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On the study of existence of sequential fractional integro-differential equations with boundary conditions

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Abstract

This paper studies a boundary value problem of sequential fractional integro-differential equations. The uniqueness results are studied using the classical contraction mapping principle; however, the existence results are proved via D.O'Regan's fixed point theorem. We also present an example to illustrate our findings.

Keywords: Mixed fractional derivative, boundary value problem, fixed point theorem, contraction mapping
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1 Introduction

Fractional differential eqautions is the most prominent approach to inovate the modelling phenomena in domains such as control, immunology, aerodynamics, viscoelasticity [10, 12, 13, 17, 20, 22, 23], etc. This unlocks the difficulties to solve the real-world issues. It mainly deals with the memory effects and dynamical systems see [1, 2, 3, 15, 18, 25, 26]. To explore the complex models with multi-stage memory, population dynamics and hereditary effects, the concept of sequential fractional differential equations is the optimal technique to carry out the task. Some recent works related to sequential and mixed fractional integro-differential equations are given in [4, 6, 7, 8, 9, 14, 16, 19, 27, 28]. In [5] Ahmad and Luca studied existence results for the nonlinear Caputo type sequential fractional integro-differential equation and inclusion:

$$
\begin{aligned}
&({}^C D^{\gamma} + \mu {}^C D^{\gamma-1}) g(x) = f(x, g(x), {}^C D^{r} g(x), I^{s} g(x)),\ x \in (0,1)\\
&({}^C D^{\gamma} + \mu {}^C D^{\gamma-1}) g(x) \in F(x, g(x), I^{s} g(x)),\ x \in (0,1)
\end{aligned}
$$

subject to the nonlocal boundary conditions

$$
g(0) = h(g),\ g'(0) = g''(0) = 0,\ aI^{\delta} g(\zeta) = \int_0^1 g(t)\,dH(t).
$$

Here the authors deals with fractional integro-differential inclusions, which is more effective when the system is uncertain. In [6] Ahmad et al. studied a sequential fractional differential equations of order $\gamma+1$ with nonlinear three-point boundary value problem and some existence results are obtained by Banach's contraction mapping principle and

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Krasnoselskii's fixed point theorem.

$$
\begin{aligned}
&{}^C D^{\gamma}(D+\mu)y(z) = f(z, y(z)),\ 0 < z < 1,\ 1 < \gamma \leq 2,\\
&y(0) = 0,\ y'(0) = 0,\ y(1) = \delta y(\tau),\ 0 < \tau < 1.
\end{aligned}
$$

Further in [21] Murad et al. discussed the existence and uniqueness using Banach fixed point theorem and Schauder fixed point theorem for the given boundary value problem

$$
\begin{cases}
{}^C D^{\alpha}\vartheta(s) = \phi(s, \vartheta(s), \int_0^s \kappa(s, r, \vartheta(r))dr, s \in \mathbb{J} = [0, S],\ 2 < \alpha \leq 3,\\
\beta\vartheta(0) + \gamma\vartheta(S) = \delta,\\
\lambda\vartheta'(0) + \upsilon\vartheta'(S) = \mu,\\
\imath\vartheta''(0) + \jmath\vartheta''(S) = \ell,
\end{cases}
$$

which is the Caputo fractional derivative involved in the fractional integro-differential equations with some boundary conditions. For instance, [11] presents the existence of solutions for a boundary value problem of nonlinear sequential fractional integro-differential equations of the form

$$
\begin{aligned}
&{}^C D^{\alpha}({}^C D^{\beta} g)(z) = f(z, g(z), \varphi g(z), \psi g(z)), 0 < z < 1,\\
&g(1) = g(0) = g'(1) = 0.
\end{aligned}
$$

By exploring these works, we got a foundation to extend our work for studying the behaviour of the solutions of fractional integro-differential equations with sequential derivative

$$
{}^C D^{\gamma}_{1-}\left\{{}^C D^{\delta}_{0+}\left[\varrho(z) - y(z, \varrho(z))\right]\right\} = \theta(z, \varrho(z), H\varrho(z)),\ z \in [0, 1], \tag{1.1}
$$

with boundary conditions

$$
\begin{aligned}
&\alpha\varrho(0) + \beta\varrho(1) = \tau\\
&\phi\varrho'(0) + \psi\varrho'(1) = \nu\\
&u\varrho''(0) + v\varrho''(1) = \eta
\end{aligned} \tag{1.2}
$$

where ${}^C D^{\gamma}, {}^C D^{\delta}$ denote the Caputo fractional derivative of order γ and δ respectively, $0 < \delta \leq 1,\ 2 < \gamma \leq 3$, $\alpha, \beta, \tau, \phi, \psi, \nu, u, v$ and η are constants and $\theta : C([0, 1], \mathbb{R} \times \mathbb{R}) \to \mathbb{R}, y : C([0, 1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions with $y(0, \varrho(0)) = 0,\ H\varrho(z) = \int_0^z f(z, w, \varrho(w))dw,\ \Lambda = \{(z, w) : 0 \leq w \leq z \leq 1\}$.

The existence and uniqueness results for (1.1) are proved using Banach's contraction principle, in addition by adopting D.O'Regan's theorem, we prove the existence of atleast one solution under several conditions.

The structure of this paper is as follows: Section 2 recalls the essential definitions and theorems that are required for establishing the main results. In Section 3, we present the existence results related to the mixed fractional integro-differential equations. A concrete example that illustrates the practical implications of this study is discussed in Section 4.

2 Preliminaries

This section presents an auxiliary lemma and some fundamental definitions related to fractional calculus. The space $(G, \|\cdot\|)$ is a Banach space if $G = \{g(v) \mid g(v) \in C([0, 1], \mathbb{R})\}$ is the space with the norm $\|g\| = \sup\{|g(v)|, v \in [0, 1]\}$.

Definition 2.1. [22] The right and left Riemann-Liouville fractional integrals of order ρ for a continuous function $h \in L_1[0, 1]$ are respectively, defined as

$$
\begin{aligned}
I^{\rho}_{1-}h(v) &= \frac{1}{\Gamma(\rho)}\int_t^1 \frac{h(u)}{(u-v)^{1-\rho}}du,\\
I^{\rho}_{0+}h(v) &= \frac{1}{\Gamma(\rho)}\int_0^t \frac{h(u)}{(v-u)^{1-\rho}}du,
\end{aligned}
$$

where $\rho > 0, \Gamma(\rho)$ is the Euler Gamma function, provided that the right-hand side is point-wise defined on $\mathbb{R}^+$.

Definition 2.2. [22] The right and the left Caputo fractional derivative of order ρ of a function $h \in AC^n[0,1]$ are respectively, defined as

$$
\begin{aligned}
{}^C D^{\rho}_{1-} h(v) &= (-1)^n I^{n-\rho}_{1-} h^{(n)}(v),\\
{}^C D^{\rho}_{0+} h(v) &= \frac{1}{\Gamma(n-\rho)} \int_0^v \frac{h^{(n)}(u)}{(v-u)^{\rho+1-n}} du = I^{n-\rho}_{0+} h^{(n)}(v),
\end{aligned}
$$

where $n = [\rho]+1, v > 0, n-1 < \rho < n, [\rho]$ represents the integer part of the real number ρ.

Definition 2.3. [22] If $\rho > 0$, let $h, {}^C D^{\rho}_{0+} h(t), {}^C D^{\rho}_{1-} h(t) \in L^1[0,1]$, then

$$
\begin{aligned}
I^{\rho}_{0+} {}^C D^{\rho}_{0+} h(v) &= h(v) + c_0 + c_1 v + c_2 v^2 + \ldots + c_{n-1} v^{n-1},\\
I^{\rho}_{1-} {}^C D^{\rho}_{1-} h(v) &= h(v) + d_0 + d_1(1-v) + d_2(1-v)^2 + \ldots + d_{n-1}(1-v)^{n-1},
\end{aligned}
$$

where $c_i, d_i \in \mathbb{R}, i = 0, 1, \ldots, n-1 (n = [\rho]+1)$.

Lemma 2.4. Let $Y \in C([0,1], \mathbb{R})$, $\mu \in C([0,1], \mathbb{R}\times\mathbb{R})$ and $Y(0) = 0$, then the linear fractional differential system's solution enhanced with boundary conditions

$$
\begin{cases}
{}^C D^{\gamma}_{1-} \left\{ {}^C D^{\delta}_{0+} [\varrho(z) - Y(z)] \right\} = \mu(z), \ z \in [0,1], \ 0 < \delta \le 1, \ 2 < \gamma \le 3,\\
\alpha \varrho(0) + \beta \varrho(1) = \tau,\\
\phi \varrho'(0) + \psi \varrho'(1) = \nu,\\
u \varrho''(0) + v \varrho''(1) = \eta,
\end{cases}
$$

is equivalent to the fractional integral equations

$$
\begin{aligned}
\varrho(z) =& \frac{1}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I^{\gamma}_{1-} \mu(w) dw - \frac{z^{\delta}}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I^{\gamma}_{1-} \mu(1) dw + \delta \Lambda_1 \right.\\
&\left. + Y''(1) - \frac{\eta}{v} \right] + \frac{z^{\delta+1}}{\delta+1} \Lambda_1 - \frac{1}{\alpha+\beta} \left\{ \frac{\beta}{\Gamma(\delta)} \int_0^z (z-w)^{\delta-1} I^{\gamma}_{1-} \mu(1) dw + \beta Y(1) - \tau \right.\\
&\left. - \frac{\beta}{\delta(\delta-1)} \left[\frac{1}{\Gamma(\delta-2)} \int_0^z (z-w)^{\delta-3} I^{\gamma}_{1-} \mu(1) dw + \delta \Lambda_1 + Y''(1) - \frac{\eta}{v} \right] + \frac{\beta}{(\delta+1)} \Lambda_1 \right\} + Y(z)
\end{aligned} \tag{2.1}
$$

Proof .: From $\mu \in C([0,1], \mathbb{R}\times\mathbb{R})$, we get

$$
\begin{aligned}
\varrho(z) &= I^{\delta}_{0+} \left(I^{\gamma}_{1-} \mu(z) + c_0 + c_1 z \right) + c_2 + Y(z)\\
&= I^{\delta}_{0+} I^{\gamma}_{1-} \mu(z) + \frac{z^{\delta}}{\Gamma(\delta+1)} c_0 + \frac{z^{\delta+1}}{\Gamma(\delta+2)} c_1 + c_2 + Y(z).
\end{aligned} \tag{2.2}
$$

Using the conditions $\alpha\varrho(0) + \beta\varrho(1) = \tau$, we get

$$
\beta I^{\delta}_{0+} I^{\gamma}_{1-} \mu(1) + \frac{\beta c_0}{\Gamma(\delta+1)} + \frac{\beta c_1}{\Gamma(\delta+2)} + (\alpha+\beta) c_2 + \beta Y(1) = \tau. \tag{2.3}
$$

Applying the boundary conditions $\phi\varrho'(0) + \psi\varrho'(1) = \nu$, we obtain

$$
\psi \left[I^{\delta-1}_{0+} I^{\gamma}_{1-} \mu(1) + \frac{\delta}{\Gamma(\delta+1)} c_0 + \frac{(\delta+1)}{\Gamma(\delta+2)} c_1 + Y'(1) \right] = \nu. \tag{2.4}
$$

Making use of the boundary condition $u\varrho''(0) + v\varrho''(1) = \eta$, we obtain

$$
v \left[I^{\delta-2}_{0+} I^{\gamma}_{1-} \mu(1) + \frac{\delta(\delta-1)}{\Gamma(\delta+1)} c_0 + \frac{(\delta+1)\delta}{\Gamma(\delta+2)} c_1 + Y''(1) \right] = \eta. \tag{2.5}
$$

Solving the equation (2.3), (2.4) and (2.5), we get

$$
c_0 = \frac{\Gamma(\delta+1)}{\delta(\delta-1)} \left[-\delta \Lambda_1 - I^{\delta-2}_{0+} I^{\gamma}_{1-} \mu(1) - Y''(1) + \frac{\eta}{v} \right]
$$

$$
\begin{aligned}
c_1 &= \frac{\Gamma(\delta+2)}{(\delta+1)}\Lambda_1 \\
c_2 &= \frac{1}{\alpha+\beta}\left\{\tau - \beta I_{0+}^{\delta} I_{1-}^{\gamma}\mu(1) - \beta Y(1) - \frac{\beta}{\delta(\delta-1)}\left[-\delta\Lambda_1 - I_{0+}^{\delta-2} I_{1-}^{\gamma}\mu(1) - Y''(1) + \frac{\eta}{v}\right] - \frac{\beta}{(\delta+1)}\Lambda_1\right\}
\end{aligned}
$$

where

$$
\Lambda_1 = (\delta-1)\left[I_{0+}^{\delta-1} I_{1-}^{\gamma}\mu(1) + Y'(1) - \frac{\nu}{\psi}\right] - I_{0+}^{\delta-2} I_{1-}^{\gamma}\mu(1) - Y''(1) + \frac{\eta}{v}.
$$

Using c_0, c_1 and c_2 values in (2.2)

$$
\begin{aligned}
\varrho(z) =& \frac{1}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma}\mu(w)dw - \frac{z^{\delta}}{\delta(\delta-1)}\left[\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}\mu(1)dw + \delta\Lambda_1\right. \\
&\left. + Y''(1) - \frac{\eta}{v}\right] + \frac{z^{\delta+1}}{\delta+1}\Lambda_1 - \frac{1}{\alpha+\beta}\left\{\frac{\beta}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma}\mu(1)dw + \beta Y(1) - \tau\right. \\
&\left. - \frac{\beta}{\delta(\delta-1)}\left[\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}\mu(1)dw + \delta\Lambda_1 + Y''(1) - \frac{\eta}{v}\right] + \frac{\beta}{(\delta+1)}\Lambda_1\right\} + Y(z)
\end{aligned}
$$

In a similar approach, the converse of the lemma can be derived. □

Lemma 2.5. [24] Let G be an open set in a closed, convex set C of a Banach space E. Assume $0 \in G$. Also assume that $T(\bar{G})$ is bounded and that $T : \bar{G} \to C$ is given by $T = T_1 + T_2$, in which $T_1 : \bar{G} \to E$ is continuous and completely continuous and $T_2 : \bar{G} \to E$ is a nonlinear contraction.
Then, either

(C1) T has a fixed point $\varrho \in \bar{G}$; or

(C2) there exist a point $\varrho \in \partial G$ and $\epsilon \in (0,1)$ with $\varrho = \epsilon T(\varrho)$, where $\bar{G}$ and ∂G respectively, represent the closure and boundary of G.

Consider $\lambda_r = \{\varrho \in C([0,1],\mathbb{R}) : \|x\| < r\}$, and indicate the highest number by

$$
B_r = \max\{|\theta(z,\varrho,H\varrho)| : (z,\varrho,H\varrho) \in [0,1]\times[-r,r]\times[-r,r]\}.
$$

Following Lemma 2.4, we express problem (2.1) as a fixed point problem $\varrho = F\varrho$, where $F : X \to X$ is determined by

$$
\begin{aligned}
F\varrho(z) =& \frac{1}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma}\theta(w,\varrho(w),H\varrho(w))dw + y(z,\varrho(z)) \\
&- \frac{z^{\delta}}{\delta(\delta-1)}\left[\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}\theta(1,\varrho(1),H\varrho(1))dw + \delta\Lambda_1 + y''(1,\varrho(1)) - \frac{\eta}{v}\right] \\
&+ \frac{z^{\delta+1}}{(\delta+1)}\Lambda_1 - \frac{\beta}{(\alpha+\beta)}\left\{\frac{1}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma}\theta(1,\varrho(1),H\varrho(1))dw\right. \\
&+ y(1,\varrho(1)) - \frac{\tau}{\beta} - \frac{1}{\delta(\delta-1)}\left[\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}\theta(1,\varrho(1),H\varrho(1))dw\right. \\
&\left.\left. + \delta\Lambda_1 + y''(1,\varrho(1)) - \frac{\eta}{v}\right] + \frac{\Lambda_1}{(\delta+1)}\right\}
\end{aligned} \tag{2.6}
$$

Let us define $F_i, i = 1,2 : X \to X$ by

$$
\begin{aligned}
F_1\varrho(z) =& \frac{1}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma}\theta(w,\varrho(w),H\varrho(w))dw - \frac{z^{\delta}}{\delta(\delta-1)}\left[\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}\theta(1,\varrho(1),H\varrho(1))dw\right] \\
&- \frac{\beta}{(\alpha+\beta)}\left\{\frac{1}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^{\gamma}\theta(1,\varrho(1),H\varrho(1))dw\right. \\
&\left. - \frac{1}{\delta(\delta-1)}\left[\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}\theta(1,\varrho(1),H\varrho(1))dw\right]\right\}
\end{aligned} \tag{2.7}
$$

and

$$F_2\varrho(z) = y(z,\varrho(z)) - \frac{z^\delta}{\delta(\delta-1)}\left[\delta\Lambda_1 + y''(1,\varrho(1)) - \frac{\eta}{v}\right] + \frac{z^{\delta+1}}{(\delta+1)}\Lambda_1 - \frac{\beta}{(\alpha+\beta)}\left\{\left[y(1,\varrho(1)) - \frac{\tau}{\beta}\right]\right. \quad (2.8)$$
$$\left. - \frac{1}{\delta(\delta-1)}\left[\delta\Lambda_1 + y''(1,\varrho(1)) - \frac{\eta}{v}\right] + \frac{\Lambda_1}{(\delta+1)}\right\}$$

Clearly,

$$F\varrho(z) = F_1\varrho(z) + F_2\varrho(z), \ z \in [0,1]. \quad (2.9)$$

3 Existence results

The existence results can be proved based on the following assumptions.

(A1)

$$|\theta(z,c(z),Hc(z)) - \theta(z,d(z),Hd(z))| \le l[|c-d| + |Hc - Hd|], \forall z \in [0,1], \ l > 0, \ c,d \in \mathbb{R};$$

(A2) For a continuous function $\zeta : \Lambda \times X \to X$, there exists a constant $l_1 > 0$, such that

$$|\zeta(z,w,c) - \zeta(z,w,d)| \le l_1|c-d|, \forall c,d \in X \ and \ z,w \in \Lambda.$$

(A3) y and y'' achieving the condition

$$|y(c) - y(d)| \le \hbar_1|c-d|, \ \hbar_1 < \lambda_2^{-1}, \ c,d \in C([0,1],\mathbb{R}), \ \hbar_1 > 0,$$
$$|y''(c) - y''(d)| \le \hbar_2|c-d|, \ y \in C^3([0,1]), \ \hbar_2 > 0.$$

(A4) $\omega := (\lambda_1(l + ll_1) + \lambda_2\hbar_1 + \lambda_3\hbar_2) < 1$.

(A5) there exist constants $e_1 > 0, e_2 \ge 0$ and $\alpha \in [0,1)$, such that

$$|\theta(z,\varrho,h)| \le e_1 + e_2|\varrho + h|^\alpha \ for \ z \in [0,1] \ and \ h = (H\varrho)(z) = \int_0^z f(k,w,\varrho(w))dw$$

(A6)

$$\sup_{r\in(0,\infty)} \frac{r}{\lambda_1(e_1 + e_2|r+h|^\alpha)} > \frac{1}{1 - \hbar_1\lambda_2 - \hbar_2\lambda_3}, \text{where}$$

$$\lambda_1 = \left(\frac{\alpha}{\alpha+\beta}\right)\int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)}\int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw$$
$$+ \left[\frac{\beta}{\delta(\delta-1)(\alpha+\beta)} - \frac{z^\delta}{\delta(\delta-1)}\right]\int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)}\int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw$$
$$\lambda_2 = \left(\frac{\alpha}{\alpha+\beta}\right) \text{ and } \lambda_3 = \left[\frac{\beta}{\delta(\delta-1)(\alpha+\beta)} - \frac{z^\delta}{\delta(\delta-1)}\right]$$

Theorem 3.1. Let $\theta : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $y : [0,1] \times \mathbb{R} \to \mathbb{R}$ represents a continuous function. Suppose that (A1)-(A4) holds, then there is a unique solution for the boundary value problem (1.1).

Proof .: For $c,d \in X$ also for all $z \in [0,1]$, based on the asssumptions (A1), (A2) and (A3), it follows that

$$\|Fc - Fd\| \le \sup_{z\in[0,1]}\left\{\frac{1}{\Gamma(\delta)}\int_0^z (z-w)^{\delta-1} I_{1-}^\gamma |\theta(w,c(w),Hc(w)) - \theta(w,d(w),Hd(w))|dw\right.$$
$$+ |y(z,c(z)) - y(z,d(z))| + \left[\frac{\beta}{\delta(\delta-1)(\alpha+\beta)} - \frac{z^\delta}{\delta(\delta-1)}\right]$$

$$
\begin{aligned}
&\frac{1}{\Gamma(\delta-2)}\int_0^z (z-w)^{\delta-3} I_{1-}^{\gamma}|\theta(1,c(1),Hc(1))-\theta(1,d(1),Hd(1))|dw\\
&+\left[\frac{\beta}{\delta(\delta-1)(\alpha+\beta)}-\frac{z^\delta}{\delta(\delta-1)}\right]|y''(1,c(1))-y''(1,d(1))|\\
&-\frac{\beta}{(\alpha+\beta)\Gamma(\delta)}\int_0^z I_{1-}^{\gamma}|\theta(1,c(1),Hc(1))-\theta(1,d(1),Hd(1))|dw\\
&-\left.\frac{\beta}{(\alpha+\beta)}|y(1,c(1))-y(1,d(1))|\right\}\\
\leq&(l+ll_1)\|c-d\|\left(\frac{\alpha}{\alpha+\beta}\right)\int_0^z\frac{(z-w)^{\delta-1}}{\Gamma(\delta)}\int_w^1\frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw\\
&+\left[\frac{\beta}{\delta(\delta-1)(\alpha+\beta)}-\frac{z^\delta}{\delta(\delta-1)}\right]\int_0^z\frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)}\int_w^1\frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw\\
&+\hbar_1\|c-d\|\left(\frac{\alpha}{\alpha+\beta}\right)+\hbar_2\|c-d\|\left[\frac{\beta}{\delta(\delta-1)(\alpha+\beta)}-\frac{z^\delta}{\delta(\delta-1)}\right]\\
\leq&(\lambda_1(l+ll_1)+\lambda_2\hbar_1+\lambda_3\hbar_2)\|c-d\|
\end{aligned}
$$

Hence,

$$\|Fc-Fd\|\leq\omega\|c-d\|.$$

By (A4) we have $\omega<1$, $F:X\to X$ is a contraction mapping. Therefore, the contraction mapping principle leads to the theorem's conclusion. □

Theorem 3.2. Let $\theta:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ denotes a continuous function. Assume (A1)-(A6) holds then there exist at least one solution on $[0,1]$ for the boundary value problem (1.1).

Proof .: (A6) gives that, there exists a number $r_0>0$ such that

$$\sup_{r_0\in(0,\infty)}\frac{r_0}{\lambda_1(e_1+e_2|r_0+h|^\alpha)}>\frac{1}{1-\hbar_1\lambda_2-\hbar_2\lambda_3}.$$

We illustrate the operators F_1 and F_2, given in (2.7) and (2.8) respectively, fulfill all of the criteria of Lemma 2.4.

Step 1. F_1 is completely continuous and continuous. Initially we show that $F_1\left(\bar{\lambda}_{r_0}\right)$ is bounded. For given any $t\in\bar{\lambda}_{r_0}$, we obtain

$$
\begin{aligned}
\|F_1\varrho\|=&\frac{1}{\Gamma(\delta)}\int_0^z(z-w)^{\delta-1}I_{1-}^{\gamma}|\theta(w,\varrho(w),H\varrho(w))|dw-\frac{z^\delta}{\delta(\delta-1)\Gamma(\delta-2)}\int_0^z(z-w)^{\delta-3}I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\\
&-\frac{\beta}{(\alpha+\beta)}\left[\frac{1}{\Gamma(\delta)}\int_0^z(z-w)^{\delta-1}I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right.\\
&\left.-\frac{1}{\delta(\delta-1)\Gamma(\delta-2)}\int_0^z(z-w)^{\delta-3}I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right]\\
\leq&B_r\int_0^z\frac{(z-w)^{\delta-1}}{\Gamma(\delta)}\int_w^1\frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw-\frac{z^\delta}{\delta(\delta-1)}\int_0^z\frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)}\int_w^1\frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw\\
&-\frac{\beta}{(\alpha+\beta)}\int_0^z\frac{(z-w)^{\delta-1}}{\Gamma(\delta)}\int_w^1\frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw+\frac{\beta}{(\alpha+\beta)\delta(\delta-1)}\int_0^z\frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)}\int_w^1\frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)}dmdw\\
\leq&B_r\lambda_1.
\end{aligned}
$$

Therefore, $F_1\left(\bar{\lambda}_{r_0}\right)$ is uniformly bounded. For any $f_1,f_2\in[0,1]$, $f_1<f_2$, we get

$$
\begin{aligned}
F_1\varrho\left(f_2\right)-F_1\varrho\left(f_1\right)=&\frac{1}{\Gamma(\delta)}\left\{\int_0^{f_1}\left[(f_2-w)^{\delta-1}-(f_1-w)^{\delta-1}\right]I_{1-}^{\gamma}|\theta(w,\varrho(w),H\varrho(w))|dw\right.\\
&\left.+\int_{f_1}^{f_2}(f_2-w)^{\delta-1}I_{1-}^{\gamma}|\theta(w,\varrho(w),H\varrho(w))|dw\right\}-\frac{1}{\delta(\delta-1)\Gamma(\delta-2)}
\end{aligned}
$$

$$
\begin{aligned}
&\left\{(f_2^\delta - f_1^\delta)\int_0^{f_1}\left[(f_2-w)^{\delta-3}-(f_1-w)^{\delta-3}\right]I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right.\\
&\left.+ f_2^\delta\int_{f_1}^{f_2}(f_2-w)^{\delta-3}I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right\}\\
&-\frac{\beta}{(\alpha+\beta)\Gamma(\delta)}\left\{\int_0^{f_1}\left[(f_2-w)^{\delta-1}-(f_1-w)^{\delta-1}\right]I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right.\\
&\left.+\int_{f_1}^{f_2}(f_2-w)^{\delta-1}I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right\}+\frac{\beta}{(\alpha+\beta)\delta(\delta-1)\Gamma(\delta-2)}\\
&\left\{\int_0^{f_1}\left[(f_2-w)^{\delta-3}-(f_1-w)^{\delta-3}\right]I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right.\\
&\left.+\int_{f_1}^{f_2}(f_2-w)^{\delta-3}I_{1-}^{\gamma}|\theta(1,\varrho(1),H\varrho(1))|dw\right\}\\
\le&\frac{B_r}{\Gamma(\delta)\Gamma(\gamma+1)}\left\{\int_0^{f_1}\left[(f_2-w)^{\delta-1}-(f_1-w)^{\delta-1}\right]dw+\int_{f_1}^{f_2}(f_2-w)^{\delta-1}dw\right\}\\
&-\frac{B_r}{\Gamma(\gamma+1)\Gamma(\delta-2)\delta(\delta-1)}\left\{(f_2^\delta-f_1^\delta)\int_0^{f_1}\left[(f_2-w)^{\delta-3}-(f_1-w)^{\delta-3}\right]dw\right.\\
&\left.+f_2^\delta\int_{f_1}^{f_2}(f_2-w)^{\delta-3}dw\right\}-\frac{B_r\ \beta}{\Gamma(\gamma+1)\Gamma(\delta)(\alpha+\beta)}\left\{\int_0^{f_1}\left[(f_2-w)^{\delta-1}-(f_1-w)^{\delta-1}\right]dw\right.\\
&\left.+\int_{f_1}^{f_2}(f_2-w)^{\delta-1}dw\right\}\\
&-\frac{B_r\ \beta}{\Gamma(\gamma+1)\Gamma(\delta-2)(\alpha+\beta)\delta(\delta-1)}\left\{\int_0^{f_1}\left[(f_2-w)^{\delta-3}-(f_1-w)^{\delta-3}\right]dw+\int_{f_1}^{f_2}(f_2-w)^{\delta-3}dw\right\}
\end{aligned}
$$

$|F_1\varrho(f_2)-F_1\varrho(f_1)|\to 0$, as $f_2-\ f_1\to 0$ and is independent of ϱ. Therefore, F_1 is equicontinuous.

By Arzel-Ascoli Theorem, the property of relatively compactness holds for the set $F_1\left(\bar{\lambda}_{r_0}\right)$. Now, assume $\varrho_n\subset\bar{\lambda}_{r_0}$ with $\|\varrho_n-\varrho\|\to 0$. Consequently, on $[0,1]$ the limit $\|\varrho_n(z)-\varrho(z)\|\to 0$ is uniformly valid. As a result of uniform continuity of $\theta(z,q,Hq)$ on the compact set $[0,1]\times[-r_0,r_0]\times[-r_0,r_0]$, $\|\theta\left(z,\varrho_n(z),H\varrho_n(z)\right)-\theta(z,\varrho(z),H\varrho(z))\|\to 0$ is uniformly valid on $[0,1]$. Thereby as $n\to\infty$, $\|F_1\varrho_n-F_1\varrho\|\to 0$, showing the continuity of F_1. Thus, the proof is complete.

Step 2. Consider $F_2:\bar{\lambda}_{r_0}\to C([0,1],\mathbb{R})$ is a contractive operator. According to (A3), for $\varrho_1,\varrho_2\in C([0,1],\mathbb{R})$, we obtain

$$
\begin{aligned}
|F_2\varrho_1(z)-F_2\varrho_2(z)| =&|y(z,\varrho_1(z))-y(z,\varrho_2(z))|+\left(\frac{\beta-(\alpha+\beta)\ z^\delta}{(\alpha+\beta)\delta(\delta-1)}\right)|y''(1,\varrho_1(1))-y''(1,\varrho_2(1))|\\
&-\frac{\beta}{(\alpha+\beta)}|y(1,\varrho_1(1))-y(z,\varrho_2(1))|\\
\le&\hbar_1\|\varrho_1-\varrho_2\|\left(\frac{\alpha}{\alpha+\beta}\right)+\hbar_2\|\varrho_1-\varrho_2\|\left(\frac{\beta-(\alpha+\beta)\ z^\delta}{(\alpha+\beta)\delta(\delta-1)}\right)\\
\le&(\hbar_1\lambda_2+\hbar_2\lambda_3)\|\varrho_1-\varrho_2\|
\end{aligned}
$$

when supremum is taken over $z\in[0,1]$, it produces

$$
|F_2\varrho_1(z)-F_2\varrho_2(z)|\le M_0\left\|\varrho_1-\varrho_2\right\|,\quad M_0=\hbar_1\lambda_2+\hbar_2\lambda_3<1.
$$

This shows that as $M_0<1$, F_2 is contraction.

Step 3. We will show that $F\left(\bar{\lambda}_{r_0}\right)$ is a bounded set. Using (A3)

$$
F_2\left(\bar{\lambda}_{r_0}\right)\le\theta\lambda_2 r_0,
$$

for any $t\in\bar{\lambda}_{r_0}$. Thus, the fact that $F_1\left(\bar{\lambda}_{r_0}\right)$ is bounded confirms that $F\left(\bar{\lambda}_{r_0}\right)$ is a bounded set.

Step 4. To show that the case (C2) is not satisfied in Lemma 2.5.

In contrast, we consider that (C2) is true. After that, we get $\delta \in (0,1)$ also $\varrho \in \partial\lambda_{r_0}$ in which $\varrho = \delta F\varrho$. Thus, we obtain $\|\varrho\| = r_0$ and

$$\begin{aligned}\varrho(z) = &\int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} \theta(m, \varrho(m), H\varrho(m))dmdw + y(z, \varrho(z)) \\ &- \frac{z^\delta}{\delta(\delta-1)}\left[\int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} \theta(1, \varrho(1), H\varrho(1))dmdw + \delta\Lambda_1 + y''(1, \varrho(1)) - \frac{\eta}{v}\right] + \frac{z^{\delta+1}}{(\delta+1)}\Lambda_1 \\ &- \frac{\beta}{(\alpha+\beta)}\left\{\int_0^z \frac{(z-w)^{\delta-1}}{\Gamma(\delta)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} \theta(1, \varrho(1), H\varrho(1))dmdw + y(1, \varrho(1)) - \frac{\tau}{\beta}\right. \\ &\left.- \frac{1}{\delta(\delta-1)}\left[\int_0^z \frac{(z-w)^{\delta-3}}{\Gamma(\delta-2)} \int_w^1 \frac{(m-w)^{\gamma-1}}{\Gamma(\gamma)} \theta(1, \varrho(1), H\varrho(1))dmdw + \delta\Lambda_1 + y''(1, \varrho(1)) - \frac{\eta}{v}\right] + \frac{\Lambda_1}{(\delta+1)}\right\},\end{aligned}$$

for $z \in [0,1]$. Based on assumptions (A4) - (A5), we obtain

$$r_0 \leq \lambda_1(e_1 + e_2|r_0 + h|^\alpha) + \hbar_1\lambda_2 r_0 + \hbar_2\lambda_3 r_0$$

Thus, we get

$$\frac{r_0}{\lambda_1(e_1 + e_2|r_0 + h|^\alpha)} \leq \frac{1}{1 - \hbar_1\lambda_2 - \hbar_2\lambda_3}.$$

Based on the assumption of (A6), (C2) does not hold. Thus the operators F_1 and F_2 fulfill all the criteria of Lemma 2.5. Therefore, there is at least one fixed point $\varrho \in \bar{\lambda}_{r_0}$ for the operator F, which is the solution of the problem (1.1). Hence the proof. □

4 Example

Consider the boundary value problem

$$\begin{cases} {}^C D_{1-}^{\frac{1}{3}} \left\{ {}^C D_{0+}^{\frac{3}{4}} [\varrho(z) - y(z, \varrho(z))] \right\} = \theta(z, \varrho(z), H\varrho(z)), \ z \in [0,1], \\ \frac{4}{5}\varrho(0) + \frac{5}{9}\varrho(1) = \frac{3}{7} \\ \frac{2}{3}\varrho'(0) + \frac{1}{4}\varrho'(1) = \frac{4}{7} \\ \frac{3}{5}\varrho''(0) + \frac{1}{2}\varrho''(1) = \frac{2}{5} \end{cases} \tag{4.1}$$

where

$$\theta(z, \varrho, M\varrho) = \frac{1}{(z^3 + 55)}\left(\cos\varrho + \int_0^z \sin\varrho \, d\varrho\right), \tag{4.2}$$

and

$$f(z, w, q) = \frac{1}{14}[(\cos z + \sin w + e^{q^2})]. \tag{4.3}$$

Here, $\gamma = \frac{1}{3}, \delta = \frac{3}{4}, \alpha = \frac{4}{5}, \beta = \frac{5}{9}, z = \frac{1}{2}, y(z, \varrho(z)) = \frac{2}{7} + \frac{|\varrho(z)|}{45(2+|\varrho(z)|)}$. Obviously,

$$|y(z, c) - y(z, d)| \leq \hbar_1|c - d|$$

with $\hbar_1 = \frac{1}{45}$ and $\hbar_2 = \frac{2}{45}$. Further, $\lambda_1 = 0.71745$, $\lambda_2 = 0.590163$, $\lambda_3 = 0.9854266$. Then

$$|\theta(z, c, Hc) - \theta(z, d, Hd)| \leq l[|c - d| + |Hc - Hd|]$$

with $l = \frac{1}{55}$,

$$|\zeta(z, b, c) - \zeta(z, b, d)| \leq l_1|c - d|$$

with $l_1 = \frac{1}{14}$ and

$$\omega := \lambda_1(l + ll_1) + \lambda_2\hbar_1 + \lambda_3\hbar_2 \approx 0.0708878 < 1.$$

Thus, every conditions of Theorem 3.1 with $\theta(z, c, Hc)$ defined by (4.2) are satisfied and as a result of this, Theorem 3.1 concludes that there is a unique solution for the problem (4.1).

5 Conclusion

In this work, we have established both the existence and uniqueness of solutions for a class of integro-differential equations involving mixed fractional derivatives under specified boundary conditions. The uniqueness of the solution was derived through the application of the classical contraction mapping principle, while the existence was ensured using D.O'Regan's fixed point theorem. These results contribute to the growing theory of fractional differential equations and provide a solid analytical foundation for further investigations into more complex systems involving mixed fractional operators.

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