitz case, and by adding the stationarity conditions for properly efficient solutions, it completes the results of [22].

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