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On Character Contractibility of Real Banach Algebras

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Abstract

In this paper, we introduce and study φ-contractibility and character contractibility of real Banach algebras. We first show that if $(A, \|\cdot\|)$ is a real Banach algebra with nonempty character space $\triangle(A)$ then A is φ-contractibile if and only if A is $\overline{\varphi}$-contractible, where $\varphi \in \triangle(A)$ and $\overline{\varphi}$ is the conjugate of φ. We next prove that if $(A, \|\cdot\|)$ is a real Banach algebra and $A_{\mathbb{C}}$ is a suitable complexification of A, then A is character contractibile if and only if $A_{\mathbb{C}}$ is character contractibile. We also give a hereditary property for 0-contractibility of A. In continue, for a real Banach algebra A we study relations between the projectivity of Banach left A-modules and left φ-contractibility of A. Finally, we investigate the left character contractibility of certain real Banach algebras.

Keywords: Banach algebra; Character contractible; Complexification; Banach left module; projectivity
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1 Introduction and preliminaries

The symbol $\mathbb{F}$ denotes a field that can be $\mathbb{R}$ or $\mathbb{C}$. Let A be an algebra and $\mathfrak{X}$ be a linear space over $\mathbb{F}$ which is an A-bimodule with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ and the right module action $(a, x) \longmapsto x \cdot a : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$. A linear map $D : A \longrightarrow \mathfrak{X}$ over $\mathbb{F}$ is called an $\mathfrak{X}$-*derivation* on A if $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. For each $x \in \mathfrak{X}$, the map $d_{A,\mathfrak{X},x} : A \longrightarrow \mathfrak{X}$ defined by $d_{A,\mathfrak{X},x}(a) = a \cdot x - x \cdot a \quad (a \in A)$, is an $\mathfrak{X}$-derivation on A. An $\mathfrak{X}$-derivation D on A is called *inner* if $D = d_{A,\mathfrak{X},x}$ for some $x \in \mathfrak{X}$.

Let $(A, \|\cdot\|)$ be a Banach algebra over $\mathbb{F}$. An A-bimodule $\mathfrak{X}$ is called a *Banach A-bimodule* if $\mathfrak{X}$ is a Banach space over $\mathbb{F}$ with a norm $\|\cdot\|$ and there exists a positive constant k such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \quad \|x \cdot a\| \leq k\|a\|\|x\|,$$

for all $a \in A$ and $x \in \mathfrak{X}$. The set of all continuous $\mathfrak{X}$-derivations on A over $\mathbb{F}$ is denoted by $Z^1_{\mathbb{F}}(A, \mathfrak{X})$. It is known that $Z^1_{\mathbb{F}}(A, \mathfrak{X})$ is a linear subspace of $\mathcal{B}_{\mathbb{F}}(A, \mathfrak{X})$, the linear space of all bounded linear operators from A to $\mathfrak{X}$ over $\mathbb{F}$. The set of all inner $\mathfrak{X}$-derivations on A over $\mathbb{F}$ is denoted by $N^1_{\mathbb{F}}(A, \mathfrak{X})$. Clearly, $N^1_{\mathbb{F}}(A, \mathfrak{X})$ is a linear subspace of $Z^1_{\mathbb{F}}(A, \mathfrak{X})$ over $\mathbb{F}$. The quotient space $Z^1_{\mathbb{F}}(A, \mathfrak{X})/N^1_{\mathbb{F}}(A, \mathfrak{X})$ is called the *first cohomology group* of A over $\mathbb{F}$ with the coefficients in $\mathfrak{X}$ and denoted by $H^1_{\mathbb{F}}(A, \mathfrak{X})$. A is called *contractible* if $H^1_{\mathbb{F}}(A, \mathfrak{X}) = \{0\}$ for all Banach A-bimodules $\mathfrak{X}$ over $\mathbb{F}$, i.e., $Z^1_{\mathbb{F}}(A, \mathfrak{X}) = N^1_{\mathbb{F}}(A, \mathfrak{X})$. For a Banach space $(\mathfrak{X}, \|\cdot\|)$ over $\mathbb{F}$, the set of all linear functionals $\lambda : A \longrightarrow \mathbb{F}$ is called the dual space of $\mathfrak{X}$ and denoted by $\mathfrak{X}^*$. We know that $\mathfrak{X}^*$ is a Banach space over $\mathbb{F}$ with the operator norm.

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Furetheremore, if $\mathfrak{X}$ is a Banach A-bimodule over $\mathbb{F}$ then $\mathfrak{X}^*$ is also a Banach A-bimodule over $\mathbb{F}$ with the left module action $(a,\lambda) \longmapsto a \cdot \lambda : A \times \mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ and with the right module action $(a,\lambda) \longmapsto \lambda \cdot a : A \times \mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ given by

$$(a \cdot \lambda)(x) = \lambda(x \cdot a), \quad (\lambda \cdot a)(x) = \lambda(a \cdot x) \qquad (a \in A, \lambda \in \mathfrak{X}^*, x \in \mathfrak{X}).$$

A is called *amenable* if $H^1_{\mathbb{F}}(A, \mathfrak{X}^*) = \{0\}$ for all Banach A-bimodules $\mathfrak{X}$ over $\mathbb{F}$, i.e., $Z^1_{\mathbb{F}}(A, \mathfrak{X}^*) = N^1_{\mathbb{F}}(A, \mathfrak{X}^*)$. Let $\varphi : A \longrightarrow \mathbb{C}$ be an algebra homomorphism from A to $\mathbb{C}$ over $\mathbb{F}$. Then φ is called a *character* of A (the zero homomorphism from A to $\mathbb{C}$, respectively) if $\varphi(a_0) \neq 0$ for some $a_0 \in A$ ($\varphi(a) = 0$ for all $a \in A$, respectively). The zero homomorphism from A to $\mathbb{C}$ is denoted by 0. The set of all characters of A is denoted by $\triangle(A)$. It is known that $\triangle(A)$ is a subset of $\mathcal{B}_{\mathbb{F}}(A, \mathbb{C})$. If A is a commutative Banach algebra with identity over $\mathbb{F}$, then $\triangle(A)$ is nonempty. It is not true whenever A is noncommutative. For example $\mathcal{H}$, the set of all quaternion numbers, is a real noncommutative Banach algebra with identity but $\triangle(\mathcal{H}) = \emptyset$ (see [15, Page 20]). Note that it is possible $\triangle(A) = \emptyset$ wherever A has not identity (see [11, Examples 2.1.6 and 2.1.7]). It is clear that $\varphi \in \triangle(A)$ if and only if $\overline{\varphi} \in \triangle(A)$, where $\overline{\varphi} : A \longrightarrow \mathbb{C}$ is defined by $\overline{\varphi}(a) = \overline{\varphi(a)}$ $(a \in A)$.

Let A be a Banach algebra over $\mathbb{F}$ and $\varphi \in \triangle(A) \cup \{0\}$. We denote by $\mathcal{M}^r_{\mathbb{F}}(A, \varphi)$ ($\mathcal{M}^l_{\mathbb{F}}(A, \varphi)$, respectively) the collection of all complex Banach spaces $\mathfrak{X}$ for which $\mathfrak{X}$ is a Banach A-bimodule with the right (left, respectively) module action defined by $x \cdot a = \varphi(a)x$ ($a \cdot x = \varphi(a)x$, respectively) for all $(a, x) \in A \times \mathfrak{X}$.

Definition 1.1. Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $\varphi \in \triangle(B) \cup \{0\}$.

(i) B is called *left* (*right*, respectively) *φ-contractible* if $H^1_{\mathbb{C}}(B, \mathfrak{X}) = \{0\}$ for all $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(B, \varphi)$ ($\mathfrak{X} \in \mathcal{M}^l_{\mathbb{C}}(B, \varphi)$, respectively).

(ii) B is called *left* (*right*, respectively) *φ-amenable* if $H^1_{\mathbb{C}}(B, \mathfrak{X}^*) = \{0\}$ for all $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(B, \varphi)$ ($\mathfrak{X} \in \mathcal{M}^l_{\mathbb{C}}(B, \varphi)$, respectively).

The concepts of left and right φ-amenabilitybility and left and right φ-contractibility of complex Banach algebras were first introduced by Hu, Monfared and Traynor in [10] wh ich modified by Nasr-Isfahani and Soltani in [17] as the definition above.

Definition 1.2. Let B be a Banach algebra over $\mathbb{F}$.

(i) For $\varphi \in \triangle(B) \cup \{0\}$, it is said that B is *φ-contractible* (*φ-amenable*, respectively) if B is left and right φ-contractible (left and right *φ-amenable*, respectively).

(ii) B is called *left* (*right*, respectively) *character contractible* if B is left (right, respectively) φ-contractible for all $\varphi \in \triangle(B) \cup \{0\}$.

(iii) B is called *left* (*right*, respectively) *character amenable* if B is left (right, respectively) φ-amenable for all $\varphi \in \triangle(B) \cup \{0\}$.

(iv) B is called *character contractible* (*character amenable*) if B is left and right character contractible (left and right character amenable).

The concepts of φ-contractibility, φ-amenability, character contractibility and character amenability of complex Banach algebras were first introduced by Hu, Monfared and Traynor in [10]. Some properties of character contractibility and character amenability of complex Banach algebras were studied in [10], [12], [13] and [17]. The concepts of left and right φ-amenability, φ-amenability and character amenability of real Banach algebras were first introduced and studied in [1]. We now introduce the concepts of left and right φ-contractibility, φ-contractibility and character contractibility for real Banach algebras as the following.

Definition 1.3. Let A be a real Banach algebra and let $\varphi \in \triangle(A) \cup \{0\}$. We say that A is *left* (*right*, respectively) *φ-contractible* if $H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}$ for all $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$ ($\mathfrak{X} \in \mathcal{M}^l_{\mathbb{R}}(A, \varphi)$, respectively) satisfying

$$i(a \cdot x) = a \cdot (ix) \quad (i(x \cdot a) = (ix) \cdot a, \text{respectively})$$

for all $(a, x) \in A \times \mathfrak{X}$.

Definition 1.4. Let A be a Banach algebra over $\mathbb{F}$.

(i) For $\varphi \in \triangle(A) \cup \{0\}$, we say that A is *φ-contractible* if A is left and right φ-contractible.

(ii) A is called *left* (*right*, respectively) *character contractible* if A is left (right, respectively) φ-contractible for all $\varphi \in \triangle(A) \cup \{0\}$.

(iii) A is called *character contractible* if A is left and right character contractible.

For a real linear space (real algebra, respectively) E, a complex linear space (complex algebra, respectively) $E_{\mathbb{C}}$ is called a *complexification of* E if there exists an injective real linear mapping (a real algebra homomorphism, respectively) $J : E \longrightarrow E_{\mathbb{C}}$ such that $E_{\mathbb{C}} = J(E) \oplus iJ(E)$.

Let $\mathfrak{X}$ be a real linear space. Then $\mathfrak{X} \times \mathfrak{X}$ with the additive operation and scalar multiplication defined by

$$\begin{aligned} &(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad (x_1, x_2, y_1, y_2 \in \mathfrak{X}),\\ &(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x) \qquad (\alpha, \beta \in \mathbb{R}, x, y \in \mathfrak{X}), \end{aligned} \tag{1}$$

is a complexification of $\mathfrak{X}$ with respect to the injective linear map $J : \mathfrak{X} \longrightarrow \mathfrak{X} \times \mathfrak{X}$ defined by $J(x) = (x, 0)$, $x \in \mathfrak{X}$.

Let A be a real algebra. Then $A \times A$ with the algebra operations

$$\begin{aligned} &(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \qquad (a_1, a_2, b_1, b_2 \in A),\\ &(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \alpha b + \beta a) \qquad (\alpha, \beta \in \mathbb{R}, a, b \in A),\\ &(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) \qquad (a_1, b_1, a_2, b_2 \in A), \end{aligned} \tag{2}$$

is a complexification of A with the algebra homomorphism $J : A \longrightarrow A \times A$ defined by $J(a) = (a, 0)$, $a \in A$.

Let $(E, \|\cdot\|)$ be a real normed linear space (real normed algebra, respectively) and let $E_{\mathbb{C}}$ be a complexification of E with respect to an injective real linear mapping (real algebra homomorphism, respectively) $J : E \longrightarrow E_{\mathbb{C}}$ that is eqquiped with a norm (an algebra norm, respectively) $|||\cdot|||$. We say that $|||\cdot|||$ satisfies in the $(*)$ condition if there exist positive constants k_1 and k_2 such that

$$\max\{\|a\|, \|b\|\} \leq k_1|||J(a) + iJ(b)||| \leq k_2 \max\{\|a\|, \|b\|\},$$

for all $a, b \in E$. By [5, Proposition I.1.13], there exists a norm (an algebra norm, respectively) $|||\cdot|||$ on $E_{\mathbb{C}}$ satisfying in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$ where $E_{\mathbb{C}} = E \times E$ and $J : E \longrightarrow E_{\mathbb{C}}$ is defined by $J(a) = (a, 0)$, $a \in E$. Note that the $(*)$ condition implies that $(E, \|\cdot\|)$ is a real Banach space (a real Banach algebra, respectively) if and only if $(E_{\mathbb{C}}, |||\cdot|||)$ is complex Banach space (a complex Banach algebra, respectively).

Let $(A, \|\cdot\|)$ be a real Banach algebra, $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ and $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition. In [4], it is shown that A is contractible (amenable, respectively) if and only if $A_{\mathbb{C}}$ is contractible (amenable, respectively). In this paper, we study left and right φ-contractibility, φ-contractibilit and character contractibilit of real Banach algebras and extend the obtained result in [1] as fallows. In Section 2, we prove that A is left (right, respectively) φ-contractible if and only if A is $\overline{\varphi}$-contractible, whenever $\varphi \in \triangle(A)$. Moreover, we give a characterization of left and right φ-contractibility of A whenever $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$. In Section 3, we show that A is left character (right character, character) contractible if and only if $A_{\mathbb{C}}$ is left character (right character, character) contractible, respectively. In Section 4, we give a characterization of the left (right, respectively) 0-contractibility of A. In Section 5, we show that if $\varphi \in \triangle(A)$ and $\mathfrak{X}$ is a complex Banach space, then A is left φ-contractible if and only if the real left Banach A-module $\mathfrak{X}$, with the left module action $a \cdot x = \varphi(a)x \quad ((a, x) \in A \times \mathfrak{X})$, is projective. In Section 6, for a complex Banach algebra A we assume that $A_{\mathbb{R}}$ is A regarded as a real Banach algebra and show that $A_{\mathbb{R}}$ is left character (right character, character) contractible if and only if A is left character (right character, character) contractible, respectively. In Section 7, applying certain known results for left character contractibility of complex Banach algebras and some obtain results in Sections 2-6, we give some results for the left character contractibility of certain real Banach algebras.

2 Relation between φ-contractibility and $\overline{\varphi}$-contractibility

We first investigate the relation between φ-contractibility and $\overline{\varphi}$-contractibility for a real Banach algebra A, where $\varphi \in \triangle(A)$.

Theorem 2.1. Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$ and let $\varphi \in \triangle(A)$. Then the following assertions hold.

(i) A is left φ-contractible if and only if A is left $\overline{\varphi}$-contractible.

(ii) A is right φ-contractible if and only if A is right $\overline{\varphi}$-contractible.

(iii) A is φ-contractible if and only if A is $\overline{\varphi}$-contractible.

Proof . (i). We first assume that A is left φ-contractible. Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \overline{\varphi})$ with the norm $\|\cdot\|$ such that $i(a \cdot x) = a \cdot (ix)$, for all $(a, x) \in A \times \mathfrak{X}$. Let $\underline{\mathfrak{X}}$ denote $\mathfrak{X}$ with the scalar multiplication $(\alpha, x) \longmapsto \alpha * x : \mathbb{C} \times \underline{\mathfrak{X}} \longrightarrow \underline{\mathfrak{X}}$ defined by

$$\alpha * x = \bar{\alpha} x \quad (\alpha \in \mathbb{C}, x \in \underline{\mathfrak{X}}).$$

It is easy to see that $\underline{\mathfrak{X}}$ is a complex Banach space with the norm $\|\cdot\|$ and a real Banach A-bimodule with the module actions $(a, x) \longmapsto a \odot x : A \times \underline{\mathfrak{X}} \longrightarrow \underline{\mathfrak{X}}$ and $(a, x) \longmapsto x \odot a : A \times \underline{\mathfrak{X}} \longrightarrow \underline{\mathfrak{X}}$ defined by

$$\begin{aligned} a \odot x &= a \cdot x \quad (a \in A, x \in \underline{\mathfrak{X}}), \\ x \odot a &= \varphi(a) * x = \overline{\varphi}(a) x \quad (x \in \underline{\mathfrak{X}}, a \in A). \end{aligned}$$

Therefore, $\underline{\mathfrak{X}} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$. Moreover, for each $(a, x) \in (A \times \underline{\mathfrak{X}})$ we have

$$i * (a \odot x) = \bar{i}(a \odot x) = -i(a \odot x) = -(a \cdot (ix)) = a \cdot (-ix) = a \cdot (\bar{i}x) = a \cdot (i * x).$$

Hence, by hypothesis we have

$$H^1_{\mathbb{R}}(A, \underline{\mathfrak{X}}) = \{0\}. \tag{1}$$

Assume that $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X})$. Define the map $\underline{d} : A \longrightarrow \underline{\mathfrak{X}}$ by

$$\underline{d}(a) = d(a) \quad (a \in A).$$

It is easy to see that $\underline{d}$ is a bounded real linear operator from A to $\underline{\mathfrak{X}}$ and $\|\underline{d}\| = \|d\|$. Similar to the argument given in [1, Theorem 2.1], we get $\underline{d}(ab) = \underline{d}(a) \odot b + a \odot \underline{d}(b)$, for all $a, b \in A$. Hence, $\underline{d} \in Z^1_{\mathbb{R}}(A, \underline{\mathfrak{X}})$ and so, by (1), there exists $x \in \underline{\mathfrak{X}}$ such that

$$\underline{d} = d_{A, \underline{\mathfrak{X}}, x}. \tag{2}$$

Applying (2), for each $a \in A$ we have

$$\begin{aligned} d(a) = \underline{d}(a) = d_{A, \underline{\mathfrak{X}}, x}(a) &= a \odot x - x \odot a = a \cdot x - (\varphi(a) * x) \\ &= a \cdot x - \overline{\varphi(a)} x = a \cdot x - \overline{\varphi}(a) x = a \cdot x - x \cdot a = d_{A, \mathfrak{X}, x}(a). \end{aligned}$$

Hence, $d = d_{A, \mathfrak{X}, x}$. Therefore, $H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}$ and so A is left φ-contractible.

We now assume that A is left $\overline{\varphi}$-contractible. By the necessity part, A is left $\overline{\overline{\varphi}}$-contractible, that is, A is left φ-contractible. Hence, (i) holds.
(ii). It follows similar to (i).
(iii). This follows from (i) and (ii). $\square$

We now characterize the φ-contractibility of a real Banach algebra A, where $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$.

Theorem 2.2. Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$ and let $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$. Then the following assertions are equivalent.

(i) $H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}$ for each real Banach A-bimodule $\mathfrak{X}$ for which the right module action of A on $\mathfrak{X}$ is defined by $x \cdot a = \varphi(a)x$, $(a, x) \in A \times \mathfrak{X}$.

(ii) There is an element $m \in A$ such that $\varphi(m) = 1$ and $am = \varphi(a)m$ for all $a \in A$.

(iii) A is left φ-contractible.

Proof . (i)$\Rightarrow$(ii). Assume that (i) holds. It is easy to see that $\mathrm{Ker}(\varphi)$ is a real Banach A-bimodule with the module actions defined by

$$\begin{aligned} a \cdot x &= ax \quad (a \in A, \quad x \in \mathrm{Ker}(\varphi)), \\ x \cdot a &= \varphi(a)x \quad (x \in \mathrm{Ker}(\varphi), \quad a \in A). \end{aligned}$$

Hence, by (i) we have

$$H^1_{\mathbb{R}}(A, \mathrm{Ker}(\varphi)) = \{0\}. \tag{3}$$

Since $\varphi \in \triangle(A)$ and $\overline{\varphi} = \varphi$, we deduce that $\varphi(A)$ is a real subalgebra of $\mathbb{C}$, $\{0\}$ is a proper subset of $\varphi(A)$ and $\varphi(A) \subseteq \mathbb{R}$. Hence, $\varphi(A) = \mathbb{R}$ and so there exists $b_1 \in A$ with $\varphi(b_1) = 1$. Define the map $d : A \longrightarrow \mathrm{Ker}(\varphi)$ by

$$d(a) = ab_1 - \varphi(a)b_1 \quad (a \in A).$$

It is easy to see that d is a bounded real linear operator from A to $\mathrm{Ker}(\varphi)$. On the other hand, we have

$$\begin{aligned} d(a)\cdot b + a\cdot d(b) &= (ab_1 - \varphi(a)b_1)\cdot b + a\cdot(bb_1 - \varphi(b)b_1)\\ &= \varphi(b)(ab_1 - \varphi(a)b_1) + a(bb_1 - \varphi(b)b_1)\\ &= a(bb_1) - \varphi(a)\varphi(b)b_1\\ &= (ab)b_1 - \varphi(ab)b_1\\ &= d(ab), \end{aligned}$$

for all $a, b \in A$. Hence, $d \in Z^1_{\mathbb{R}}(A, \mathrm{Ker}(\varphi))$ and so, by (3), there exists $x \in \mathrm{Ker}(\varphi)$ such that

$$d = d_{A,\mathrm{Ker}(\varphi),x}. \tag{4}$$

Set $m = b_1 - x$. Then $m \in A$ and

$$\varphi(m) = \varphi(b_1 - x) = \varphi(b_1) - \varphi(x) = 1 - 0 = 1.$$

Let $a \in A$. By the definition of d and applying (4), we have

$$\begin{aligned} am &= a(b_1 - x) = ab_1 - ax = d(a) + \varphi(a)b_1 - ax = d_{A,\mathrm{Ker}(\varphi),x}(a) + \varphi(a)b_1 - x\\ &= ax - xa - \varphi(a)b_1 - ax = -\varphi(a)x + \varphi(a)b_1 = \varphi(a)(b_1 - x) = \varphi(a)m. \end{aligned}$$

Hence, (ii) holds.
(ii)⇒(iii). Assume that (ii) holds. Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$ and

$$a\cdot(ix) = i(a\cdot x),$$

for all $(a, x) \in A \times \mathfrak{X}$. Assume that $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X})$. By (ii), there exists $m \in A$ such that $\varphi(m) = 1$ and $am = \varphi(a)m$ for all $a \in A$. This implies that $-d(m) \in \mathfrak{X}$ and for each $a \in A$ we have

$$\begin{aligned} d_{A,\mathfrak{X},-d(m)}(a) &= a\cdot(-d(m)) - (-d(m))\cdot a = -(a\cdot d(m)) + d(m)\cdot a\\ &= -(a\cdot d(m)) + \varphi(a)d(m) = -(a\cdot d(m)) + d(\varphi(a)m)\\ &= -(a\cdot d(m)) + d(am) = -(a\cdot d(m)) + d(a)\cdot m + a\cdot d(m)\\ &= d(a)\cdot m = \varphi(m)d(a)\\ &= d(a). \end{aligned}$$

Thus, $d = d_{A,\mathfrak{X},-d(m)}$. Therefore, $H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}$ and so A is left φ-contractible. Hence, (iii) holds.
(iii)⇒ (i): Assume that (iii) holds. Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach A-bimodule with the module actions $(a, x) \longmapsto a\cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ and $(a, x) \longmapsto x\cdot a : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ such that

$$x\cdot a = \varphi(a)x \quad (x \in \mathfrak{X}, \quad a \in A).$$

Set $\mathfrak{X}_{\mathbb{C}} = \mathfrak{X}\times\mathfrak{X}$. Then $\mathfrak{X}_{\mathbb{C}}$ is a complex linear space with the additive and scalar multiplication defined by (1). Moreover, $\mathfrak{X}_{\mathbb{C}}$ is a complexification of $\mathfrak{X}$ with the injective real linear mapping $J : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ defined by $J(x) = (x, 0) \quad (x \in \mathfrak{X})$. It is known that there exists a norm $|||\cdot|||$ on $\mathfrak{X}_{\mathbb{C}}$ satisfying in the $(*)$ condition with the positive constant $k_1 = 1$ and $k_2 = 2$. Hence, $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$ is a complex Banach space. It is easy to see that $\mathfrak{X}_{\mathbb{C}}$ is a real Banach A-bimodule with the module actions $(a, (x, y)) \longmapsto a \boxdot (x, y) : A \times \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ and $(a, (x, y)) \longmapsto (x, y) \boxdot a : A \times \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ defined by

$$\begin{aligned} a \boxdot (x, y) &= (a\cdot x, a\cdot y) \quad (a \in A, \quad (x, y) \in \mathfrak{X}_{\mathbb{C}}),\\ (x, y) \boxdot a &= (x\cdot a, y\cdot a) \quad ((x, y) \in \mathfrak{X}_{\mathbb{C}}, \quad a \in A). \end{aligned}$$

On the other hand, for all $(a, (x, y)) \in A \times \mathfrak{X}_{\mathbb{C}}$ we have

$$\begin{aligned} i(a \boxdot (x, y)) &= i(a\cdot x, a\cdot y) = (-(a\cdot y), a\cdot x) = (a\cdot(-y), a\cdot x)\\ &= a \boxdot (-y, x) = a \boxdot i(x, y). \end{aligned}$$

Hence, by (iii) we have
$$H^1_{\mathbb{R}}(A, \mathfrak{X}_{\mathbb{C}}) = \{0\}. \tag{5}$$
Assume that $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X})$. Define the map $D : A \longrightarrow \mathfrak{X}_{\mathbb{C}}$ by
$$D(a) = (d(a), 0) \quad (a \in A).$$
It is easy to see that D is a real linear mapping from A to $\mathfrak{X}_{\mathbb{C}}$. Moreover, for each $(a, b) \in A \times A$ we have
$$\begin{aligned} D(ab) &= (d(ab), 0) = (d(a) \cdot b + a \cdot d(b), 0) = (d(a) \cdot b, 0) + (a \cdot d(b), 0) \\ &= (d(a), 0) \boxdot b + a \boxdot (d(b), 0) = D(a) \boxdot b + a \boxdot D(b). \end{aligned}$$
Thus, D is an $\mathfrak{X}_{\mathbb{C}}$-derivation on A over $\mathbb{R}$. On the other hand,
$$|||D(a)||| = |||(d(a), 0)||| = 2 \leqslant \|d(a)\| \leqslant 2\|d\|\|a\|,$$
for all $a \in A$. Hence, D is bounded and $|||D||| \leqslant 2\|d\|$. Therefore, $D \in Z^1_{\mathbb{R}}(A, \mathfrak{X}_{\mathbb{C}})$ and so, by (5), there exists $(x, y) \in \mathfrak{X}_{\mathbb{C}}$ such that
$$D = d_{A,\mathfrak{X}_{\mathbb{C}},(x,y)}. \tag{6}$$
We claim that
$$d = d_{A,\mathfrak{X},x}. \tag{7}$$
Let $a \in A$. By the definition of D and (6), we have
$$\begin{aligned} (d(a), 0) = D(a) &= d_{A,\mathfrak{X}_{\mathbb{C}},(x,y)}(a) \\ &= a \boxdot (x, y) - (x, y) \boxdot a = (a \cdot x, a \cdot y) - (x \cdot a, y \cdot a) \\ &= (a \cdot x - x \cdot a, a \cdot y - y \cdot a) = (d_{A,\mathfrak{X},x}(a), d_{A,\mathfrak{X},y}(a)). \end{aligned}$$
This implies that
$$d(a) = d_{A,\mathfrak{X},x}(a). \tag{8}$$
Since (8) holds for all $a \in A$, we deduce that (7) holds. Therefore, $H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}$ and so (i) holds. Hence the proof is complete. □

Similarly, we obtain the following result.

Theorem 2.3. Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$ and let $\varphi \in \triangle(A)$ with $\overline{\varphi} = \varphi$. Then the following assertions are equivalent.

(i) $H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}$ for each real Banach A-bimodule $\mathfrak{X}$ for which the right module action of A on $\mathfrak{X}$ is defined by $a \cdot x = \varphi(a)x$, $(a, x) \in A \times \mathfrak{X}$.

(ii) There is an element $m \in A$ such that $\varphi(m) = 1$ and $ma = \varphi(a)m$ for all $a \in A$.

(iii) A is right φ-contractible.

3 Relation between character contractibility of A and $A_{\mathbb{C}}$

Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ and let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition. For $\varphi \in \triangle(A) \cup \{0\}$, we define the map $\varphi_C : A_{\mathbb{C}} \longrightarrow \mathbb{C}$ by
$$\varphi_C(J(a) + iJ(b)) = \varphi(a) + i\varphi(b) \qquad (a, b \in A). \tag{1}$$
Clearly, $\varphi_C \in \triangle(A_{\mathbb{C}})$ if $\varphi \in \triangle(A)$ and $\varphi_C = 0$ if $\varphi = 0$. Moreover, the map $\Psi : \triangle(A) \cup \{0\} \longrightarrow \triangle(A_{\mathbb{C}}) \cup \{0\}$ defined by
$$\Psi(\varphi) = \varphi_C \quad (\varphi \in \triangle(A) \cup \{0\}), \tag{2}$$
is bijection and $\Psi(0) = 0$. For $\varphi \in \triangle(A)$, φ_C is called the character of $A_{\mathbb{C}}$ induced by φ.

Here, we show that character contractibility of a real Banach algebra $(A, \|\cdot\|)$ is equivalent to character contractibility of complex Banach algebra $(A_{\mathbb{C}}, |||\cdot|||)$.

Theorem 3.1. Let $(A, \|\cdot\|)$ be a real Banach algebra, let $A_{\mathbb{C}}$ be a complexification of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, and let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition. Then the following hold.

(i) For $\varphi \in \triangle(A) \cup \{0\}$, A is left (right, respectively) φ-contractible if and only if $A_{\mathbb{C}}$ is left (right, respectively) φ_C-contractible.

(ii) A is left (right, respectively) character contractible if and only if $A_{\mathbb{C}}$ is left (right, respectively) character contractible.

(iii) A is character contractible if and only if $A_{\mathbb{C}}$ is character contractible.

Proof . (i). Since the algebra norm $|||\cdot|||$ satisfies in the $(*)$ condition, there exist positive constants k_1 and k_2 such that

$$\max\{\|a\|, \|b\|\} \leqslant k_1 |||J(a) + iJ(b)||| \leqslant k_2 \max\{\|a\|, \|b\|\}, \tag{3}$$

for all $a, b \in A$. We first assume that $\varphi \in \triangle(A) \cup \{0\}$ and A is left φ-contractible. Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(A_{\mathbb{C}}, \varphi_C)$ with the norm $\|\cdot\|$ and the module actions $(c, x) \longmapsto c \cdot x$ and $(c, x) \longmapsto x \cdot c$. Similar to the argument given in [1, Theorem 3.1], we can show that $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$. On the other hand, for each $(a, x) \in A \times \mathfrak{X}$ we have

$$i(a \odot x) = i(J(a) \cdot x) = J(a) \cdot (ix) = a \odot (ix).$$

Therefore,

$$H^1_{\mathbb{R}}(A, \mathfrak{X}) = \{0\}. \tag{4}$$

Let $D \in Z^1_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X})$. Define the map $d : A \longrightarrow \mathfrak{X}$ by

$$d(a) = D(J(a)) \quad (a \in A).$$

It is easy to see that d is a real linear operator. On the other hand, we have

$$\begin{aligned} d(ab) &= D(J(ab)) = D(J(a)J(b)) = J(a) \cdot D(J(b)) + D(J(a)) \cdot J(b) \\ &= J(a) \cdot d(b) + d(a) \cdot J(b) = d(a) \odot b + a \odot d(b), \end{aligned}$$

for all $a, b \in A$. Hence, d is an $\mathfrak{X}$-derivation on the real algebra A. Since

$$\|d(a)\| = \|D(J(a)\| \leqslant \|D\| \ \ |||J(a)||| \leqslant \frac{k_2}{k_1}\|D\| \ \ \|a\|,$$

for all $a \in A$, we deduce that d is bounded and $\|d\| \leqslant \frac{k_2}{k_1}\|D\|$. Thus $d \in Z^1_{\mathbb{R}}(A, \mathfrak{X})$. According to (4), there exists $x \in \mathfrak{X}$ such that

$$d = d_{A,\mathfrak{X},x}. \tag{5}$$

By the definition of d and applying (5), similar to the argument given in [1, Theorem 3.1] we can show that $D = d_{A_{\mathbb{C}},\mathfrak{X},x}$. Hence, $H^1_{\mathbb{C}}(A_{\mathbb{C}}, \mathfrak{X}) = \{0\}$. Therefore, $A_{\mathbb{C}}$ is left φ_C-contractible.

We now assume that $\varphi \in \triangle(A) \cup \{0\}$ and $A_{\mathbb{C}}$ is left φ_C-contractible. Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(A, \varphi)$ with the norm $\|\cdot\|$ and with the module actions $(a, x) \longmapsto a \cdot x$ and $(a, x) \longmapsto x \cdot a$ such that

$$i(a \cdot x) = a \cdot (ix), \tag{6}$$

for all $(a, x) \in A \times \mathfrak{X}$. Define the map $(J(a) + iJ(b), x) \longmapsto (J(a) + iJ(b)) \boxdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ by

$$(J(a) + iJ(b)) \boxdot x = (a \cdot x) + i(b \cdot x) \quad (a, b \in A, x \in \mathfrak{X}), \tag{7}$$

and the map $(J(a) + iJ(b), x) \longmapsto x \boxdot (J(a) + iJ(b)) : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ by

$$x \boxdot (J(a) + iJ(b)) = (x \cdot a) + i(x \cdot b) \quad (x \in \mathfrak{X}, a, b \in A). \tag{8}$$

Applying (6) and (7), we can show that

$$\begin{aligned} (\alpha + i\beta)((J(a) + iJ(b)) \boxdot x) &= ((\alpha + i\beta)(J(a) + iJ(b))) \boxdot x \\ &= (J(a) + iJ(b)) \boxdot ((\alpha + i\beta)x) \end{aligned}$$

for all $(\alpha, \beta, a, b, x) \in \mathbb{R} \times \mathbb{R} \times A \times A \times \mathfrak{X}$. Since $\varphi \in \triangle(A) \cup \{0\}$ and $\mathfrak{X} \in \mathcal{M}_{\mathbb{R}}^{r}(A, \varphi)$, we have $x \cdot a = \varphi(a)x$ for all $(x, a) \in \mathfrak{X} \times A$. This implies that

$$x \boxdot (J(a) + iJ(b)) = \varphi_C(J(a) + iJ(b))x, \tag{9}$$

for all $x \in \mathfrak{X}$ and $a, b \in A$. Applying (9) and (8), we get

$$\begin{aligned}(\alpha + i\beta)(x \boxdot (J(a) + iJ(b))) &= ((\alpha + i\beta)x) \boxdot (J(a) + iJ(b)) \\ &= x \boxdot ((\alpha + i\beta)(J(a) + iJ(b))),\end{aligned}$$

for all $(\alpha, \beta, x, a, b) \in \mathbb{R} \times \mathbb{R} \times \mathfrak{X} \times A \times A$. Hence, $\mathfrak{X}$ is a complex $A_{\mathbb{C}}$-bimodule. Since $\mathfrak{X}$ is a real Banach A-bimodule, there exists a positive constant k such that

$$\|a \cdot x\| \leqslant k\|a\| \ \ \|x\| \quad (a \in A, x \in \mathfrak{X}), \tag{10}$$
$$\|x \cdot a\| \leqslant k\|a\| \ \ \|x\| \quad (a \in A, x \in \mathfrak{X}). \tag{11}$$

Applying (3), (7) and (10), we get

$$\|x \cdot (J(a) + iJ(b))\| \leqslant 2kk_1 |||J(a) + iJ(b)||| \ \ \|x\|,$$

for all $(a, b, x) \in A \times A \times \mathfrak{X}$. Hence, $\mathfrak{X}$ is a complex Banach $A_{\mathbb{C}}$-bimodule and so, by (9), $\mathfrak{X} \in \mathcal{M}_{\mathbb{C}}^{r}(A_{\mathbb{C}}, \varphi_C)$. Therefore,

$$H_{\mathbb{C}}^{1}(A_{\mathbb{C}}, \mathfrak{X}) = \{0\}. \tag{12}$$

Let $d \in Z_{\mathbb{R}}^{1}(A, \mathfrak{X})$. Define the map $D : A_{\mathbb{C}} \longrightarrow \mathfrak{X}$ by

$$D(J(a) + iJ(b)) = d(a) + id(b) \quad (a, b \in A). \tag{13}$$

It is easy to show that D is a complex linear operator. According to $d \in Z_{\mathbb{R}}^{1}(A, \mathfrak{X})$ and applying (7), (8) and (13), we can easily show that

$$\begin{aligned}D((J(a) + iJ(b))(J(a') + iJ(b'))) &= D((J(a) + iJ(b))) \boxdot (J(a') + iJ(b')) \\ &\quad + (J(a) + iJ(b)) \boxdot D(J(a') + iJ(b')),\end{aligned}$$

for all $a, b, a', b' \in A$. Hence, D is a complex $\mathfrak{X}$-derivation on $A_{\mathbb{C}}$. By (13) and (3), we have

$$\begin{aligned}\|D(J(a) + iJ(b))\| &= \|d(a) + id(b)\| \leqslant ||d(a)|| + \|d(b)\| \\ &\leqslant \|d\| \ \ \|a\| + \|d\| \ \ \|b\| \leqslant 2\|d\| \max\{\|a\|, \|b\|\} \\ &\leqslant 2k_1\|d\| \ \ |||J(a) + iJ(b)|||,\end{aligned}$$

for all $a, b \in A$. This implies that D is bounded and $\|D\| \leqslant 2k_1\|d\|$. Hence, $D \in Z_{\mathbb{C}}^{1}(A_{\mathbb{C}}, \mathfrak{X})$. According to (12), there exists $x \in \mathfrak{X}$ such that

$$D = d_{A_{\mathbb{C}}, \mathfrak{X}, x}. \tag{14}$$

Applying (14), (13), (7) and (8), we have

$$d(a) = D(J(a)) = d_{A_{\mathbb{C}}, \mathfrak{X}, x}(J(a)) = J(a) \boxdot x - x \boxdot J(a) = a \cdot x - x \cdot a = d_{A, \mathfrak{X}, x}(a),$$

for all $a \in A$. Hence, $d = d_{A, \mathfrak{X}, x}$ and so $H_{\mathbb{R}}^{1}(A, \mathfrak{X}) = \{0\}$. Therefore, A is left φ-contractible.

Similarly, we can show that if $\varphi \in \triangle(A) \cup \{0\}$ then A is right φ-contractible if and only if $A_{\mathbb{C}}$ is right φ_C-contractible. Hence, (i) holds.
(ii). Since the map $\Psi : \triangle(A) \cup \{0\} \longrightarrow \triangle(A_{\mathbb{C}}) \cup \{0\}$ defined by (2) is bijection, (ii) follows from (i).
(iii). Clearly, (ii) implies that (iii) holds. $\square$

4 A hereditary property of left and right 0-contractibility

A hereditary property of the left 0-contractibility of complex Banach algebras studied by Nasr-Isfahani and Soltani [17, Proposition 3.4(ii)] which is modified as the following.

Proposition 4.1. Let $(B, \|\cdot\|)$ be a complex Banach algebra. Then B is left (right, respectively) 0-contractible if and only if B has a right (left, respectively) identity.

Applying Proposition 4.1 and part (i) of Theorem 3.1 for $\varphi = 0$, we obtain a hereditary property of left and right 0-contractibility of real Banach algebras as following.

Proposition 4.2. Let $(A, \|\cdot\|)$ be a real Banach algebra. Then A is left (right, respectively) 0-contractible if and only if A has a right (left, respectively) identity.

Proof . Take $A_{\mathbb{C}} = A \times A$. Recall that $A_{\mathbb{C}}$ is a complex algebra with the algebra operations defined by (2) and so it is a complexification of A with respect to the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0), \quad (a \in A)$. By [5, Proposition, I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$.

We first assume that A is left (right, respectively) 0-contractible. By part (i) of Theorem 3.1 for $\varphi = 0$, the complex Banach algebra $(A_{\mathbb{C}}, |||\cdot|||)$ is left (right, respectively) 0-contractible. Hence, $A_{\mathbb{C}}$ has a right (left, respectively) identity (e_1, e_2) by Proposition 4.1. It is easy to see that e_1 is a right (left, respectively) identity for A.

We now assume that A has a right (left, respectively) identity e. It is easy to see that $(e, 0)$ is a right (left, respectively) identity for $A_{\mathbb{C}}$. Hence, $A_{\mathbb{C}}$ is left (right, respectively) 0-contractible by Proposition 4.1. Therefore, A is left (right, respectively) 0-contractible by part (i) of Theorem 3.1 for $\varphi = 0$. □

As consequences of Propositions 4.1 and 4.2, we obtain the following results.

Corollary 4.3. Let $(A, \|\cdot\|)$ be a commutative Banach algebra over $\mathbb{F}$. Then A is left 0-contractible if and only if A is right 0-contractible.

Corollary 4.4. Let $(A, \|\cdot\|)$ be a Banach algebra over $\mathbb{F}$. Then A is 0-contractible if and only if A has identity.

Corollary 4.5. Let $(A, \|\cdot\|)$ be a complex Banach algebra and let $A_{\mathbb{R}}$ denote A regarded as a real algebra. Then A is left (right, respectively) 0-contractible if and only if $A_{\mathbb{R}}$ is left (right, respectively) 0-contractible.

5 Left φ-contractibility and projectivity

In this section, we assume that A is a real Banach algebra with $\triangle(A) \neq \emptyset$ and $\varphi \in \triangle(A)$. We discuss the relation between left φ-contractibility of A and projectivity of real Banach left A-modules.

Let A be a Banach algebra and $\mathfrak{X}$ be a left Banach A-module over $\mathbb{F}$. We say that $\mathfrak{X}$ is *essential* if the linear span $A \cdot \mathfrak{X}$ over $\mathbb{F}$ is dense in $\mathfrak{X}$, where $A \cdot \mathfrak{X} = \{a \cdot x : \quad a \in A, \ x \in \mathfrak{X}\}$.

The following result is a modification of [17, Proposition 4.1] which is useful in the sequel.

Proposition 5.1. Let A be a Banach algebra over $\mathbb{F}$ with $\triangle(A) \neq \emptyset$, let $\varphi \in \triangle(A)$ and let $\mathfrak{X}$ be a complex Banach space. Then $\mathfrak{X}$ is an essential Banach left A-module over $\mathbb{F}$ with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a, x) \in A \times \mathfrak{X}$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces over $\mathbb{F}$. We denote by $\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ the Banach space of all bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$ over $\mathbb{F}$ with the operator norm. We say that $T \in \mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ is *admissible* if $T \circ S \circ T = T$ for some $S \in \mathcal{B}_{\mathbb{F}}(\mathfrak{Y}, \mathfrak{X})$.

Let A be a Banach algebra over $\mathbb{F}$ and let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach left A-modules over $\mathbb{F}$. We denote by ${}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ the set of all $T \in \mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ for which T is an A-module morphism. Clearly, ${}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ is a closed subspace of $\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ over $\mathbb{F}$. An operator $T \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ is called a *retraction* if there exists $S \in {}_A\mathcal{B}(\mathfrak{Y}, \mathfrak{X})$ with $T \circ S = I_{\mathfrak{Y}}$, the identity self-map on $\mathfrak{Y}$. In this case, $\mathfrak{Y}$ is called a *retract* of $\mathfrak{X}$.

Let A be a Banach algebra and let $\mathcal{P}$ be a Banach left A-module over $\mathbb{F}$. We say that $\mathcal{P}$ is *projective* if for any Banach left A-modules $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathbb{F}$, each admissible epimorphism $T \in {}_A\mathcal{B}_{\mathbb{F}}(\mathfrak{X}, \mathfrak{Y})$ and each $S \in {}_A\mathcal{B}_{\mathbb{F}}(\mathcal{P}, \mathfrak{Y})$, there exists $R \in {}_A\mathcal{B}_{\mathbb{F}}(\mathcal{P}, \mathfrak{X})$ such that $T \circ R = S$.

Let A be a Banach algebra and let $\mathfrak{X}$ be a Banach space over $\mathbb{F}$. Then the projective tensor product $A \widehat{\otimes}_{\mathbb{F}} \mathfrak{X}$ is a Banach left A-module over $\mathbb{F}$ with the left module action determined by

$$a \cdot (b \otimes x) = ab \otimes x \qquad (a, b \in A, x \in \mathfrak{X}).$$

It is easy to see that if $\mathfrak{X}$ is a Banach left A-module over $\mathbb{F}$, then the linear map $\pi_{\mathbb{F}} : A\widehat{\otimes}_{\mathbb{F}}\mathfrak{X} \longrightarrow \mathfrak{X}$ determined by

$$\pi_{\mathbb{F}}(a \otimes x) = a \cdot x \quad (a \in A, x \in \mathfrak{X}),$$

is an element of ${}_A\mathcal{B}_{\mathbb{F}}(A\widehat{\otimes}_{\mathbb{F}}\mathfrak{X}, \mathfrak{X})$ which is called the *canonical morphism*.

The following result is due to Helemskii which is useful in the sequel.

Proposition 5.2. [8, Proposition I.V.11] Let A be a Banach algebra and let $\mathfrak{X}$ be an essential left A-bimodule over $\mathbb{F}$. Then $\mathfrak{X}$ is projective if and only if the canonical morphism $\pi_{\mathbb{F}} \in {}_A\mathcal{B}_{\mathbb{F}}(A\widehat{\otimes}_{\mathbb{F}}\mathfrak{X}, \mathfrak{X})$ is a retraction.

For a real Banach algebra $(A, \|\cdot\|)$ and a complex Banach space $(\mathfrak{X}, \|\cdot\|)$, we show that $\mathfrak{X}$ is a projective real Banach left A-module with a suitable left module action if and only if $\mathfrak{X}$ is a projective complex Banach $A_{\mathbb{C}}$-module with a suitable left module action.

Lemma 5.3. Let A be a real Banach algebra and let $\mathfrak{X}$ be a complex Banach space. Then the followings hold.

(i) $A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$ with the projective norm $\|\cdot\|_p$ is a complex Banach space whenever the scalar multiplication is determined by

$$\alpha(a \otimes x) = a \otimes \alpha x \qquad (\alpha \in \mathbb{C}, \quad a \in A, \quad x \in \mathfrak{X}). \tag{1}$$

(ii) $A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$ is a real Banach left A-module whenever the left module action is determined by

$$a \cdot (b \otimes x) = ab \otimes x \qquad (a, b \in A, x \in \mathfrak{X}).$$

Moreover,

$$i(a \cdot \theta) = a \cdot (i\theta),$$

for all $a \in A$ and $\theta \in A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$.

Theorem 5.4. Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$, let $\varphi \in \triangle(A)$, let $A_{\mathbb{C}}$ be a complexifiction of A with respect to an injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$, let $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying the $(*)$ condition and let $\mathfrak{X}$ be a complex Banach space. Then the following assertions are equivalent.

(i) $\mathfrak{X}$ is a projective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a, x) \in A \times \mathfrak{X}$.

(ii) $\mathfrak{X}$ is a projective complex Banach left $A_{\mathbb{C}}$-module with the left module action $(J(a)+iJ(b), x) \longmapsto (J(a)+iJ(b)) \cdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $(J(a) + iJ(b)) \cdot x = \varphi_C(J(a) + iJ(b))x, \quad (a, b, x) \in A \times A \times \mathfrak{X}$.

Proof . Clearly, $\mathfrak{X}$ is a real Banach left A-module (a complex Banach left $A_{\mathbb{C}}$-module, respectively) with the left module action defined in (i) (in (ii), respectively). Hence, $\mathfrak{X}$ is an essential real (complex) Banach left $A(A_{\mathbb{C}})$-module by Proposition 5.1. Let $\pi_{\mathbb{R}} : A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X} \longrightarrow \mathfrak{X}$ be the real canonical morphism determined by

$$\pi_{\mathbb{R}}(a \otimes x) = a \cdot x \quad (a \in A, x \in \mathfrak{X}),$$

and let $\pi_{\mathbb{C}} : A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X} \longrightarrow \mathfrak{X}$ be the complex canonical morphism determined by

$$\pi_{\mathbb{C}}((J(a) + iJ(b)) \otimes x) = (J(a) + iJ(b)) \cdot x \quad (a, b \in A, x \in \mathfrak{X}).$$

It is easy to see that $A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}$ is a real Banach left A-module with the left module action determined by

$$a \cdot ((J(b) + iJ(c)) \otimes x) = J(a) \cdot ((J(b) + iJ(c)) \otimes x) \quad (a, b, c \in A, x \in \mathfrak{X}).$$

Define the map $T : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$ by

$$T(J(a) + iJ(b), x) = (a \otimes x) + (b \otimes (ix)) \quad (a, b \in A, x \in \mathfrak{X}).$$

It is easy to see that T is a complex bilinear mapping. Moreover,

$$\begin{aligned}
\|T((J(a) + iJ(b)), x)\|_p &= \|(a \otimes x) + (b \otimes (ix))\|_p \\
&\leq \|a \otimes x\|_p + \|b \otimes (ix)\|_p \\
&= \|a\|\|x\| + \|b\|\|x\| \\
&\leq 2\|x\| \max\{\|a\|, \|\|b\|\} \\
&\leq 2k_1\|x\||||J(a) + iJ(b)|||,
\end{aligned}$$

for all $(a,b,x) \in A \times A \times \mathfrak{X}$. Thus, the real bilinear mapping T is bounded and $\|T\| \leqslant 2k_1$. Hence, there exists a bounded complex linear mapping $\tilde{T} : A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X} \longrightarrow A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$ determined by

$$\tilde{T}(J(a)+iJ(b))\otimes x) = T(J(a)+iJ(b),x) \quad (a,b \in A, x \in \mathfrak{X}).$$

Moreover, $\|\tilde{T}\| = \|T\|$. It is easy to see that $A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}$ is a real Banach left A-module with the left module action determined by

$$a\cdot((J(b)+iJ(c))\otimes x) = (J(ab)+iJ(ac))\otimes x \quad (a,b,c \in A, x \in \mathfrak{X}).$$

Since

$$\begin{aligned}\tilde{T}(a\cdot((J(b)+iJ(c))\otimes x) &= \tilde{T}((J(ab)+iJ(ac))\otimes x) = T((J(ab)+iJ(ac)),x)\\ &= ab\otimes x + ac\otimes(ix) = a\cdot(b\otimes x + c\otimes(ix))\\ &= a\cdot T((J(b)+iJ(c)),x) = a\cdot\tilde{T}((J(b)+iJ(c))\otimes x)\end{aligned}$$

for all $(a,b,c,x) \in A\times A\times A\times\mathfrak{X}$, we deduce that $\tilde{T} \in {}_A\mathcal{B}_{\mathbb{R}}(A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}, A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X})$. By part (i) of Lemma 5.3, $(A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}, \|\cdot\|_p)$ is a complex Banach space with the scalar multiplication determined by

$$\alpha(a\otimes x) = a\otimes\alpha x \qquad (\alpha\in\mathbb{C}, a\in A, x\in\mathfrak{X}). \tag{2}$$

Moreover,

$$i(a\cdot\theta) = a\cdot(i\theta), \tag{3}$$

for all $a \in A$ and $\theta \in A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$. It is easy to see that $A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$ is a complex Banach left $A_{\mathbb{C}}$-module with the left module action determined by

$$(J(a)+iJ(b))\cdot(c\otimes x) = (ac)\otimes x + (bc)\otimes(ix) \quad (a,c,b\in A, x\in\mathfrak{X}).$$

Define the map $S : A\times\mathfrak{X} \longrightarrow A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}$ by

$$S(a,x) = J(a)\otimes x \quad (a\in A, x\in\mathfrak{X}).$$

It is easy to see that S is a real bilinear mapping. Moreover

$$\|(S(a),x)\|_p = \|J(a)\otimes x\|_p \le \||J(a)\|| \;\; \|x\| \le \frac{k_2}{k_1}\|a\| \;\; \|x\|.$$

for all $a \in A$ and $x \in \mathfrak{X}$. Thus, S is bounded and $\|S\| \le \frac{k_2}{k_1}$. Hence, there exists a bounded real linear map $\tilde{S} : A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X} \longrightarrow A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}$ determined by

$$\tilde{S}(a\otimes x) = S(a,x) = J(a)\otimes x \quad (a\in A, x\in\mathfrak{X}).$$

Moreover, $\|\tilde{S}\| = \|S\|$. Since

$$\begin{aligned}\tilde{S}(i(a\otimes x)) &= \tilde{S}(a\otimes(ix)) = S(a,ix) = J(a)\otimes(ix)\\ &= i(J(a)\otimes x) = iS(a,x) = i\tilde{S}(a\otimes x),\end{aligned}$$

for all $a\in A$ and $x\in\mathfrak{X}$, we deduce that $\tilde{S}$ is a complex linear mapping. Therefore, $\tilde{S} \in \mathcal{B}_{\mathbb{C}}(A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}, A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X})$. Since

$$\begin{aligned}\tilde{S}((J(a)+iJ(b))\cdot(c\otimes x)) &= S((ac)\otimes x + i(bc)\otimes(ix))\\ &= \tilde{S}((ac)\otimes x) + \tilde{S}((bc)\otimes(ix))\\ &= \tilde{S}((ac)\otimes x) + \tilde{S}(i(bc\otimes x))\\ &= \tilde{S}((ac)\otimes x) + i\tilde{S}(bc\otimes x)\\ &= S(ac\otimes x) + iS(bc\otimes x)\\ &= (J(a)J(c))\otimes x + i(J(b)J(c))\otimes x\\ &= J(a)\cdot(J(c)\otimes x) + i(J(b)\cdot(J(c)\otimes x))\\ &= J(a)\cdot(J(c)\otimes x) + (iJ(b))\cdot(J(c)\otimes x)\\ &= (J(a)+iJ(b))\cdot(J(c)\otimes x) = (J(a)+iJ(b))\cdot\tilde{S}(c\otimes x).\end{aligned}$$

for all $(a,b,c,x) \in A \times A \times A \times \mathfrak{X}$, we deduce that $\tilde{S} \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}, A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X})$.
It is easy to see that

$$\tilde{S} \circ \tilde{T}((J(a) + iJ(b)) \otimes x) = (J(a) + iJ(b)) \otimes x,$$

for all $a, b \in A$ and $x \in \mathfrak{X}$. This implies that

$$\tilde{S} \circ \tilde{T} = I_{A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}}. \tag{4}$$

Similarly, we can show that

$$\tilde{T} \circ \tilde{S} = I_{A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}}. \tag{5}$$

Since

$$\begin{aligned}
\pi_{\mathbb{R}} \circ \tilde{T}((J(a) + iJ(b)) \otimes x) &= \pi_{\mathbb{R}}(a \otimes x + b \otimes (ix)) = a \cdot x + b \cdot (ix) \\
&= \varphi(a)x + \varphi(b)(ix) = \varphi_C(J(a))x + \varphi_C(J(b))(ix) \\
&= J(a) \cdot x + J(b) \cdot (ix) = J(a) \cdot x + (iJ(b)) \cdot x \\
&= J(a) + iJ(b)) \cdot x = \pi_{\mathbb{C}}((J(a) + iJ(b)) \otimes x),
\end{aligned}$$

for all $a, b \in A$ and $x \in \mathfrak{X}$, we deduce that $(\pi_{\mathbb{R}} \circ \tilde{T})(\theta) = \pi_{\mathbb{C}}(\theta)$, for all $\theta \in A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X}$. This implies that

$$\pi_{\mathbb{R}} \circ \tilde{T} = \pi_{\mathbb{C}}. \tag{6}$$

From (5) and (6), we get

$$\pi_{\mathbb{R}} = \pi_{\mathbb{C}} \circ \tilde{S} \tag{7}$$

By (7) and the complex linearity of $\pi_{\mathbb{C}}$ and $\tilde{S}$, we deduce that $\pi_{\mathbb{R}}$ is complex linear. Assume that $\mathfrak{X}$ is a projective real Banach left A-module with the left module action

$$a \cdot x = \varphi(a)x \quad (a \in A, x \in \mathfrak{X}).$$

By Proposition 5.1, $\mathfrak{X}$ is essential. Therefore, by Proposition 5.2 for $\mathbb{F} = \mathbb{R}$, we deduce that $\pi_{\mathbb{R}}$ is a retraction. Hence, there exists $\Lambda \in {}_A\mathcal{B}_{\mathbb{R}}(\mathfrak{X}, A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X})$ such that

$$\pi_{\mathbb{R}} \circ \Lambda = I_{\mathfrak{X}}. \tag{8}$$

Define the map $\Lambda_C : \mathfrak{X} \longrightarrow A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}$ by

$$\Lambda_C(x) = \Lambda(x) - i\Lambda(ix) \quad (x \in \mathfrak{X}).$$

It is easy to see that $\Lambda_C \in \mathcal{B}_{\mathbb{C}}(\mathfrak{X}, A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X})$. Applying (8), we get

$$\begin{aligned}
\Lambda_C(J(a) \cdot x) &= \Lambda_C(\varphi_C(a)x) = \Lambda_C(\varphi(a)x) \\
&= \Lambda_C(a \cdot x) = \Lambda(a \cdot x) - i\Lambda(i(a \cdot x)) \\
&= \Lambda(a \cdot x) - i\Lambda(i(J(a) \cdot x)) = \Lambda(a \cdot x) - i\Lambda((J(a) \cdot (ix)) \\
&= a \cdot \Lambda(x) - i(a \cdot \Lambda(ix)) = a \cdot \Lambda(x) - a \cdot (i\Lambda(ix)) \\
&= a \cdot (\Lambda(x) - i\Lambda(ix)) = J(a) \cdot (\Lambda(x) - i\Lambda(ix)) \\
&= J(a) \cdot \Lambda_C(x),
\end{aligned}$$

for all $a, b \in A$ and $x \in \mathfrak{X}$. This implies that

$$\Lambda_C((J(a) + iJ(b)) \cdot x) = (J(a) + iJ(b)) \cdot \Lambda_C(x),$$

for all $a \in A$ and $x \in \mathfrak{X}$. Hence, $\Lambda_C \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(\mathfrak{X}, A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X})$. Therefore, $\frac{1}{2}\tilde{S} \circ \Lambda_C \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(\mathfrak{X}, A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X})$ since $\tilde{S} \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(A\widehat{\otimes}_{\mathbb{R}}\mathfrak{X}, A_{\mathbb{C}}\widehat{\otimes}_{\mathbb{C}}\mathfrak{X})$. Applying (7) and (8) and according to the complex linearity of $\pi_{\mathbb{R}}$, we have

$$\begin{aligned}
\pi_{\mathbb{C}} \circ \frac{1}{2}(\tilde{S} \circ \Lambda_C)(x) &= \frac{1}{2}(\pi_{\mathbb{C}} \circ \tilde{S} \circ \Lambda_C)(x) = \frac{1}{2}(\pi_{\mathbb{R}} \circ \Lambda_C)(x) = \frac{1}{2}\pi_{\mathbb{R}}(\Lambda(x) - i\Lambda(ix)) \\
&= \frac{1}{2}(\pi_{\mathbb{R}}(\Lambda(x)) - \pi_{\mathbb{R}}(i\Lambda(ix))) = \frac{1}{2}(\pi_{\mathbb{R}}(\Lambda(x)) - i\pi_{\mathbb{R}}(\Lambda(ix))) \\
&= \frac{1}{2}(x - i(ix)) = I_{\mathfrak{X}}(x),
\end{aligned}$$

for all $x \in \mathfrak{X}$. This implies that $\pi_{\mathbb{C}} \circ \frac{1}{2}(\tilde{S} \circ \Lambda_C) = I_{\mathfrak{X}}$. Hence, $\pi_{\mathbb{C}}$ is a retraction. Since $\mathfrak{X}$ is an essential complex Banach left $A_{\mathbb{C}}$-module, we deduce that $\mathfrak{X}$ is a projective complex Banach left $A_{\mathbb{C}}$-module by Proposition 5.2 for $\mathbb{F} = \mathbb{C}$. Hence, (i) implies (ii).

We now assume that $\mathfrak{X}$ is projective complex Banach left $A_{\mathbb{C}}$-module. Since the left module action $(J(a) + iJ(b), x) \longmapsto (J(a) + iJ(b)) \cdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ is defined by

$$(J(a) + iJ(b)) \cdot x = \varphi_C(J(a) + iJ(b))x \quad (a, b \in A, x \in \mathfrak{X}),$$

we deduce that $\mathfrak{X}$ is an essential complex Banach left $A_{\mathbb{C}}$-module by Proposition 5.1 for $\mathbb{F} = \mathbb{C}$. Therefore, the complex canonical morphism $\pi_{\mathbb{C}} : A_{\mathbb{C}} \widehat{\otimes}_{\mathbb{C}} \mathfrak{X} \longrightarrow \mathfrak{X}$ is a retraction by Proposition 5.2 for $\mathbb{F} = \mathbb{C}$. Thus, there exists $\Gamma \in {}_{A_{\mathbb{C}}}\mathcal{B}_{\mathbb{C}}(\mathfrak{X}, A_{\mathbb{C}} \widehat{\otimes}_{\mathbb{C}} \mathfrak{X})$ such that

$$\pi_{\mathbb{R}} \circ \Gamma = I_{\mathfrak{X}}. \tag{9}$$

Define tha map $\Gamma' : \mathfrak{X} \longrightarrow A \widehat{\otimes}_{\mathbb{R}} \mathfrak{X}$ by

$$\Gamma' = \tilde{T} \circ \Gamma. \tag{10}$$

Applying (9), (10) and (6), we get

$$\begin{aligned}(\pi_{\mathbb{R}} \circ \Gamma')(x) &= \pi_{\mathbb{R}}(\Gamma'(x)) = \pi_{\mathbb{R}}(\tilde{T}(\Gamma(x))) = (\pi_{\mathbb{R}} \circ \tilde{T})(\Gamma(x)) \\ &= \pi_{\mathbb{C}}(\Gamma(x)) = (\pi_{\mathbb{C}} \circ \Gamma)(x) = I_{\mathfrak{X}}(x)\end{aligned}$$

for all $x \in \mathfrak{X}$. This implies that $\pi_{\mathbb{R}} \circ \Gamma' = I_{\mathfrak{X}}$. Hence, $\pi_{\mathbb{R}}$ is a retraction. Since $\mathfrak{X}$ is an essential real Banach left A-module, we deduce that $\mathfrak{X}$ is a projective real Banach left $A_{\mathbb{C}}$-module by Proposition 5.2 for $\mathbb{F} = \mathbb{R}$. Hence, (ii) implies (i). $\square$

A relation between φ-contractibility of a complex Banach algebra B and the projectivity of certain Banach left B-modules is given in [17, Theorem 4.3]. We obtain similar result for real Banach algebras as the following.

Theorem 5.5. Let $(A, \|\cdot\|)$ be a real Banach algebra with $\triangle(A) \neq \emptyset$, let $\varphi \in \triangle(A)$. Then the following assertions are equivalent.

(i) If $\mathfrak{X}$ is a complex Banach space, then $\mathfrak{X}$ is a projective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$.

(ii) There is a complex Banach space $\mathfrak{X}$ such that $\mathfrak{X}$ is a projective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$.

(iii) $\mathbb{C}$ is a projective real Banach left A-module with the left module action $(a, z) \longmapsto a \cdot z : A \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $a \cdot z = \varphi(a)z, \quad (a \in A, x \in \mathbb{C})$.

(iv) A is left φ-contractible.

Proof . (i)$\Rightarrow$ (ii). Suppose that (i) holds. To prove (ii), take $\mathfrak{X} = A \times A$. Then $\mathfrak{X}$ is a complex algebra with the algebra operations defined by (2). By [5, Proposition I.1.13], there exists an algebra norm $|||\cdot|||$ on $\mathfrak{X}$ satisfying

$$\max\{\|a\|, \|b\|\} \leq |||(a, b)||| \leq 2\max\{\|a\|, \|b\|\},$$

for all $a, b \in A$. Clearly, $(\mathfrak{X}, |||\cdot|||)$ is a complex Banach algebra and so complex Banach space. Define the left module action $(a, (b, c)) \longmapsto a \cdot (b, c) : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ by

$$a \cdot (b, c) = \varphi(a)(b, c) \quad (a \in A, (b, c) \in \mathfrak{X}).$$

It is easy to see that $\mathfrak{X}$ is a real Banach left A-module with the mentioned left module action. By (i), $\mathfrak{X}$ is projective. Hence (ii) holds.

(ii)$\Rightarrow$ (iii). Clearly, $\mathbb{C}$ is a real Banach A-module with the Euclidean norm and the left module action defined in (iii). Set $A_{\mathbb{C}} = A \times A$. Then $A_{\mathbb{C}}$ is a complex algebra with the algebra operations defined in (2) and it is a complexification with the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0), \quad a \in A$. By [5, Proposition I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition with $k_1 = 1$ and $k_2 = 2$. By (ii), there exists a complex Banach space $\mathfrak{X}$ such that $\mathfrak{X}$ is a projective real Banach left A-module with the left module action $(a, x) \longmapsto a \cdot x : A \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$. By Theorem 5.4, $\mathfrak{X}$ is a projective complex Banach left $A_{\mathbb{C}}$-module with the left module action $(J(a) + iJ(b), x) \longmapsto (J(a) + iJ(b)) \cdot x : A_{\mathbb{C}} \times \mathfrak{X} \longrightarrow \mathfrak{X}$ defined by $(J(a) + iJ(b)) \cdot x = \varphi_C(J(a) + iJ(b))x, \quad (a, b \in A, x \in \mathfrak{X})$. Therefore, $\mathbb{C}$ is a projective complex Banach left $A_{\mathbb{C}}$-module with the left module action

$$(J(a) + iJ(b)) \cdot z = \varphi_C(J(a) + iJ(b))z \quad (a, b \in A, z \in \mathbb{C}),$$

by [17, Theorem 4.3], Hence, $\mathbb{C}$ is a projective real Banach left A-module with the left module action

$$a \cdot z = \varphi(a)z, \quad (a \in A, z \in \mathbb{C}),$$

by Theorem 5.4. Hence, (iii) hilds.

(iii)$\Rightarrow$ (iv). By (iii) and Theorem 5.4, $\mathbb{C}$ is a projective complex Banach left $A_{\mathbb{C}}$-module with the left module action

$$(J(a) + iJ(b)) \cdot z = \varphi_C(J(a) + iJ(b))z \quad (a, b \in A, z \in \mathbb{C}).$$

This implies that $A_{\mathbb{C}}$ is left φ_C-contractible by [17, Theorem 4.3]. Hence, A is left φ-contractible by part (i) of Theorem 3.1 and so (iv) holds.

(iv)$\Rightarrow$ (i). Let $\mathfrak{X}$ be a complex Banach space. Clearly, $\mathfrak{X}$ is a real (complex, respectively) Banach left A-module ($A_{\mathbb{C}}$-module, respectively), with the left module action $a \cdot x = \varphi(a)x$ for all $a \in A, x \in \mathfrak{X}$, $((J(a) + iJ(b)) \cdot x = \varphi_C(J(a) + iJ(b))x$ for all $a, b \in A, x \in \mathfrak{X}$, respectively). By (iv) and part (i) of Theorem 3.1, we deduce that $A_{\mathbb{C}}$ is left φ_C-contractible. Therefore, $\mathfrak{X}$ is a projective complex Banach left $A_{\mathbb{C}}$-module with the mentioned left module action by [17, Theorem 4.3]. Hence, by Theorem 5.4, $\mathfrak{X}$ is a projective real Banach left A-module with the left module action defined by $a \cdot x = \varphi(a)x, \quad (a \in A, x \in \mathfrak{X})$. Thus (i) holds. $\square$

6 Relation between character contractibility of B and $B_{\mathbb{R}}$

Let $(B, \|\cdot\|)$ be a complex Banach algebra with $\triangle(B) \neq \emptyset$ and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Clearly,

$$\triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\} \subseteq \triangle(B_{\mathbb{R}}).$$

For each $\varphi \in \triangle(B)$, we give a characterization of left φ-contractibility of B as the following.

Theorem 6.1. Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then the following assertions are equivalent.

(i) B is left φ-contractible.

(ii) $\mathbb{C}$ is a projective complex Banach left B-module with the left module action $(b, z) \longmapsto b \cdot z : B \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $b \cdot z = \varphi(b)z, \quad (b \in B, \quad z \in \mathbb{C})$.

(iii) $\mathbb{C}$ is a projective real Banach left $B_{\mathbb{R}}$-module with the left module action $(b, z) \longmapsto b \odot z : B_{\mathbb{R}} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $b \odot z = \varphi(b)z, \quad (b \in B, \quad z \in \mathbb{C})$.

(iv) $B_{\mathbb{R}}$ is left φ-contractible.

Proof . (i)$\Rightarrow$ (ii): It follows from [17, Theorem 4.3].

(ii)$\Rightarrow$ (iii): By (ii), there exists $\rho \in {}_B\mathcal{B}_{_\mathbb{C}}(\mathbb{C}, B\widehat{\otimes}_{\mathbb{C}}\mathbb{C})$ such that

$$\pi_{\mathbb{C}} \circ \rho = I_{\mathbb{C}}, \tag{1}$$

where $\pi_{\mathbb{C}} : B\widehat{\otimes}_{\mathbb{C}}\mathbb{C} \longrightarrow \mathbb{C}$ is the canonical morphism. Define the map $S : B\widehat{\otimes}_{\mathbb{C}}\mathbb{C} \longrightarrow B_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathbb{C}$ by

$$S(a \otimes 1) = a \otimes 1, \quad (a \in B).$$

It is easy to see that $S \in \mathcal{B}_{_\mathbb{R}}(B\widehat{\otimes}_{\mathbb{C}}\mathbb{C}, B_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathbb{C})$. On the other hand, $B\widehat{\otimes}_{\mathbb{C}}\mathbb{C}$ and $B_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathbb{C}$ are real Banach left $B_{\mathbb{R}}$-module with the left module actions defined by

$$\begin{aligned}
&a \cdot (b \otimes 1) = (ab) \otimes 1 \quad (a \in B_{\mathbb{R}}, b \in B), \\
&a \odot (b \otimes 1 + c \otimes i) = (ab) \otimes 1 + (ac) \otimes i \quad (a, b, c \in B_{\mathbb{R}}).
\end{aligned}$$

Since

$$S(a \cdot (b \otimes 1)) = S((ab) \otimes 1) = (ab) \otimes 1 = a \cdot (b \otimes 1) = a \odot S(b \otimes 1),$$

for all $a \in B_{\mathbb{R}}$, $b \in B$, we deduce that $S \in {}_{B_{\mathbb{R}}}\mathcal{B}_{_\mathbb{R}}(B\widehat{\otimes}_{\mathbb{C}}\mathbb{C}, B_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathbb{C})$. Let $\pi_{\mathbb{R}} : B_{\mathbb{R}}\widehat{\otimes}_{\mathbb{R}}\mathbb{C} \longrightarrow \mathbb{C}$ be the canonical morphism. Since

$$\begin{aligned}
(\pi_{\mathbb{R}} \circ S)(a \otimes 1) &= \pi_{\mathbb{R}}(S(a \otimes 1)) = \pi_{\mathbb{R}}(a \otimes 1) = a \odot 1 \\
&= \varphi(a)1 = a \cdot 1 = \pi_{\mathbb{C}}(a \otimes 1),
\end{aligned}$$

for all $a \in B$, we have
$$\pi_{\mathbb{R}} \circ S = \pi_{\mathbb{C}}. \tag{2}$$
It is easy to see that $\rho \in {}_{B_{\mathbb{R}}}\mathcal{B}_{\mathbb{R}}(\mathbb{C}, B\widehat{\otimes}_{\mathbb{C}}\mathbb{C})$. Hence, $S \circ \rho \in {}_{B_{\mathbb{R}}}\mathcal{B}_{\mathbb{R}}(\mathbb{C}, B\widehat{\otimes}_{\mathbb{R}}\mathbb{C})$. By (1) and (2), we have
$$\pi_{\mathbb{R}} \circ (S \circ \rho) = (\pi_{\mathbb{R}} \circ S) \circ \rho = \pi_{\mathbb{C}} \circ \rho = I_{\mathbb{C}}.$$
Therefore, $\pi_{\mathbb{R}}$ is a retraction. Since $\varphi \in \triangle(B_{\mathbb{R}})$ and $\mathbb{C}$ is a real Banach left $B_{\mathbb{R}}$-module with the left module action $(a, z) \longmapsto a \boxdot z : B_{\mathbb{R}} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $a \boxdot z = \varphi(a)z \quad (a \in B_{\mathbb{R}}, z \in \mathbb{C})$, we deduce that $\mathbb{C}$ is an essential real Banach left $B_{\mathbb{R}}$-module by Proposition 4.1 for $\mathbb{F} = \mathbb{R}$. Hence, $\mathbb{C}$ is a projective real Banach left $B_{\mathbb{R}}$-module with the left module action defined in (iii) and so (iii) holds.

(iii)$\Rightarrow$ (iv). It follows from Theorem 5.5.

(iv)$\Rightarrow$ (i). Let $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{C}}(B, \varphi)$ with the module actions $(a, x) \longmapsto a \cdot x$ and $(a, x) \longmapsto x \cdot a$. Clearly, $\mathfrak{X}$ is a real Banach $B_{\mathbb{R}}$-module with the module actions $(a, x) \longmapsto a \odot x$ and $(a, x) \longmapsto x \odot a$ defined by
$$\begin{aligned} &a \odot x = a \cdot x = \varphi(a)x \quad (a \in B_{\mathbb{R}}, \quad x \in \mathfrak{X}),\\ &x \odot a = x \cdot a \quad (x \in \mathfrak{X}, \quad a \in B_{\mathbb{R}}). \end{aligned}$$
Since $\varphi \in \triangle(B_{\mathbb{R}})$, we deduce that $\mathfrak{X} \in \mathcal{M}^r_{\mathbb{R}}(B_{\mathbb{R}}, \varphi)$. On the other hand,
$$i(x \odot a) = i(x \cdot a) = (ix) \cdot a = ix \odot a$$
for all $x \in \mathfrak{X}$ and $a \in B$. Hence,
$$H^1_{\mathbb{R}}(B_{\mathbb{R}}, \mathfrak{X}) = \{0\}, \tag{3}$$
by (iv). Let $D \in Z^1_{\mathbb{C}}(B, \mathfrak{X})$. Define the map $d : B_{\mathbb{R}} \longrightarrow \mathfrak{X}$ by
$$d(a) = D(a) \quad (a \in B_{\mathbb{R}}).$$
It is easy to see that $d \in Z^1_{\mathbb{R}}(B_{\mathbb{R}}, \mathfrak{X})$. According to (3), there exists $x \in \mathfrak{X}$ such that
$$d = d_{B_{\mathbb{R}}, \mathfrak{X}, x}. \tag{4}$$
Since $B = B_{\mathbb{R}}$, by (4) we have
$$D(a) = d(a) = d_{B_{\mathbb{R}}, \mathfrak{X}, x}(a) = a \odot x - x \odot a = a \cdot x - x \cdot a = d_{B, \mathfrak{X}, x}(a),$$
for all $a \in B$. Hence, $D = d_{B, \mathfrak{X}, x}$ and so $H^1_{\mathbb{C}}(B, \mathfrak{X}) = \{0\}$. Therefore, (i) holds. □

It is known [15, Remark 1.2.8] that if B is a complex commutative Banach algebra with identity, then
$$\triangle(B_{\mathbb{R}}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}. \tag{5}$$
Here, we give an extension of the mentioned result as the following.

Proposition 6.2. Let $(B, \|\cdot\|)$ be a complex Banach algebra with $\triangle(B) \neq \emptyset$ and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then $\triangle(B_{\mathbb{R}}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}$.

Proof . Clearly,
$$\triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\} \subseteq \triangle(B_{\mathbb{R}}). \tag{6}$$
Suppose that $\psi \in \triangle(B_{\mathbb{R}})$. Then $\psi(B)$ is a real subalgebra of $\mathbb{C}$ and $\{0\}$ is a proper subset of $\psi(B)$. Thus, $\psi(B) = \mathbb{R}$ or $\psi(B) = \mathbb{C}$. Therefore, $1 \in \psi(B)$ and so there exists $a_1 \in B$ with $\psi(a_1) = 1$. It follows that
$$(\psi(ia_1))^2 = (\psi(ia_1)^2) = \psi(-a_1^2) = -\psi(a_1^2) = -(\psi(a_1))^2 = -1.$$
Therefore, $\psi(ia_1) = i$ or $\psi(ia_1) = -i$. If $\psi(ia_1) = i$, then for each $a \in B$ we have
$$\psi(ia) = \psi(a_1)\psi(ia) = \psi(ia_1 a) = \psi(ia_1)\psi(a) = i\psi(a).$$
This implies that $\psi((\alpha + i\beta)a) = (\alpha + i\beta)\psi(a)$ for all $\alpha, \beta \in \mathbb{R}$ and $a \in B$. Hence, $\psi \in \triangle(B)$. If $\psi(ia_1) = -i$, then by a similar calculation we get $\overline{\psi}(a) = i\overline{\psi}(a)$ for all $a \in B$ which implies that $\overline{\psi} \in \triangle(B)$. Therefore, $\psi \in \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}$. Thus,
$$\triangle(B_{\mathbb{R}}) \subseteq \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\}. \tag{7}$$
From (6) and (7), we have
$$\triangle(B_{\mathbb{R}}) = \triangle(B) \cup \{\overline{\varphi} : \ \varphi \in \triangle(B)\},$$
and so the proof is complete. □

Theorem 6.3. Let $(B, \|\cdot\|)$ be a complex Banach algebra and let $B_{\mathbb{R}}$ denote B regarded as a real algebra. Then B is left character contractible if and only if $B_{\mathbb{R}}$ left character contractible.

Proof . We firt assume that B is left character contractible. Let $\varphi \in \triangle(B_{\mathbb{R}})$. Then $\varphi \in \triangle(B)$ or $\overline{\varphi} \in \triangle(B)$ by Proposition 6.2. If $\varphi \in \triangle(B)$, then B is left φ-contractible and so $B_{\mathbb{R}}$ is left φ-contractible by Theorem 6.1. If $\overline{\varphi} \in \triangle(B)$, then B is left $\overline{\varphi}$-contractible and so $B_{\mathbb{R}}$ is left $\overline{\varphi}$-contractible by Theorem 6.1. Therefore, $B_{\mathbb{R}}$ is left φ-contractible by part (i) of Theorem 2.1. Suppose that $\varphi = 0$. Then B is left 0-contractible and so by Corollary 4.5, $B_{\mathbb{R}}$ is left 0-contractible. Therefore, $B_{\mathbb{R}}$ is left character contractible.

We now assume that $B_{\mathbb{R}}$ is left character contractible. Let $\varphi \in \triangle(B)$. Then $\varphi \in \triangle(B_{\mathbb{R}})$ and so $B_{\mathbb{R}}$ is left φ-contractible. Hence, B is left φ-contractible by Theorem 6.1. Suppose that $\varphi = 0$. Then $B_{\mathbb{R}}$ is left 0-contractible and so B is left 0-contractible by Corollary 4.5. Therefore, B is left character contractible. □

7 Applications and examples

Applying some results in previous sections and some known results about the character contractibility for complex commutative Banach algebras, we obtain the following theorems.

Theorem 7.1. Let $(A, \|\cdot\|)$ be a commutative real Banach algebra. If A is character contractible, then A is a finite dimensional real linear space.

Proof . Let A be character contractible. Set $A_{\mathbb{C}} = A \times A$. Then $A_{\mathbb{C}}$ with the algebra operations defined by (2) is complex algebra which is complexification of A with respect to the injective real algebra homomorphism $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(a) = (a, 0) \quad a \in A$. Moreover by [5, Proposition I.1.13], there exists an algebra norm $|||\cdot|||$ on $A_{\mathbb{C}}$ satisfying in the $(*)$ condition with the positive constants $k_1 = 1$ and $k_2 = 2$. Hence, $A_{\mathbb{C}}$ is character contractible by part (iii) of Theorem 3.1. Therefore, $A_{\mathbb{C}}$ is finite dimensional by [10, Theorem 6.10] and so there exists a finite subset $\{(a_1, b_1), \cdots, (a_n, b_n)\}$ of $A_{\mathbb{C}}$ which generates $A_{\mathbb{C}}$. It is easy to see that A generates by the finite set $\{a_1, b_1, \cdots, a_n, b_n\}$. Hence, A is a finite dimensional real linear space. □

Theorem 7.2. Let $(A, \|\cdot\|)$ be a commutative real Banach algebra with $\triangle(A) \neq \emptyset$ and $\varphi \in \triangle(A)$.

(i) If A is φ-contractible, then $\{\varphi\}$ is an open set in $\triangle(A)$ with the Gelfand topology.

(ii) If A is semisimple and if $\{\varphi\}$ is an open set in $\triangle(A)$ with the Gelfand topology, then A is φ-contractible.

Proof . Let $A_{\mathbb{C}} = A \times A$ and $|||\cdot|||$ be an algebra norm on $A_{\mathbb{C}}$ satisfying in the $(*)$-condition. Recall that for each $\psi \in \triangle(A)$, the map $\psi_C : A \times A \longrightarrow \mathbb{C}$ defined by

$$\psi_C(a, b) = \psi(a) + i\psi(b) \quad (a, b \in A),$$

is an element of $\triangle(A_{\mathbb{C}})$ and the map $\Theta : \triangle(A) \longrightarrow \triangle(A_{\mathbb{C}})$ defined by $\Theta(\psi) = \psi_C,\ \psi \in \triangle(A)$, is a homeomorphism from $\triangle(A)$ with the Gelfand topology onto $\triangle(A_{\mathbb{C}})$ with the Gelfand topology.

To prove (i), assume that A is φ-contractible. Then $A_{\mathbb{C}}$ is φ_C-contractible by part (i) of Theorem 3.1. Hence, $\{\varphi_C\}$ is an open set in $\triangle(A_{\mathbb{C}})$ with the Gelfand topology by [7, Theorem 2.1(i)]. Since $\{\varphi\} = \Theta^{-1}(\{\varphi_C\})$ and Θ is a homeomorphism, we deduce that $\{\varphi\}$ is an open set in $\triangle(A)$ with the Gelfand topology. Hence (i) holds.

To prove (ii), assume that A is semisimple and $\{\varphi\}$ is an open set in $\triangle(A)$ with the Gelfand topology. Then $A_{\mathbb{C}}$ is semisimple by [16, Proposition 2.7.6] and $\{\varphi_C\}$ is an open set in $\triangle(A_{\mathbb{C}})$ with the Gelfand topology since $\{\varphi_C\} = \Theta(\{\varphi\})$ and Θ is a homeomorphism. Thus, $A_{\mathbb{C}}$ is φ_C-contractible by [7, Theorem 2.1(ii)]. Therefore, A is φ-contractible by part (i) of Theorem 3.1. Hence, (ii) holds. □

In continue we study character contractibility of certain real Banach algebras.

Let X be a compact Hausdorff space. We denote by $C_{\mathbb{F}}(X)$ the set of all continuous $\mathbb{F}$-valued functions on X. Then $C_{\mathbb{F}}(X)$ is a commutative Banach algebra over $\mathbb{F}$ with unit 1_X, the constant function on X with value 1, and with th uniform norm $\|\cdot\|_X$ on X defined by

$$\|f\|_X = \sup\{f(x) : \ x \in X\} \qquad (f \in C_{\mathbb{F}}(X)).$$

We write $C(X)$ instead of $C_{\mathbb{C}}(X)$. A complex subalgebra B of $C(X)$ is called a *Banach function algebra* on X if B separates the points of X, $1_X \in B$ and B is a unital Banach algebra under an algebra norm $\|\cdot\|$.

Let B be a Banach function algebra on X. For each $x \in X$, the map $e_{B,x} : B \longrightarrow \mathbb{C}$ defined by $e_{B,x}(f) = f(x) \quad (f \in B)$, is a character of B which is called the *evaluation character* on B at x. It is known that the map $E_{B,X} : X \longrightarrow \triangle(B)$ defined by $E_{B,X}(x) = e_{B,x}$, $x \in X$, is an injective continuous mapping. We say that B is *natural* if $E_{B,X}$ is surjective.

Let X be a compact Hausdorff space. A self-map $\tau : X \longrightarrow X$ is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Let $\tau : X \longrightarrow X$ be an topological involution on X. Then the map $\tau^* : C(X) \longrightarrow C(X)$ defined by $\tau^*(f) = \bar{f} \circ \tau \quad (f \in C(X))$, is an algebra involution on $C(X)$ which is called the *algebra involution on* $C(X)$ *induced by* τ. Set

$$C(X,\tau) = \{f \in C(X) : \quad \tau^*(f) = f\}.$$

Then $C(X,\tau)$ is a self-adjoint real uniformly closed subalgebra of $C(X)$ containing 1_X and separating the points of X. Moreover, $C(X) = C(X,\tau) \oplus iC(X,\tau)$ and

$$\max\{\|f\|_X, \|g\|_X\} \leqslant \|f + ig\|_X \leqslant 2\max\{\|f\|_X, \|g\|_X\},$$

for all $f, g \in C(X,\tau)$. Furthermore, $C(X,\tau) = C_{\mathbb{R}}(X)$ is and only if τ is the identity map on X. A real subalgebra A of $C(X,\tau)$ is called a *real Banach function algebra* on (X,τ) if A separates the points of X, $1_X \in A$ and A is a unital real Banach algebra with an algebra norm $\|\cdot\|$ on A.

Let A be a real Banach function algebra on (X,τ). For each $x \in X$, the map $e_{A,x} : A \longrightarrow \mathbb{C}$ defined by $e_{A,x}(f) = f(x) \quad (f \in A)$ is a character of A. It is easy to see that the map $E_{A,X} : X \longrightarrow \triangle(A)$ defined by $E_{A,X}(x) = e_{A,x}$ $(x \in X)$, is an injective continuous mapping from X to $\triangle(A)$ with the Gelfand topology. We say that A is natural if $E_{A,X}$ is surjective.

If $(B, \|\cdot\|)$ is a complex Banach algebra on X with $\tau^*(B) = B$ and $A = \{f \in B : \quad \tau^*(f) = f\}$, then A is a real subalgebra of B, $B = A \oplus iA$ and there exists a positive constant C such that

$$\max\{\|f\|, \|g\|\} \leqslant C\|f + ig\| \leqslant 2C\max\{\|f\|, \|g\|\},$$

for all $f, g \in A$. Moreover, $(A, \|\cdot\|)$ is a real Banach function algebra on (X,τ) (see [3, Theorem 2.1]). Note that $(e_{A,x})_C = e_{B,x}$ for all $x \in X$ and so A is natural if and only if B is natural.

Theorem 7.3. Let X be a compact Hausdorff space, $\tau : X \longrightarrow X$ be a topological involution on X, let $(B, \|\cdot\|)$ is a natural complex Banach function algebra on X with $\tau^*(B) = B$ and let $A = \{f \in B : \quad \tau^*(f) = f\}$.

(i) If $x \in X$, then the real Banach algebra $(A, \|\cdot\|)$ is $e_{A,x}$-contractible if and only if x is an isolated point of X.
(ii) The real Banach algebra A is character contractible if and only if X is finite.

Proof . (i). Let $x \in X$. Since B is a complexification of A with the injective real algebra homomorphism $J : A \longrightarrow B$ defined by $J(f) = f \quad (f \in A)$ and there exists a positive constant C such that

$$\max\{\|f\|, \|g\|\} \leqslant C\|f + ig\| \leqslant 2C\max\{\|f\|, \|g\|\}$$

for all $f, g \in A$. According to $e_{B,x} = (e_{A,x})_C$, we deduce that A is $e_{A,x}$-contractible if and only if B is $e_{B,x}$-contractible by part (i) of Theorem 3.1. Since B is a natural complex Banach function algebra on X, by [7, Theorem 2.3], B is $e_{B,x}$-contractible is and only if x is an isolated point of X. Hence, (i) holds.

(ii). By part (iii) of Theorem 3.1, A is character contractible if and only if B is character contractible. On the other hand, by [7, Corollary 2.4], B is character contractible if and only if X is finite. Hence, (ii) holds. □

Corollary 7.4. Let X be a compact Hausdorff space, let $\tau : X \longrightarrow X$ be a topological involution on X and let $A = C(X,\tau)$.

(i) If $x \in X$, then A is $e_{A,x}$-contractible if and only if x is an isolated point of X.
(ii) A is character contractible if and only if X is finite.

Proof . Set $B = C(X)$. Since $(B, \|\cdot\|_X)$ is a natural complex Banach function algebra on X, $\tau^*(B) = B$ and $A = \{f \in B : \quad \tau^*(f) = f\}$, by Theorem 7.3 the result holds. □ Let (X, d) be a compact metric space and $\alpha \in (0, 1]$. We denote by $\mathrm{Lip}_{\mathbb{F}}(X, d^\alpha)$ the set of all $\mathbb{F}$-valued functions f on X for which

$$p_{(X,d^\alpha)}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x,y)} : \quad x, y \in X, x \neq y\} < \infty.$$

Then $\mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})$ is an algebra over $\mathbb{F}$ containing 1_X and separates the points of X. $\mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})$ is called Lipschitz algebra of order α over $\mathbb{F}$. For $\alpha \in (0,1)$, we denote by $\mathrm{lip}_{\mathbb{F}}(X, d^{\alpha})$ the set of all $f \in \mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})$ for which

$$\lim_{d(x,y) \longmapsto 0} \frac{|f(x) - f(y)|}{d^{\alpha}(x,y)} = 0.$$

Then $\mathrm{lip}_{\mathbb{F}}(X, d^{\alpha})$ is a subalgebra of $\mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})$ over $\mathbb{F}$. The algebra $\mathrm{lip}_{\mathbb{F}}(X, d^{\alpha})$ is called little Lipschitz algebra of order α over $\mathbb{F}$. It is known that $\mathrm{Lip}_{\mathbb{F}}(X, d^{\beta}) \subseteq \mathrm{lip}_{\mathbb{F}}(X, d^{\alpha}) \subseteq \mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha}) \subseteq C_{\mathbb{F}}(X)$ whenever $0 < \alpha \leq \beta$. The Lipschitz algebra $\mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})$ and the little Lipschitz algebra $\mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})$ were first introduced by Sherbert in [18, 19]. We write $\mathrm{Lip}(X, d^{\alpha})$ ($\mathrm{lip}(X, d^{\alpha})$, respectively) instead of $\mathrm{Lip}_{\mathbb{C}}(X, d^{\alpha})$ ($\mathrm{lip}_{\mathbb{C}}(X, d^{\alpha})$, respectively). It is known that $\mathrm{Lip}(X, d^{\alpha})$ is a natural complex Banach function algebra on (X, d) under the algebra Lipschitz norm $\| \cdot \|_{\mathrm{Lip}(X,d^{\alpha})}$ defined by

$$\|f\|_{\mathrm{Lip}(X,d^{\alpha})} = \|f\|_X + p_{(X,d^{\alpha})}(f) \quad (f \in \mathrm{Lip}_{\mathbb{F}}(X, d^{\alpha})).$$

Moreover, $\mathrm{lip}(\mathrm{X}, \mathrm{d}^{\alpha})$ is a closed subalgebra of $(\mathrm{Lip}(X, d^{\alpha}), \| \cdot \|_{\mathrm{Lip}(X,d^{\alpha})})$ whenever $\alpha \in (0,1)$. Also, $(\mathrm{lip}(X, d^{\alpha}), \| \cdot \|_{\mathrm{Lip}(X,d^{\alpha})})$ is a natural complex Banach function algebra on (X, d). A self-map $\tau : X \longrightarrow X$ is called a *Lipschitz mapping* on (X, d) if

$$p(\tau) = \sup\{\frac{d(\tau(x), \tau(y))}{d(x,y)} : \quad x, y \in X, \quad x \neq y\} < \infty.$$

A Lipschitz mapping τ on (X, d) is called a *Lipschitz involution* on (X, d) if $\tau(\tau(x)) = x$ for all $x \in X$. It is easy to see that if $\tau : X \longrightarrow X$ is a Lipschitz involution on (X, d) then $\tau^*(\mathrm{Lip}(X, d^{\alpha})) = \mathrm{Lip}(X, d^{\alpha})$ for $\alpha \in (0,1]$ and $\tau^*(\mathrm{lip}(X, d^{\alpha})) = \mathrm{lip}(X, d^{\alpha})$ for $\alpha \in (0,1)$. Define

$$\begin{aligned} \mathrm{Lip}(X, d^{\alpha}, \tau) &= \{f \in \mathrm{Lip}(X, d^{\alpha}) : \ \tau^*(f) = f\} \quad (\alpha \in (0,]), \\ \mathrm{lip}(X, d^{\alpha}, \tau) &= \{f \in \mathrm{lip}(X, d^{\alpha}) : \ \tau^*(f) = f\} \quad (\alpha \in (0,1)). \end{aligned}$$

It is known [3, Theorem 2.7] that if $B = \mathrm{Lip}(X, d^{\alpha})$ ($B = \mathrm{lip}(X, d^{\alpha})$, respectively) and $A = \mathrm{Lip}(X, d^{\alpha}, \tau)$ ($A = \mathrm{lip}(X, d^{\alpha}, \tau)$, respectively), then $B = A \oplus iA$,

$$\begin{aligned} \max\{\|f\|_{\mathrm{Lip}(X,d^{\alpha})}, \|g\|_{\mathrm{Lip}(X,d^{\alpha})}\} &\leq (p(\tau))^{\alpha} \|f + ig\|_{\mathrm{Lip}(X,d^{\alpha})} \\ &\leq 2(p(\tau))^{\alpha} \max\{\|f\|_{\mathrm{Lip}(X,d^{\alpha})}, \|g\|_{\mathrm{Lip}(X,d^{\alpha})}\} \end{aligned}$$

for all $f, g \in A$ and $(A, \| \cdot \|_{\mathrm{Lip}(X,d^{\alpha})})$ is a natural real Banach function algebra on $((X, d), \tau)$. The real Lipschitz algebras $\mathrm{Lip}(X, d^{\alpha}, \tau)$ and $\mathrm{lip}(X, d^{\alpha}, \tau)$ were first introduced in [3].

As an application of Theorem 7.3, we give the following result.

Theorem 7.5. Let (X, d) be a compact metric space, $\tau : X \longrightarrow X$ be a Lipschitz involution on (X, d) and let $A = \mathrm{Lip}(X, d^{\alpha}, \tau)$ for $\alpha \in (0,1]$ or $A = \mathrm{lip}(X, d^{\alpha}, \tau)$ for $\alpha \in (0,1)$.

(i) If $x \in X$, then A is $e_{A,x}$-contractible if and only if x is an isolated point in (X, d).
(ii) A is character contractible if and only if X is finite.

Proof . Let $B = \mathrm{Lip}(X, d^{\alpha})$ for $\alpha \in (0,1]$ ($B = \mathrm{lip}(X, d^{\alpha})$ for $\alpha \in (0,1)$, respectively). Then $(B, \| \cdot \|_{\mathrm{Lip}(X,d^{\alpha})})$ is a natural complex Banach function algebra on (X, d) with $\tau^*(B) = B$ and $A = \{f \in B : \quad \tau^*(f) = f\}$. Hence, the result holds by Theorem 7.3. □

Let G be a locally compact group. We denote by $M(G)$ the set of all complex Borel measures on G. It is known that $M(G)$ is a complex Banach algebra with the norm

$$\|\mu\| = |\mu|(G) \qquad (\mu \in M(G)).$$

Let λ be a left Haar measure on G and $L^1(G) = L^1(G, \lambda)$, the group algebra on G with respect to the measure λ, endowed with the $L^1(G)$-norm

$$\|f\|_{L^1(G)} = \int_G |f| \ d\lambda \qquad (f \in L^1(G)).$$

A map $\tau : G \longrightarrow G$ is called a *topological group involution* on G if τ is a continuous group automorphism on G and $\tau(\tau(x)) = x$ for all $x \in G$.

Let G be a locally compact group and let $\tau : G \longrightarrow G$ be a topological group involution on G. It is easy to see that $\mu \circ \tau \in M(G)$ for all $\mu \in M(G)$. Define

$$M(G,\tau) = \{\mu \in M(G) : \ \mu \circ \tau = \bar{\mu}\}.$$

It is shown [6, Proposition 2.2] that $M(G,\tau)$ is a closed real subalgebra of $M(G)$, $M(G) = M(G,\tau) \oplus iM(G,\tau)$ and

$$\max\{\|\mu\|, \|\nu\|\} \leqslant \|\mu + i\nu\| \leqslant 2\max\{\|\mu\|, \|\nu\|\},$$

for all $\mu, \nu \in M(G,\tau)$. Let λ be a Haar measure on G. By [6, Theorem 2.4], $\lambda \circ \tau = \lambda$. Define

$$L^1(G,\tau) = \{f \in L^1(G) : \ f \circ \tau = \bar{f}\}.$$

By [6, Theorem 2.5], $L^1(G,\tau)$ is a closed real subalgebra of $L^1(G)$,

$$L^1(G) = L^1(G,\tau) \oplus iL^1(G,\tau),$$

and

$$\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\} \leqslant \|f + ig\|_{L^1(G)} \leqslant 2\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\},$$

for all $f, g \in L^1(G,\tau)$.

Let $\hat{G}$ be the dual group of G, the set of all continuous group homomorphisms from G to $\mathbb{T} = \{z \in \mathbb{C} : \ \ |z| = 1\}$. For each $p \in \hat{G}$, define the map $\varphi_p : L^1(G) \longrightarrow \mathbb{C}$ by

$$\varphi_p(f) = \int_G \overline{p(x)} \ \ f(x) \ \ d\lambda(x) \quad (f \in L^1(G)).$$

Then $\varphi_p \in \triangle(L^1(G))$, which is called the *induced character* by p. It is known [9, Theorem 23.7] that $\triangle(L^1(G)) = \{\varphi_p : \ \ p \in \hat{G}\}$. For $p \in \hat{G}$, define the map $\psi_p : M(G) \longrightarrow \mathbb{C}$ by

$$\psi_p(\mu) = \int_G \overline{p(x)} \ \ d\mu(x) \quad (\mu \in M(G)).$$

Then $\psi_p \in \triangle(M(G))$ and $\psi_p|_{L^1(G)} = \varphi_p$. It is known [9] that if G is discrete, then $\triangle(M(G)) = \{\psi_p : \ \ p \in \hat{G}\}$.

Let $\tau : G \longrightarrow G$ be a topological group involution on G. For each $p \in \hat{G}$, define the maps $\varphi_{p,\tau} : L^1(G,\tau) \longrightarrow \mathbb{C}$ and $\psi_{p,\tau} : M(G,\tau) \longrightarrow \mathbb{C}$ by

$$\varphi_{p,\tau}(f) = \int_G \overline{p(x)} \ \ f(x) \ \ d\lambda(x) \quad (f \in L^1(G,\tau)),$$
$$\psi_{p,\tau}(\mu) = \int_G \overline{p(x)} \ \ d\mu(x) \quad (\mu \in M(G,\tau)).$$

We can show that $\varphi_{p,\tau} \in \triangle(L^1(G,\tau))$ and $\psi_{p,\tau} \in \triangle(M(G,\tau))$ for all $p \in \hat{G}$. Moreover, $(\varphi_{p,\tau})_C = \varphi_p$ and $(\psi_{p,\tau})_C = \psi_p$ for all $p \in \hat{G}$.

Theorem 7.6. Let G be a locally compact group with a left Haar measure λ, let $\tau : G \longrightarrow G$ be a topological group involution on G and let $p \in \hat{G}$. Then the following assertions are equivalent.

(i) $L^1(G,\tau)$ is left $\varphi_{p,\tau}$-contractible.
(ii) $M(G,\tau)$ is left $\psi_{p,\tau}$-contractible.
(iii) G is compact.

Proof . Since $L^1(G) = L^1(G,\tau) \oplus iL^1(G,\tau)$,

$$\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\} \leqslant \|f + ig\|_{L^1(G)} \leqslant 2\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\},$$

for all $f, g \in L^1(G,\tau)$, $\varphi_{p,\tau} \in \triangle(L^1(G,\tau))$ and $\varphi_p = (\varphi_{p,\tau})_C$, we deduce that (i) is equivalent to $L^1(G)$ is left φ_p-contractible by part (i) of Theorem 3.1. Since $M(G) = M(G,\tau) \oplus iM(G,\tau)$,

$$\max\{\|\mu\|, \|\nu\|\} \leqslant \|\mu + i\nu\| \leqslant 2\max\{\|\mu\|, \|\nu\|\},$$

for all $\mu, \nu \in M(G,\tau)$ and $\psi_p = (\psi_{p,\tau})_C$, we deduce that (ii) is equivalent to $M(G)$ is left ψ_p-contractible by part (i) of Theorem 3.1. Therefore, the result holds by [17, Theorem 6.2]. □

Theorem 7.7. Let G be a locally compact group with a left Haar measure λ, let $L^1(G) = L^1(G, \lambda)$ and $\tau : G \longrightarrow G$ be a topological group involution on G. Then the following assertions are equivalent.

(i) $L^1(G, \tau)$ is left character contractible.

(ii) $M(G, \tau)$ is left character contractible.

(iii) G is finite.

Proof . By the given argument in the proof of Theorem 7.6 and applying part (iii) of Theorem 3.1, we deduce that (i) ((ii), respectively) is equivalent to $L^1(G)$ ($M(G)$, respectively) is left character contractible. Hence, the result holds by [17, Corollary 6.5]. □

Let G be locally compact group, λ be a left Haar measure on G and $L^1(G) = L^1(G, \lambda)$. For $p \in \hat{G}$, we define the map $\tilde{\varphi}_p : (L^1(G))^{**} \longrightarrow \mathbb{C}$ by

$$\tilde{\varphi}_p(\Lambda) = \Lambda(\varphi_p) \quad (\Lambda \in (L^1(G))^{**}).$$

Then $\tilde{\varphi}_p \in (L^1(G))^{**}$ and $\tilde{\varphi}_p$ is the unique extension of φ_p to $(L^1(G))^{**}$ (see [12, Section 2]). Let $\tau : G \longrightarrow G$ be a topological group involution on G. Since $L^1(G)$ is a complexification of $L^1(G, \tau)$ with respect to the injective real algebra homomorphism $J : L^1(G, \tau) \longrightarrow L^1(G)$ defined by $J(f) = f \quad (f \in L^1(G, \tau))$ and

$$\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\} \leqslant \|f + ig\|_{L^1(G)} \leqslant 2\max\{\|f\|_{L^1(G)}, \|g\|_{L^1(G)}\},$$

for all $f, g \in L^1(G, \tau)$, by [2, Lemmas 2.3 and 2.4], $(L^1(G))^{**}$ is a complexification of $(L^1(G, \tau))^{**}$ with respect to the injective algebra homomorphism $J_2 : (L^1(G, \tau))^{**} \longrightarrow (L^1(G))^{**}$ defined by $J_2(\Phi) = \Phi_C \quad (\Phi \in (L^1(G, \tau))^{**})$ and

$$\max\{\|\Phi\|, \|\Psi\|\} \leqslant 4\|J_2(\Phi) + iJ_2(\Psi)\| \leqslant 32\max\{\|\Phi\|, \|\Psi\|\},$$

for all $\Phi, \Psi \in (L^1(G, \tau))^{**}$. For $p \in \hat{G}$, define the function $\tilde{\varphi}_{p,\tau} : (L^1(G, \tau))^{**} \longrightarrow \mathbb{C}$ by

$$\tilde{\varphi}_{p,\tau}(\Phi) = \tilde{\varphi}_p(J_2(\Phi)) \quad (\Phi \in (L^1(G, \tau))^{**}).$$

Then $\tilde{\varphi}_{p,\tau} \in \triangle((L^1(G, \tau))^{**})$ and $(\tilde{\varphi}_{p,\tau})_C = \tilde{\varphi}_p$. This implies that $\tilde{\varphi}_{p,\tau} \in \triangle((L^1(G, \tau))^{**})$ is the unique extension $\varphi_{p,\tau} \in \triangle((L^1(G, \tau)))$ on $(L^1(G, \tau))^{**}$.

Applying part (i) of Theorem 3.1 and [17, Proposition 6.8], we obtain the following result.

Theorem 7.8. Let G be a locally compact group with a left Haar measure λ, let $L^1(G) = L^1(G, \lambda)$, let $\tau : G \longrightarrow G$ be a topological group involution on G and let $p \in \hat{G}$. Then $(L^1(G, \tau))^{**}$ is left $\tilde{\varphi}_{p,\tau}$-contractible if and only if G is amenable.

Applying part (iii) of Theorem 3.1 and [17, Proposition 6.10], we get the following result.

Theorem 7.9. Let G be a locally compact group with a left Haar measure λ, let $L^1(G) = L^1(G, \lambda)$ and let $\tau : G \longrightarrow G$ be a topological group involution on G. Then $(L^1(G, \tau))^{**}$ is left character contractible if and only if G is finite.

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