Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2025.38600.5516

Multiplicity result for anisotropic Dirichlet problems involving $\overrightarrow{p}(x)-$Laplacian operator

Dofyniwassouani Alain Houede[a,*], Abdoul Raouf Pougma[b], Adama Ouédraogo[a]

[a] *Département de Mathématiques, Université Nazi BONI, Bobo Dioulasso, Houet, Burkina Faso*

[b] *Laboratoire de Mathématiques LaMIA, Université Nazi BONI, Bobo Dioulasso, Houet, Burkina Faso*

(Communicated by Abdolrahman Razani)

Abstract

We give existence and multiplicity results of nontrivial solutions for a class of Dirichlet problems involving the $\overrightarrow{p}(x)-$Laplacian operator. The proof is based on the use of adequate variational methods and a variant of the Mountain Pass lemma.

Keywords: Anisotropic equations, anisotropic Sobolev spaces, variational method, multiple solutions
2020 MSC: Primary 35A15; Secondary 37B30

1 Introduction

Recently, the study of problems involving anisotropic variable exponent has been motivated by the description of models in image restoration [1, 9], elastic mechanics [38], porous media [5, 17], electromagnetism [6, 37], nonstandard growth [2, 8, 10, 23, 27, 39] and electrorheological fluids [3, 4, 29, 36]. The natural framework for solving these kinds of problems is that of Sobolev spaces with variable exponents, whose origins date back to the pioneering work of Orlicz [26]. In this direction, we can find many previous works (see [13, 25, 30, 31, 32, 33, 34, 35]). Let us mention below those who motivated our investigation:

- In [12], the authors considered the isotropic $p(x)-$Laplacian operator and establish existence and multiplicity results of solutions for the following problem

$$\begin{cases} -\Delta_{p(x)}(u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

- In [21], the authors generalize the previous work [12] by considering the anisotropic $\overrightarrow{p}(.)-$Laplacian operator which possesses more complicated nonlinearities, mainly due to the fact that it is not homogeneous. In fact,

*Corresponding author
Email addresses: dofynia@yahoo.fr (Dofyniwassouani Alain Houede), pougmaabdoulraouf@gmail.com (Abdoul Raouf Pougma), adam_ouedraogo3@yahoo.fr (Adama Ouédraogo)

Received: August 2025 *Accepted:* November 2025

they provided an existence and multiplicity result of solutions for the following problem

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)}(u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In this manuscript, we extend the previous results obtained in [12] and [21] to a more general problem that incorporates the nonlinearity term $\lambda(x)|u|^{p_0(x)-2}$. Indeed, this nonlinearity generates several additional difficulties, which makes this work more complex and interesting. To overcome these difficulties, we combine tools, techniques and ideas developed in [18, 19, 20, 28] and reference therein. More precisely, we deals with existence and multiplicity of solutions to the following nonlinear anisotropic elliptic problem

$$(\mathcal{P}) \begin{cases} -\Delta_{\overrightarrow{p}(x)}(u) + \lambda(x)|u|^{p_0(x)-2} = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here, we assume that Ω is a bounded, open subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary denoted by $\partial\Omega$ and meas$(\Omega) > 0$. $\Delta_{\overrightarrow{p}(.)}$ represents the $\overrightarrow{p}(.)-$Laplace operator, acting from $W_0^{1,\vec{p}(.)}(\Omega)$ into its dual denote $W_0^{-1,\vec{p}'(.)}(\Omega)$, defined by

$$\Delta_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}\big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\big). \tag{1.4}$$

$f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function satisfying the following conditions

$$|f(x,t)| \leq \alpha(1 + |t|^{q(x)-1}), \text{for some } \alpha > 0, \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega,\ q \in C(\overline{\Omega}) \text{ and } 1 < q(x) < P_{-,\infty}, \forall x \in \overline{\Omega}. \tag{1.5}$$

$$\begin{gathered}\text{There exist a constant } M > 0 \text{ and } \theta > p_0^+ \text{ such that} \\ 0 < \theta F(x,t) \leq tf(x,t), \text{ for all } x \in \Omega \text{ and } |t| \geq M, \text{where } F(x,t) = \int_0^t f(x,s)ds.\end{gathered} \tag{1.6}$$

$$f(x,t) = o(|t|^{p_0^+-1}) \text{ as } t \to 0 \text{ and uniformly for } x \in \Omega, \text{with } q^- > p_0^+. \tag{1.7}$$

$$\lambda \in L^\infty(\Omega) \text{ and there exists } \lambda_0 > 0 \text{ such that } \quad \lambda(x) \geq \lambda_0 \text{ for all } x \in \Omega. \tag{1.8}$$

Let us recall that a function $u \in W_0^{1,\vec{p}(.)}(\Omega)$ is a weak solution to problem (1.3) if the following identity holds

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}v dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uv dx = \int_\Omega f(x,u)v dx, \quad \text{for all } v \in W_0^{1,\vec{p}(.)}(\Omega). \tag{1.9}$$

Related to the problem (1.3), we define the so-called energy functional $\Phi : W_0^{1,\vec{p}(.)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{\lambda(x)}{p_0(x)}|u|^{p_0(x)} dx - \int_\Omega F(x,u) dx. \tag{1.10}$$

It is well known that under (1.5), Φ is well defined and is a $\mathcal{C}^1$ functional with derivative given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\partial_{x_i}v dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2}uv dx - \int_\Omega f(x,u)v dx, \text{ for all } u,v \in W_0^{1,\vec{p}(.)}(\Omega).$$

Hence, from (1.9), it follows that u is a weak solution of (1.3) if and only if u is a critical point of Φ, i.e.

$$u \in W^{1,\vec{p}(.)}(\Omega) \text{ weak solution of (1.3) } \iff \langle \Phi'(u), v\rangle = 0\ \forall v \in W^{1,\vec{p}(.)}(\Omega). \tag{1.11}$$

Hence with the previous definition, we state as follows our main result, namely the existence at least two nontrivial solutions to Problem (1.3):

Theorem 1.1. Assume that (1.5)−(1.7) hold true and $f(x,0) = 0$ for a.e. $x \in \Omega$. Then the problem (1.3) has at least two nontrivial solutions, in which one is non-negative and one is non-positive.

The remaining part of this paper is organized as follows: In Section (2), we recall some notations and preliminaries about the anisotropic variable exponent Sobolev spaces. In Section (3), we give the main steps of our method to stablish the proof of our result.

2 Variational framework

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$ and $\text{meas}(\Omega) > 0$. We recall some know results on variable exponent Lebesgue and Sobolev spaces. See [7, 11, 14, 15, 22] and reference therein for more detail. Define

$$C_+(\overline{\Omega}) = \left\{p : \overline{\Omega} \longrightarrow \mathbb{R} \text{ measurable, such that } 1 < p^- \leq p^+ < N\right\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) \mid x \in \overline{\Omega}\} \qquad \text{and} \qquad p^+ = \operatorname{ess\,sup}\{p(x) \mid x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(.)}(\Omega) := \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable, such that } \int_\Omega |u(x)|^{p(x)}dx \ < \ \infty\right\},$$

endowed with the Luxemburg norm

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \ \rho_{p(.)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

where the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ is the convex modular of the $L^{p(.)}(\Omega)$ space defined by

$$\rho_{p(.)}(u) := \int_\Omega |u(x)|^{p(x)}dx,$$

and satisfies the following inequalities (see [11, 14]): for any $u \in L^{p(.)}(\Omega)$,

$$\min\left\{|u|^{p^-}_{p(.)};\ |u|^{p^+}_{p(.)}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{|u|^{p^-}_{p(.)};\ |u|^{p^+}_{p(.)}\right\} \leq |u|^{p^+}_{p(.)} + 1 \tag{2.1}$$

and

$$\min\left\{\rho^{1/p^+}_{p(.)}(u);\ \rho^{1/p^-}_{p(.)}(u)\right\} \leq |u|_{p(.)} \leq \max\left\{\rho^{1/p^+}_{p(.)}(u);\ \rho^{1/p^-}_{p(.)}(u)\right\} \leq \left(\rho_{p(.)}(u) + 1\right)^{1/p^-}. \tag{2.2}$$

The space $\left(L^{p(.)}(\Omega), |u|_{p(.)}\right)$ is a separable reflexive Banach space (see [22]), and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for all $x \in \Omega$. For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, the following Hölder type inequality holds true:

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(.)}|v|_{p'(.)}. \tag{2.3}$$

Let $\vec{p}(.) = \left(p_0(.), p_1(.), \dots, p_N(.)\right)$ such that for any $i = 0, 1, \dots, N$, $p_i(.) \in C_+(\Omega)$ with

$$p_0(x) := \max\{p_i(x)\} \ \text{ for all } x \in \Omega \text{ and } i = 1, \dots N,$$

and put

$$1 < p_i^- := ess \inf_{x\in\Omega} p_i(x) \leq ess \sup_{x\in\Omega} p_i(x) := p_i^+ < \infty,$$

$$p_m(x) := \min\{p_1(x), \dots, p_N(x)\} \ \text{ for all } x \in \Omega,$$

$$p_m^- := \min\{p_1^-, \dots, p_N^-\}, \quad p_m^+ := \max\{p_1^-, \dots, p_N^-\}, \quad p_0^+ := \max\{p_1^+, \dots, p_N^+\}.$$

We define the anisotropic variable exponent Sobolev space by

$$W^{1,\vec{p}(.)}(\Omega) := \Big\{u \in L^{p_0}(\Omega) \;\text{ and }\; \partial_{x_i} u \in L^{p_i(.)}(\Omega),\ \forall\, i = 1, \dots, N\Big\}$$

and $W_0^{1,\vec{p}(.)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}(.)}(\Omega)$. Equipped with the following norm

$$||u||_{\vec{p}(.)} = |u|_{p_0(.)} + \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}, \tag{2.4}$$

the spaces $W^{1,\vec{p}(.)}(\Omega)$ and $W_0^{1,\vec{p}(.)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [24, 16]). We note that if $q \in C(\overline{\Omega})$ and $q(x) < P_{-,\infty}, \forall x \in \overline{\Omega}$, then the embedding $W_0^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.

3 Proof of main results

Let us denote by $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ the positive and negative parts of u. Before going further, we recall the following technical and useful lemma:

Lemma 3.1 (See [21]).

1. If $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then $u^+, u^- \in W_0^{1,\overrightarrow{p}}(\Omega)$ and
$$\nabla u^+ = \begin{cases} \nabla u & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u \leq 0, \end{cases} \qquad \nabla u^- = \begin{cases} 0 & \text{if} \quad u \geq 0, \\ \nabla u & \text{if} \quad u < 0. \end{cases} \tag{3.1}$$
2. The mappings $u \mapsto u^\pm$ are continuous on $W_0^{1,\overrightarrow{p}}(\Omega)$.

Next, we consider the following truncated problem

$$(\mathcal{P}_\pm) \begin{cases} -\Delta_{\overrightarrow{p}(x)}(u) + \lambda(x)|u|^{p_0(x)-2} = f_\pm(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where

$$f_\pm(x, t) = \begin{cases} f(x, t) & \text{if } \pm t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.2. All solutions of $(\mathcal{P}_+)$ (resp. $(\mathcal{P}_-)$ are non-negative (resp. non-positive) solutions of $(\mathcal{P})$.

Proof . Let us define $\Phi_\pm : W_0^{1,\overrightarrow{p}}(\Omega) \longrightarrow \mathbb{R}$, by

$$\begin{aligned} \Phi_\pm(u) &= \int_\Omega \sum_{i=1}^{N} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - \int_\Omega F_\pm(x, u) dx \\ &= \int_\Omega \sum_{i=1}^{N} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - \int_\Omega F(x, u^\pm) dx, \end{aligned} \tag{3.4}$$

where $F_\pm(x, s) = \displaystyle\int_0^s f_\pm(x, t) dt$. In view of Lemma 3.1 and Condition (1.5), $\Phi_\pm$ is well defined on $W_0^{1,\overrightarrow{p}}(\Omega)$, weakly lower semi-continuous and $\mathcal{C}^1-$functionals. From (1.11), we derive the following equivalence:

$$u \in W_0^{1,\vec{p}(.)}(\Omega) \text{ weak solution of } (\mathcal{P}_+) \iff \langle \Phi'_+(u), v\rangle = 0\ \forall v \in W_0^{1,\vec{p}(.)}(\Omega). \tag{3.5}$$

Replacing v by u^- in (3.5), we obtain

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} u^- dx + \int_\Omega \lambda(x)|u|^{p_0(x)-2} u u^- dx - \int_\Omega f_+(x,u)u^- dx = 0.$$

Next, it follows that

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u^-|^{p_i(x)} dx + \int_\Omega \lambda(x)|u^-|^{p_0(x)} dx = 0.$$

From (1.8), we have

$$0 \leq \min(1;\lambda_0) \int_\Omega \sum_{i=0}^N |\partial_{x_i} u^-|^{p_i(x)} dx \leq \int_\Omega \sum_{i=1}^N |\partial_{x_i} u^-|^{p_i(x)} dx + \int_\Omega \lambda(x)|u^-|^{p_0(x)} dx = 0.$$

Which imply that

$$\int_\Omega \sum_{i=0}^N |\partial_{x_i} u^-|^{p_i(x)} dx = 0.$$

Thanks to (2.1), we have $\|u^-\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = 0$. Then, we deduce that $u^- = 0$ and $u = u^+$. Similarly, nontrivial critical points of Φ_- are non-positive solutions of $(\mathcal{P})$. $\square$

We prove the following result on the required properties of the energy functional Φ related to problem (1.3).

Lemma 3.3. Let conditions (1.5), (1.6) and (1.7) be satisfied. Then, the energy functional Φ_+ given in (3.4) satisfies the (PS)-condition.

Proof . Let $\{u_n\} \subseteq W_0^{1,\vec{p}(.)}(\Omega)$ be a (PS) sequence for the functional Φ_+: $\Phi_+(u_n)$ is bounded and $\Phi'_+(u_n) \longrightarrow 0$. Let us show that $(u_n)_n$ is bounded in $W_0^{1,\vec{p}(.)}(\Omega)$. Using (1.6), since $\Phi_+(u_n)$ is bounded, we observe that:

$$\begin{aligned} C_1 &\geq \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{\lambda(x)}{p_0(x)} |u_n|^{p_0(x)} dx - \int_\Omega F(x,u_n^+) dx \\ &\geq \frac{1}{p_0^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \frac{1}{p_0^+} \int_\Omega \lambda(x)|u_n|^{p_0(x)} dx - \int_\Omega \frac{u_n^+}{\theta} f(x,u_n^+) dx + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\begin{aligned} \langle \Phi'_+(u_n), u_n\rangle &= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)} dx - \int_\Omega f(x,u_n^+) u_n dx \\ &= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)} dx - \int_\Omega f(x,u_n^+) u_n^+ dx, \end{aligned}$$

which implies

$$C_1 \geq \left(\frac{1}{p_0^+} - \frac{1}{\theta}\right)\left(\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)} dx\right) + \frac{1}{\theta}\langle \Phi'_+(u_n), u_n\rangle + C_2.$$

Suppose, by contradiction that $(u_n)_n$ unbounded in $\in W_0^{1,\vec{p}(.)}(\Omega)$, so $\|u_n\|_{W_0^{1,\vec{p}(.)}} \geq 1$ for rather large values of n. Using relation (2.1) we have

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)} dx \geq \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(.)}^{p_m^-} - N + \lambda_0 \int_\Omega |u_n|^{p_0(x)} dx,$$

then, according to the convexity of the application $t \in \mathbb{R}^+ \longmapsto t^{p_m^-}$, $p_m^- > 1$, we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}dx + \int_\Omega \lambda(x)|u_n|^{p_0(x)}dx &\geq \frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(.)}\right)^{p_m^-} + \lambda_0 \int_\Omega |u_n|^{p_0(x)}dx - N \\ &\geq \frac{1}{2^{p_m^- -1}} \min\left\{\frac{1}{N^{p_m^- -1}}, \lambda_0\right\}\|u_n\|^{p_m^-}_{\overrightarrow{p}(.)} - N - \lambda_0. \end{aligned} \tag{3.6}$$

Moreover, $\Phi'_+(u_n) \longrightarrow 0$ assure that there exists $C_3 > 0$ such that

$$-C_3\|u_n\|_{\overrightarrow{p}(.)} \leq \langle \Phi'_+(u_n), u_n\rangle \leq C_3\|u_n\|_{\overrightarrow{p}(.)}$$

for rather large values of n. Consequently,

$$C_1 \geq \left(\frac{1}{p_0^+} - \frac{1}{\theta}\right)\left(\frac{1}{2^{p_m^- -1}} \min\left\{\frac{1}{N^{p_m^- -1}}, \lambda_0\right\}\|u_n\|^{p_m^-}_{\overrightarrow{p}(.)} - N - \lambda_0\right) - \frac{C_3}{\theta}\|u_n\|_{\overrightarrow{p}(.)} + C_2.$$

Since $p_m^- > 1$ and $\left(\frac{1}{p_0^+} - \frac{1}{\theta}\right) > 0$, we end up with

$$\left(\frac{1}{p_0^+} - \frac{1}{\theta}\right)\left(\frac{1}{2^{p_m^- -1}} \min\left\{\frac{1}{N^{p_m^- -1}}, \lambda_0\right\}\|u_n\|^{p_m^-}_{\overrightarrow{p}(.)} - N - \lambda_0\right) - \frac{C_3}{\theta}\|u_n\|_{\overrightarrow{p}(.)} + C_2 \longrightarrow +\infty$$

as $\|u_n\|_{\overrightarrow{p}(.)} \to +\infty$, what is a contradiction. So $(u_n)_n$ is a bounded sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. $\square$

Lemma 3.4. There exist $r > 0$ and $\alpha > 0$ such that $\Phi_+(u) \geq \alpha$, for all $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(.)} = r$.

Proof . Combining Conditions (1.5) and (1.7), we deduce the following inequality

$$|F(x,t)| \leq \varepsilon|t|^{p_0^+} + C(\varepsilon)|t|^{q(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.7}$$

Taking $\|u\|_{\overrightarrow{p}(.)}$ small enough, we have

$$\Phi_+(u) \geq \frac{1}{p_0^+}\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}dx + \frac{1}{p_0^+}\int_\Omega \lambda(x)|u|^{p_0(x)}dx - \int_\Omega F(x,u^+)dx.$$

For such an element u we have $|\partial_{x_i} u|_{p_i(.)} < 1$ and, by relation (2.1), we obtain

$$\begin{aligned}\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}dx + \int_\Omega \lambda(x)|u|^{p_0(x)}dx &\geq \sum_{i=1}^N |\partial_{x_i} u|^{p_0^+}_{p_i(.)} + \lambda_0\int_\Omega |u|^{p_0(x)}dx \\ &\geq \frac{1}{N^{p_0^+ -1}}\left(\sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)}\right)^{p_0^+} + \lambda_0|u|^{p_0^+}_{p_0^+} \\ &\geq \frac{1}{2^{p_0^+ -1}} \min\left\{\frac{1}{N^{p_0^+ -1}}, \lambda_0\right\}\|u\|^{p_0^+}_{\overrightarrow{p}(.)}. \end{aligned} \tag{3.8}$$

Relations (3.7) and (3.8) imply

$$\begin{aligned}\Phi_+(u) &\geq \frac{1}{p_0^+ 2^{p_0^+ -1}} \min\left\{\frac{1}{N^{p_0^+ -1}}, \lambda_0\right\}\|u\|^{p_0^+}_{\overrightarrow{p}(.)} - \varepsilon\int_\Omega |u^+|^{p_0^+}dx - C(\varepsilon)\int_\Omega |u^+|^{q(x)}dx \\ &\geq \frac{1}{p_0^+ 2^{p_0^+ -1}} \min\left\{\frac{1}{N^{p_0^+ -1}}, \lambda_0\right\}\|u\|^{p_0^+}_{\overrightarrow{p}(.)} - \varepsilon\int_\Omega |u|^{p_0^+}dx - C(\varepsilon)\int_\Omega |u|^{q(x)}dx. \end{aligned} \tag{3.9}$$

From Condition (1.5), it follows $p_0^+ < q^- \leq q(x) < P_{-,\infty}$ $\forall x \in \overline{\Omega}$, then we have the following continuous and compact embedding:

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) \subset L^{p_0^+} \quad \text{and} \quad W_0^{1,\overrightarrow{p}(.)}(\Omega) \subset L^{q(x)},$$

which implies the existence of $C_4, C_5 > 0$ such that

$$\|u\|_{p_0^+} \leq C_4 \|u\|_{\overrightarrow{p}(.)} \quad \text{and} \quad \|u\|_{q(x)} \leq C_5 \|u\|_{\overrightarrow{p}(.)} \quad \text{for all } u \in W_0^{1,\overrightarrow{p}(.)}(\Omega).$$

Since $\|u\|_{\overrightarrow{p}(.)}$ is small enough, we deduce that

$$\int_\Omega |u|^{q(x)} dx \leq \|u\|_{q(x)}^{q^-} \leq C_6 \|u\|_{\overrightarrow{p}(.)}^{q^-}. \tag{3.10}$$

Combining (3.9) and (3.10) it results that

$$\Phi_+(u) \geq \frac{1}{p_0^+ 2^{p_0^+-1}} \min\left\{\frac{1}{N^{p_0^+-1}}, \lambda_0\right\} \|u\|_{\overrightarrow{p}(.)}^{p_0^+} - \varepsilon C_4^{p_0^+} \|u\|_{\overrightarrow{p}(.)}^{p_0^+} - C_7(\varepsilon) \|u\|_{\overrightarrow{p}(.)}^{q^-}$$

where C_i are positives constants. Let us choose $\varepsilon > 0$ such that

$$\varepsilon C_4^{p_0^+} \leq \frac{1}{p_0^+ 2^{p_0^+}} \min\left\{\frac{1}{N^{p_0^+-1}}, \lambda_0\right\},$$

we obtain

$$\begin{aligned}
\Phi_+(u) &\geq \frac{1}{p_0^+ 2^{p_0^+}} \min\left\{\frac{1}{N^{p_0^+-1}}, \lambda_0\right\} \|u\|_{\overrightarrow{p}(.)}^{p_0^+} - C_7(\varepsilon) \|u\|_{\overrightarrow{p}(.)}^{q^-} \\
&\geq \|u\|_{\overrightarrow{p}(.)}^{p_0^+} \left(\frac{1}{p_0^+ 2^{p_0^+}} \min\left\{\frac{1}{N^{p_0^+-1}}, \lambda_0\right\} - C_7 \|u\|_{\overrightarrow{p}(.)}^{q^- - p_0^+} \right).
\end{aligned} \tag{3.11}$$

Since $p_0^+ < q^-$, the function $t \longmapsto \left(\frac{1}{p_0^+ 2^{p_0^+}} \min\left\{\frac{1}{N^{p_0^+-1}}, \lambda_0\right\} - C_7 t^{q^- - p_0^+} \right)$ is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $\alpha > 0$ such that

$$\Phi_+(u) \geq \alpha, \quad \forall u \in W_0^{1,\overrightarrow{p}(.)}(\Omega) : \|u\|_{\overrightarrow{p}} = r. \tag{3.12}$$

□

In order to apply the Mountain Pass Theorem, it remains to show the following property on the energy functional Φ_+:

Lemma 3.5. There exists $e \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $\|e\| > r$ and $\Phi_+(e) < 0$.

Proof . We prove that

$$\Phi_+(su) \longrightarrow -\infty \text{ as } s \to +\infty,$$

for a certain $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$. From Condition (1.6), we obtain

$$F(x,t) \geq c|t|^\theta \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Let $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and $s > 1$ we have

$$\begin{aligned}
\Phi_+(su) &= \int_\Omega \sum_{i=1}^N \frac{|s|^{p_i(x)}}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} dx + \int_\Omega \frac{\lambda(x)|s|^{p_0(x)}}{p_0(x)} |u|^{p_0(x)} dx - \int_\Omega F(x,(su)^+) dx \\
&\leq s^{p_0^+} \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + s^{p_0^+} \int_\Omega \frac{\lambda(x)}{p_0(x)} |u|^{p_0(x)} dx - c s^\theta \int_\Omega |u^+|^\theta dx.
\end{aligned}$$

The fact $\theta > p_0^+$, gives that

$$\Phi_+(su) \longrightarrow -\infty \text{ as } s \to +\infty.$$

This implies that there exists $e \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $\|e\| > r$ and $\Phi_+(e) < 0$. □

Proof .[Proof of Theorem 1.1] In view of Lemma 3.3, Lemma 3.4, Lemma 3.5 and the Mountain Pass Theorem, Φ_+ admits a critical value $\mu \geq \alpha$ which is characterized by

$$\mu = \inf_{h\in\Lambda} \sup_{t\in[0,1]} \Phi_+(h(t)) \tag{3.13}$$

where

$$\Lambda = \{h \in C\big([0,1], W_0^{1,\overrightarrow{p}(.)}(\Omega)\big) : h(0) = 0 \text{ and } h(1) = e\}. \tag{3.14}$$

Then, the functional Φ_+ has a critical point u^+ with $\Phi_+(u^+) \geq \alpha$. But, $\Phi_+(0) = 0$, that is, $u^+ \neq 0$. Therefore, the problem $(\mathcal{P}_+)$ has a nontrivial solution which, by Lemma 3.2, is a non-negative solution of the problem $(\mathcal{P})$. Similarly, using Φ_-, we show that there exists furthermore a non-positive solution. □

References

[1] R. Aboulaich, D. Meskine, and A. Souissi, *New diffusion models in image processing*, Comput. Math. Appl. **56** (2008), 874–882.

[2] E. Acerbi and G. Mingione, *Regularity results for a class of functionals with nonstandard growth*, Arch. Ration. Mech. Anal. **156** (2001), 121–140.

[3] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), no. 3, 213–259.

[4] S. Antontsev and J.F. Rodrigues, *On stationary thermorheological viscous flows*, Univ. Ferrara Sez. VII Sci. Mat. **52** (2007), 19–36.

[5] S. Antontsev and S. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: Existence, uniqueness and localization properties of solutions*, Nonlinear Anal. **60** (2005), 515–545.

[6] A. Bahrouni and D. Repovš, *Existence and nonexistence of solutions for p(x)-curl systems arising in electromagnetism*, Complex Var. Elliptic Equ. **62** (2018), no. 2, 292–301.

[7] M.M. Boureanu and V. Radulescu, *Anisotropic Neumann problems in Sobolev spaces with variable exponent*, Nonlinear Anal. **75** (2012), 4471–4482.

[8] G. Butakin, E. Piskin, and E. Çelik, *Blowup and global solutions of a fourth-order parabolic equation with variable exponent logarithmic nonlinearity*, J. Funct. Spaces **2024** (2024), no. 1, 2847533.

[9] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functional in image restoration*, SIAM J. Appl. Math. **66** (2006), 1383–1406.

[10] N. Chems Eddine, M.A. Ragusa, and D.D. Repovs, *Correction to: On the concentration-compactness principle for anisotropic variable exponent Sobolev spaces and its applications*, Fract. Calc. Appl. Anal. **27** (2024), no. 2, 725–756.

[11] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag, vol. 2017, 2011.

[12] A.R. El Amrouss and F. Kissi, *Multiplicity of solutions for a general p(x)-Laplacian Dirichlet problem*, Arab J. Math. Sci. **19** (2013), no. 2, 205–216.

[13] S.A. Fallahshams, A. Razani, and S. Baraket, *Existence of nonnegative solutions for an anisotropic elliptic equation with singular weights and critical growth*, Bol. Soc. Mat. Mex. **31** (2025), 95.

[14] X. Fan, *Solution for p(x)-Laplacian Dirichlet problems with singular coefficient*, J. Math. Anal. Appl. **312** (2005), 464–477.

[15] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\vec{p}(x)$-Laplacian equations*, Complex Var. Elliptic Equ. **46** (2011), 623–642.

[16] X. Fan and D. Zhao, *On spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424–446.

[17] E. Henriques and J.M. Urbano, *Intrinsic scaling for PDEs with an exponential nonlinearity*, Indiana Univ. Math. J. **55** (2006), 1701–1722.

[18] D.A. Houede, I. Ibrango, and A. Ouedraogo, *Entropy solutions for some elliptic anisotropic problems involving variable exponent with Fourier boundary conditions and measure data*, J. Elliptic Parabol. Equ. **10** (2024), no. 1, 237–277.

[19] D.A. Houede, I. Sakandé, and A. Ouédraogo, *Existence of renormalized solutions for Fourier boundary-value problems involving variable exponent and measure data*, Ann. Math. Comput. Sci. **29** (2025), 28–71.

[20] D.A. Houede, I. Sakande, and A. Ouedraogo, *Entropy solutions for Leray-Lions anisotropic elliptic problem via the L^1 version of Minty's lemma*, Gulf J. Math. **21** (2025), no. 1, 596–609.

[21] F. Kissi and A. El Amrouss, *Multiplicity of solutions for anisotropic Dirichlet problem with variable exponent*, Bol. Soc. Parana. Mat. (3) **42** (2024), 143, 8p.

[22] O. Kovacik and J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

[23] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Ration. Mech. Anal. **105** (1989), 267–284.

[24] M. Mihailescu, P. Pucci, and V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl. **340** (2008), 687–698.

[25] D. Motreanu, A. Razani, and E. Tornatore, *An anisotropic Dirichlet system including unbounded coefficients*, Discrete Contin. Dyn. Syst. Ser. S **18** (2025), no. 12, 3799–3817.

[26] W. Orlicz, *Über konjugierte Exponentenfolgen*, Stud. Math. **3** (1931), no. 1, 200–211.

[27] A. Ouaziz and A. Aberqi, *Singular fractional double-phase problems with variable exponent via Morse's theory*, Filomat **38** (2024), no. 21, 7579–7595.

[28] A. Ourraoui, *On $\overrightarrow{p}(x)$-Anisotropic problems with Neumann boundary conditions*, Int. J. Differ. Equ. **2015** (2015).

[29] K.R. Rajagopal and M. Růžička, *Mathematical modeling of electrorheological materials*, Contin. Mech. Thermodyn. **13** (2001), no. 1, 59–78.

[30] A. Razani, *Competing Kohn-Spencer Laplacian systems with convection in non-isotropic Folland-Stein space*, Complex Var. Elliptic Equ. **70** (2025), no. 4, 667–680.

[31] A. Razani, *Entire weak solutions for an anisotropic equation in the Heisenberg group*, Proc. Amer. Math. Soc. **151** (2023), no. 11, 4771–4779.

[32] A. Razani and G.M. Figueiredo, *Infinitely many solutions for an anisotropic differential inclusion on unbounded domains*, Electron. J. Qual. Theory Differ. Equ. **33** (2024), 1–17.

[33] A. Razani and G.M. Figueiredo, *Degenerated and competing anisotropic (p,q)-Laplacians with weights*, Appl. Anal. **102** (2023), no. 16, 4471–4488.

[34] A. Razani and G.M. Figueiredo, *A positive solution for an anisotropic p & q-Laplacian*, Discrete Contin. Dyn. Syst. Ser. S **16** (2023), no. 6, 1629–1643.

[35] A. Razani, L.S. Tavares, and J.V.d.C. Sousa, *Existence and multiplicity results for a system involving an anisotropic $(\overrightarrow{p}, \overrightarrow{q})$-Laplacian type operator*, J. Fixed Point Theory Appl. **27** (2025), 66.

[36] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, vol. 1748, 2000.

[37] M. Xiang, F. Wang, and B. Zhang, *Existence and multiplicity of solutions for p(x)-curl systems arising in electromagnetism*, J. Math. Anal. Appl. **448** (2017), 1600–1617.

[38] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv. **29**

(1987), no. 1, 33–66.

[39] V.V. Zhikov, *On variational problems and nonlinear elliptic equations with nonstandard growth conditions*, J. Math. Sci. **173** (2011), no. 5, 463–570.