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Existence of weak solutions to boundary problem for elliptic equations with variable exponents under $p(\cdot)$-uniformly convexity condition

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Abstract

The main goal of this work is to establish the existence of a weak solution in $W_{1,0}^{p(\cdot)}(\Omega)$ to the boundary problem

$$\begin{aligned}&-div\,(a\,(x,\ \nabla u)) + d\,(x)\,u\,|u|^{p(x)-2} = f\,(x,\ u)\,,\\&u|_{\partial\Omega} = 0,\end{aligned}$$

where Ω is a bounded domain in R^n, $n \geq 3$. The main term $a\,(x,\ \xi) = \frac{\partial A(x,\ \xi)}{\partial \xi}$ satisfies: $p(\cdot)$-uniformly convexity and $p(\cdot)$- sub-homogeneousness conditions; $d \in L^{\infty}(\Omega)$ and $d_m > 0$; and f is a real Caratheodory function on $\Omega \times R$, which satisfies the following non-standard growth condition $|f\,(x,\ s)| \leq A_1\left(1 + |s|^{\gamma(x)-1}\right)$, where $p_S < \gamma_m \leq \gamma_S < p_m^*$, c_1 is a positive constant, and $p^*(x) = \frac{np(x)}{n-p(x)}$ for $p(x) < n$.

Keywords: elliptic equation, Dirichlet problem, variable exponent space, variational method, variable Lebesgue space, uniformly convex condition, variable subhomogeneous, weak solution
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1 Introduction

We consider the following elliptic boundary problem

$$\begin{aligned}&-div\,(a\,(x,\ \nabla u)) + d\,(x)\,u\,|u|^{p(x)-2} = f\,(x,\ u)\,,\\&u|_{\partial\Omega} = 0,\end{aligned} \tag{1.1}$$

where Ω is an open bounded domain in R^n, $n \geq 3$, and $\partial\Omega$ is a C^2-smooth boundary of Ω. We assume that variable exponent $p(\cdot)$ is log-Holder continuous function on Ω. We assume that $p_m = \inf p(x) > 1$ and $p_S = \sup p(x) < \infty$. We define a function $p^*(x) = \frac{np(x)}{n-p(x)}$ for $p(x) < n$ and $p^*(x) = \infty$ for $p(x) \geq n$. We assume that $d \in L^{\infty}(\Omega)$ and $d_m > 0$. A complete set of conditions is gathered in Section 2.

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Variable exponent growth conditions, when applied to elliptic differential equations and variational problems, have found widespread application in many recent research areas [1, 4, 5, 6, 7, 8, 9, 16, 19]. Background information on variable exponents and their applications is available in [3, 4]. In [1], the authors analyzed a nonlinear partial differential equation using anisotropic variable exponent Sobolev spaces. The study aims to determine the number of positive weak solutions for an elliptic-Kirchhoff equation, contributing to the understanding of complex physical phenomena like electrorheological fluids. In [8, 9], the authors proved the existence of a nontrivial weak solution to the Schrodinger-Kirchhoff type equations with a fractional magnetic field, utilizing the mountain pass theorem under specific assumptions on the external force and without assuming the Ambrosetti-Rabinowitz condition. They also demonstrate that the problem possesses infinitely many large- or small-energy solutions, authors used a Cerami condition. S.D. Zeng, A.A. Khan, and S. Migorski investigated a generalized quasi-variational inequality that involves two multivalued operators and two bifunctions within a Banach space framework. They showed the existence of solutions by coupling the Tychonov fixed point principle with the Kakutani-Ky Fan theorem for multivalued maps. For additional information, consult the reference list [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the sources cited within those works. For some additional interested works, see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

The present paper investigates the existence of weak solutions to the boundary problem (1.1) under conditions of Section 2. The main term satisfies following variable exponent growth restriction $|a(x,\ \xi)| \leq c_0\left(\delta_0(x) + \delta_1(x)\,|\xi|^{p(x)-1}\right)$. The nonnegative functions δ_0 and δ_1 are such that δ_0 belongs to $L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)$, and $1 \leq \delta_1(\cdot)$ is locally integrable. Also, we presume the existence of a measurable function $A : \Omega \times R^n \to R$ such that $a(x,\ \xi) = \frac{\partial A(x,\ \xi)}{\partial \xi}$. Our main assumptions on function A are that A is $p(\cdot)$-uniformly convex and $p(\cdot)$-subhomogeneous. A source term $f : \Omega \times R \to R$ is a real Caratheodory function, which satisfies the following growth condition $|f(x,\ s)| \leq A_1\left(1 + |s|^{\gamma(x)-1}\right)$, where $p_S < \gamma_m \leq \gamma_S < p_m^*$, and c_1 is a positive constant, and Sobolev conjugation $p^*(x) = \frac{np(x)}{n-p(x)}$ for $p(x) < n$. Employing variational framework, we establish the existence of a weak solution to problem (1.1) under the condition of Section 2.

2 Preliminary information and main assumptions

We recall some essential information pertaining to variable exponent spaces. The variable Lebesgue space $L^{p(\cdot)}$ is a Banach space consisting of all measurable real-valued functions with the finite norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0\ :\ \rho\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where the modular is given by

$$\rho(f) = \int_\Omega |f(x)|^{p(x)}\, dx.$$

The generalized Holder inequality is given by

$$\int |f(x)\, g(x)|\, dx \leq k_{Hol}\, \|f\|_{L^{p(\cdot)}}\, \|g\|_{L^{q(\cdot)}},$$

for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, with $p_m = ess\ \inf p(x)$ and $p_S = ess\ \sup p(x)$, where $q(x) = \frac{p(x)}{p(x)-1}$, and variable Holder constant $k_{Hol} = 1 + \frac{1}{p_m} - \frac{1}{p_S}$.

We formulate main assumptions as follows:

1) there is a measurable function $A : \Omega \times R^n \to R$ such that $A(x,\ 0) = 0$, and the function $a(x,\ \xi) = \frac{\partial A(x,\ \xi)}{\partial \xi}$ is such that the mapping $x \mapsto a(x,\ \xi)$ is a Lebesgue measurable function for all $\xi \in R^n$, and the mapping $\xi \mapsto a(x,\ \xi)$ is continuous for almost all $x \in \Omega$;

2) function A is $p(\cdot)$-uniformly convex, namely,

$$\begin{aligned} &A(x,\ \tau\xi + (1-\tau)\eta) + \nu_1 \delta_1(x)\,|\xi - \eta|^{p(x)} \leq \\ &\leq \tau A(x,\ \xi) + (1-\tau)\, A(x,\ \eta) \end{aligned}$$

for all $\tau \in [0,\ 1]$, almost all $x \in \Omega$, and all $\xi,\ \eta \in R^n$;

3) function A is $p(\cdot)$- subhomogeneous, namely

$$0 \leq \sum_{i=1,\ \ldots,\ n} a_i(x,\ \xi)\,\xi_i < p(x)\,A(x,\ \xi)$$

for almost all $x \in \Omega$ and all $\xi \in R^n$;

4) $|a(x,\ \xi)| \leq c_0\left(\delta_0(x) + \delta_1(x)\,|\xi|^{p(x)-1}\right)$ for almost all $x \in \Omega$ and all $\xi \in R^n$;

5) $A(x,\ \xi) \geq \nu_0 \delta_1(x)\,|\xi|^{p(x)}$ for almost all $x \in \Omega$ and all $\xi \in R^n$;

where we denoted $c_0,\ \nu_0,\ \nu_1$ are positive real numbers; and $\delta_0,\ \delta_1$ are a pair of nonnegative measurable functions on Ω, such that $\delta_0 \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)$ and $\delta_1 \in L^1_{loc}(\Omega)$, $\delta_1(\cdot) \geq 1$.

6) the function $\tau \mapsto f(x,\ \tau)$ is continuous for almost all $x \in clos(\Omega)$ and $x \to f(x,\ \tau)$ is Lebesgue measurable for all $\tau \in R$, and such that

$$|f(x,\ s)| \leq A_1\left(1 + |s|^{\gamma(x)-1}\right)$$

for all $s \in R$ and almost all $x \in clos(\Omega)$, where $p_S < \gamma_m \leq \gamma_S < p^*_m$, and c_1 is a positive constant, where function $p^*(x) = \frac{np(x)}{n-p(x)}$ if $p(x) < n$;

7) there are a constant $M > 0$ and $\theta > p_S$ such that

$$0 < \theta F(x,\ s) \leq sf(x,\ s)$$

for all $|s| \geq M$ and almost all $x \in clos(\Omega)$, $F(x,\ s) = \int_{[0,\ s]} f(x,\ t)\,dt$;

8) there are positive numbers μ and β such that

$$\frac{f(x,\ s)}{s\,|s|^{p(x)-1}} \leq \mu$$

for all $s \in (-\beta,\ \beta) \setminus \{0\}$ and for almost all $x \in \Omega$.

3 Proof of a weak solution to boundary problems for an elliptic equation with variable exponents

We will prove the following existence theorem.

Definition 3.1. Let X be a Banach space. For functional $J : X \to R$, a weakly continuously differential $DJ(u)(v)$ is define by

$$DJ(u)(v) = \lim_{t \to 0} \frac{J(u+tv) - J(u)}{t}$$

for all $v \in X$.

The main theorem can be formulated as follows.

Theorem 3.2. Assume that conditions 1)-8) holds. There exists at least one nontrivial weak solution in $W^{p(\cdot)}_{1,0}(\Omega)$ to the elliptic boundary value problem (1.1).

Proof . We define some spaces and numbers:

$$X = \left\{u \in W^{p(\cdot)}_{1,0}(\Omega)\ :\ \int_\Omega \delta_1(x)\,|\nabla u|^{p(x)}\,dx < \infty\right\},$$

$$\|u\| = \inf\left\{\lambda > 0\ :\ \rho\left(\frac{\nabla u}{\lambda}\right) \leq 1\right\},$$

$$\|u\|_X = \inf\left\{\lambda > 0\ :\ \int_\Omega \delta_1(x)\left|\frac{\nabla u}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

and

$$\rho_1(u) = \int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx$$

for all $u \in X$, and we define a number

$$\sigma_1 = \inf \left\{ \frac{\int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx} \ : \ u \in X \backslash \{0\} \right\}.$$

For all $u \in W_{1,0}^{p(\cdot)}(\Omega)$, we denote

$$I(u) = \int_\Omega A(x,\ \nabla u)\, dx,$$

$$T(u) = \int_\Omega F(x,\ u)\, dx,$$

$$K(u) = \int_\Omega \frac{d(x)}{p(x)} |u|^{p(x)} dx$$

and

$$J(u) = I(u) + K(u) - T(u),$$

with the norm

$$\|DJ(u)\| = \sup\left\{|DJ(u)(v)| \ : \ v \in X,\ \|v\|_X = 1\right\}.$$

Statement (Poincare inequality) 1. For all $\sigma_1 > 0$, the inequality

$$\rho(u) \leq \frac{1}{\sigma_1} \int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx$$

for all $u \in X$. The proof is like the classical one. We prove several lemmas.

Lemma 3.3. We have

1)

$$I(x,\ \tau\xi + (1-\tau)\eta) + \nu_1 \int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx \leq \tau I(x,\ \xi) + (1-\tau) I(x,\ \eta),$$

for all $\tau \in [0,\ 1]$, almost all $x \in \Omega$, and all $\xi,\ \eta \in R^n$;

2) $|A(x,\ \xi)| \leq c_0 \left(\delta_0(x)|\xi| + \delta_1(x)|\xi|^{p(x)}\right)$ for almost all $x \in \Omega$ and all $\xi \in R^n$;

3) $|F(x,\ s)| \leq A_2 \left(1 + |s|^{\gamma(x)}\right)$ for all $s \in R$ and almost all $x \in clos(\Omega)$, where $p_S < \gamma_m \leq \gamma_S < p_m^*$, and c_2 is a positive constant, where function $p^*(x) = \frac{np(x)}{n-p(x)}$ if $p(x) < n$;

4) for $\theta > p_S$, there is a functions $\psi \in L^\infty(\Omega)$ and a positive number $\tilde{M}$ such that

$$\psi(x) s^\theta \leq F(x,\ s)$$

for all $s \geq \tilde{M}$ and almost all $x \in clos(\Omega)$.

Proof . The statement 1) straightforwardly follows from assumption 2).

2) We have

$$\begin{aligned} |A(x,\ \xi)| = \left| \int_{[0,\ 1]} \frac{d}{d\tau} A(x,\ \tau\xi)\, d\tau \right| &= \left| \int_{[0,\ 1]} a_i(x,\ \tau\xi)\, \xi_i d\tau \right| \\ &\leq \int_{[0,\ 1]} |a(x,\ \tau\xi)|\, |\xi|\, d\tau \\ &\leq \int_{[0,\ 1]} c_0 \left(\delta_0(x) + \delta_1(x)\, \tau^{p(x)-1} |\xi|^{p(x)-1}\right) |\xi|\, d \\ &\leq \tau_0 \left(\delta_0(x)|\xi| + \delta_1(x)|\xi|^{p(x)}\right). \end{aligned}$$

3) is a consequence of 8).

To prove statement 4), we use condition 7). For almost all $x \in \overline{\Omega}$ and all $s \geq \tilde{M}$ (where $\tilde{M} \geq M$), we have

$$0 < \theta F(x,s) \leq s f(x,s).$$

Rewriting this inequality, we obtain for $s \geq \tilde{M}$:

$$\frac{\theta}{s} < \frac{f(x,s)}{F(x,s)} = \frac{1}{F(x,s)} \frac{\partial}{\partial s} F(x,s).$$

This differential inequality can be integrated with respect to s from $\tilde{M}$ to s (where $s > \tilde{M}$):

$$\int_{\tilde{M}}^{s} \frac{\theta}{t}\, dt \leq \int_{\tilde{M}}^{s} \frac{1}{F(x,t)} \frac{\partial F(x,t)}{\partial t}\, dt.$$

Evaluating the integrals yields:

$$\theta \ln\left(\frac{s}{\tilde{M}}\right) \leq \ln\left(\frac{F(x,s)}{F(x,\tilde{M})}\right).$$

Exponentiating both sides gives:

$$F(x,s) \geq \left(\frac{s}{\tilde{M}}\right)^{\theta} F(x,\tilde{M}) = \psi(x) s^{\theta},$$

where $\psi(x) := F(x,\tilde{M})/\tilde{M}^{\theta} > 0$. This completes the proof of statement (4). The lemma is proven.

From these statements, we obtain the following properties:

1) For a sequence $\{u_k\}$ weakly convergent in $W_{1,0}^{p(\cdot)}(\Omega)$, we obtain that

$$\lim_{k\to\infty} T(u_k) = T(u)$$

and

$$I(u) + K(u) \leq \liminf_{k\to\infty} \left(I(u_k) + K(u_k)\right);$$

2) $I + K$ is continuous in X;

3)

$$D(I(u) + K(u))(v) = \int_{\Omega} a(x,\ \nabla u)\nabla v dx + \int_{\Omega} d(x)\, u\, |u|^{p(x)-2}\, v dx$$

for all $u,\ v \in X$;

4) $I(u) - I(v) \geq DI(v)(u - v)$ for all $u,\ v \in X$. We show that 4) holds. Indeed, we have

$$\begin{aligned}\frac{I(u_k + \tau(u - u_k)) - I(u_k)}{\tau} &= \frac{I((1-\tau)u_k + \tau u) - I(u_k)}{\tau} \\ &\leq \frac{(1-\tau)I(u_k) + \tau I(u) - I(u_k)}{\tau} \\ &= I(u) - I(u_k),\end{aligned}$$

for all $\tau \in (0,\ 1)$. Taking the limit as $\tau \to 0$, we obtain

$$DI(u_k)(u - u_k) \leq I(u) - I(u_k).$$

The proof is proven. Now, from (8), we have

$$|f(x,\ s)| \leq \mu |s|^{p(x)-1}$$

with some positive numbers μ and β where $s \in (-\beta,\ \beta) \setminus \{0\}$, and for almost all $x \in \Omega$. Therefore, we have

$$F(x,\ s) \leq \frac{\mu}{p(x)} |s|^{p(x)}$$

for all $s \in (-\beta,\ \beta) \setminus \{0\}$. From $|F(x,\ s)| \leq A_2 \left(1 + |s|^{\gamma(x)}\right)$, we have

$$|F(x,\ s)| \leq A_3 |s|^{\gamma(x)}$$

for all $s \in R \backslash (-\beta,\ \beta)$ and almost all $x \in \Omega$. Thence, we have

$$|F(x,\ s)| \leq A_3 |s|^{\gamma(x)} + \frac{\mu}{p(x)} |s|^{p(x)}.$$

Therefore, we estimate

$$\begin{aligned} T(u) &= \int_\Omega F(x,u)\,dx \\ &\leq \int_\Omega \left(A_3 |u|^{\gamma(x)} + \frac{\mu}{p(x)} |u|^{p(x)}\right) dx \\ &\leq A_3 \int_\Omega |u|^{\gamma(x)} dx + \frac{\mu}{p_m} \int_\Omega |u|^{p(x)} dx. \end{aligned}$$

Now, we need to estimate $\int_\Omega |u|^{\gamma(x)} dx$ in terms of $\rho_1(u) = \int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx$. Since $\delta_1(x) \geq 1$ almost everywhere in Ω, we have:

$$\int_\Omega |u|^{\gamma(x)} dx \leq \sup_{x \in \Omega} \left(\frac{|u(x)|^{\gamma(x)}}{\delta_1(x) |\nabla u(x)|^{\gamma(x)}} \right) \cdot \int_\Omega \delta_1(x) |\nabla u|^{\gamma(x)} dx.$$

However, a more precise approach uses the variable exponent Sobolev embedding. Since $\gamma_S < p_m^*$, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\gamma(\cdot)}(\Omega)$ is continuous. Moreover, by the Poincaré inequality (Statement 1), we have:

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq \frac{1}{\sigma_1} \|\delta_1^{1/p(\cdot)} \nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

For the $\gamma(\cdot)$-norm, we use an interpolation inequality. Since $p(x) < \gamma(x) < p^*(x)$ and $\delta_1(x) \geq 1$, there exists $\theta(x) \in (0,1)$ such that $\gamma(x) = \theta(x) p(x) + (1 - \theta(x)) p^*(x)$. By Hölder's inequality with variable exponents and the Sobolev embedding, we obtain:

$$\int_\Omega |u|^{\gamma(x)} dx \leq C \left(\int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx \right)^\alpha + C \left(\int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx \right)^\beta$$

for some constants $\alpha, \beta > 0$ and $C > 0$ depending on Ω, $p(\cdot)$, $\gamma(\cdot)$, and the embedding constants.

For simplicity in the mountain pass argument, we use a weaker but sufficient estimate: for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\int_\Omega |u|^{\gamma(x)} dx \leq \epsilon \int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx + C_\epsilon \int_\Omega \delta_1(x) |\nabla u|^{\gamma(x)} dx.$$

This follows from Young's inequality with variable exponents applied pointwise: for almost every $x \in \Omega$,

$$|u(x)|^{\gamma(x)} \leq \epsilon \delta_1(x) |\nabla u(x)|^{p(x)} + C_\epsilon \delta_1(x) |\nabla u(x)|^{\gamma(x)}.$$

Thus, we have:

$$\begin{aligned} J(u) &= I(u) + K(u) - T(u) \\ &\geq \nu_0 \int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx + \frac{d_m}{p_S} \int_\Omega |u|^{p(x)} dx - A_3 \int_\Omega |u|^{\gamma(x)} dx - \frac{\mu}{p_m} \int_\Omega |u|^{p(x)} dx \\ &\geq \nu_0 \rho_1(u) + \left(\frac{d_m}{p_S} - \frac{\mu}{p_m} \right) \int_\Omega |u|^{p(x)} dx - A_3 \left(\epsilon \rho_1(u) + C_\epsilon \int_\Omega \delta_1(x) |\nabla u|^{\gamma(x)} dx \right) \\ &= (\nu_0 - A_3 \epsilon)\, \rho_1(u) + \left(\frac{d_m}{p_S} - \frac{\mu}{p_m} \right) \int_\Omega |u|^{p(x)} dx - A_3 C_\epsilon \int_\Omega \delta_1(x) |\nabla u|^{\gamma(x)} dx. \end{aligned}$$

Choosing $\epsilon > 0$ small enough so that $\tilde{c} := \nu_0 - A_3 \epsilon > 0$, and setting $\tilde{c}_1 := A_3 C_\epsilon$, we obtain:

$$J(u) \geq \tilde{c} \rho_1(u) - \tilde{c}_1 \int_\Omega \delta_1(x) |\nabla u|^{\gamma(x)} dx + \left(\frac{d_m}{p_S} - \frac{\mu}{p_m} \right) \int_\Omega |u|^{p(x)} dx.$$

For the mountain pass geometry, it suffices to use the weaker estimate (since $\int_\Omega |u|^{p(x)}dx \geq 0$):

$$J(u) \geq \tilde{c}\rho_1(u) - \tilde{c}_1 \int_\Omega \delta_1(x)|\nabla u|^{\gamma(x)}dx.$$

This explains the appearance of the term $\tilde{c}_1 \int_\Omega \delta_1(x)|\nabla u|^{\gamma(x)}dx$, where $\tilde{c}_1$ depends on A_3, ϵ, and the constant from the Young inequality with variable exponents. Thus, we have

$$\begin{aligned} J(u) =& I(u) + K(u) - T(u) \\ =& \int_\Omega A(x,\ \nabla u)\,dx + \int_\Omega \frac{d(x)}{p(x)}|u|^{p(x)}\,dx - \int_\Omega F(x,\ u)\,dx \\ \geq& \nu_0 \int_\Omega \delta_1(x)|\nabla u|^{p(x)}\,dx + \frac{d_m}{p_S}\int_\Omega |u|^{p(x)}\,dx - \left(A_3\int_\Omega |u|^{\gamma(x)}\,dx + \frac{\mu}{p_m}\int_\Omega |u|^{p(x)}\,dx\right) \\ \geq& \tilde{c}\rho_1(u) - \tilde{c}_1\int_\Omega \delta_1(x)|\nabla u|^{\gamma(x)}\,dx, \end{aligned}$$

where $\tilde{c} = \nu_0 - \frac{\mu}{p_m\sigma_1}$ and $\tilde{c}_1 = A_3 C$. Further, we have

$$\begin{aligned} I(u) - \frac{1}{\theta}DI(u)(u) =& I(u) \\ =& \int_\Omega A(x,\ \nabla u)\,dx - \frac{1}{\theta}\int_\Omega a_i(x,\ \nabla u)\nabla_i u dx \\ \geq& \left(1 - \frac{p_S}{\theta}\right)\int_\Omega A(x,\ \nabla u)\,dx \\ \geq& \nu_0\left(1 - \frac{p_S}{\theta}\right)\int_\Omega \delta_1(x)|\nabla u|^{p(x)}\,dx \\ =& A_4\rho_1(u) \end{aligned}$$

for almost all $u \in X$. We denote a set $\tilde{\Omega} = \left\{x \in \Omega\ :\ |u(x)| > \tilde{M}\right\}$ for all $u \in X$. We have

$$\frac{1}{\theta}f(x,\ u) - F(x,\ u) \geq 0$$

for almost all $x \in \tilde{\Omega}$, and

$$\left|\frac{1}{\theta}f(x,\ u) - F(x,\ u)\right| \leq M_1$$

for almost all $x \in \Omega\backslash\tilde{\Omega}$. Hence, it is

$$\begin{aligned} \frac{1}{\theta}DT(u)(u) - T(u) =& \int_{\Omega\backslash\tilde{\Omega}}\left(\frac{1}{\theta}f(x,\ u) - F(x,\ u)\right)dx + \int_{\tilde{\Omega}}\left(\frac{1}{\theta}f(x,\ u) - F(x,\ u)\right)dx \\ \geq& -M_1 meas\left(\Omega\backslash\tilde{\Omega}\right) \geq -M_1 meas(\Omega) = c_6(\Omega). \end{aligned}$$

We have

$$\begin{aligned} K(u) - \frac{1}{\theta}DK(u)(u) =& \int_\Omega\left(\frac{d(x)}{p(x)}|u|^{p(x)} - \frac{1}{\theta}d(x)|u|^{p(x)}\right)dx \\ =& \int_\Omega\left(\frac{1}{p(x)} - \frac{1}{\theta}\right)d(x)|u|^{p(x)}\,dx \\ \geq& \left(\frac{1}{p_S(x)} - \frac{1}{\theta}\right)\int_\Omega d(x)|u|^{p(x)}\,dx \geq 0 \end{aligned}$$

since $\theta > p_S$. We estimate

$$\begin{aligned} J(u) - \frac{1}{\theta}DJ(u)(u) =& \left(I(u) - \frac{1}{\theta}DI(u)(u)\right) + \left(K(u) - \frac{1}{\theta}DK(u)(u)\right) + \left(\frac{1}{\theta}DT(u)(u) - T(u)\right) \\ \geq& c_5\rho_1(u) + c_6(\Omega). \end{aligned}$$

□

Definition 3.4 (Palais-Smale condition). The functional $J \in C^1(X)$ is said to satisfy the Palais-Smale condition if each sequence $\{u_k\} \subset E$ such that the sequence $\{J(u_k)\}$ is bounded, and $J(u_k) \overset{k\to\infty}{\longrightarrow} \varpi$, $J'(u_k) \overset{k\to\infty}{\longrightarrow} 0$, then we have that the sequence $\{u_k\}$ contains a convergent subsequence.

We recall the statement of the mountain-pass lemma.

Lemma 3.5 (Mountain-Pass Lemma). Let J be a continuous functional on a Banach space X. Let J be weakly continuously differentiable and the Palais-Smale condition. Assume $J(0) = 0$ and there is a positive number r and $\tilde{w} \in X$ such that $\|\tilde{w}\|_X > r$, $J(\tilde{w}) \leq J(0)$ and

$$\alpha = \inf\{J(u) \ : \ u \in X, \ \|u\|_X = r\} > 0.$$

We denote $\Gamma = \{g \in C([0,\ 1],\ X) \ : \ g(0) = 0,\ g(1) = \tilde{w}\}$. Assume that $\Gamma \neq \emptyset$. We denote

$$\varpi = \inf\{\sup J(g([0,\ 1])) \ : \ g \in \Gamma\}.$$

Then, $\varpi$is a critical value of functional J and $\varpi \geq \alpha$.

The statement of the existence theorem follows from the next considerations. There exists a $\tilde{u} \in X$ such that

$$\begin{array}{l} 0 < \alpha \leq J(\tilde{u}) = \inf\{\sup J(g([0,\ 1])) \ : \ g \in \Gamma\}, \\ DJ(\tilde{u})(v) = 0 \end{array}$$

for all $v \in X$. Thus, we deduce

$$\int_\Omega a_i(x,\ \nabla\tilde{u}) \nabla_i v dx + \int_\Omega d(x)\, u\, |u|^{p(x)-2}\, v dx = \int_\Omega f(x,\ \nabla\tilde{u})\, v dx.$$

We obtain that $\tilde{u} \neq 0$ is a general solution. Thus, the existence theorem will be proven if we prove following Lemmas.

Lemma 3.6. Let assumption 1-8 be satisfied. Then, X is a Banach space.

Proof . We assume that a sequence $\{u_k\}$ is a fundamental in X. We have that it is bounded sequence and

$$\lim_{l\to\infty} \liminf_{k\to\infty} \int_\Omega \delta_1(x) |\nabla u_k - \nabla u_l|^{p(x)} dx = 0.$$

By Fatou's lemma, we obtain

$$\int_\Omega \delta_1(x) |\nabla u|^{p(x)} dx \leq \lim_{l\to\infty} \inf \int_\Omega \delta_1(x) |\nabla u_l|^{p(x)} dx = \lim_{l\to\infty} \inf \rho_1(u_l) < \infty,$$

and, applying Fatou's lemma once more,

$$\lim_{l\to\infty} \inf \int_\Omega \delta_1(x) |\nabla u - \nabla u_l|^{p(x)} dx \leq \lim_{l\to\infty} \lim_{k\to\infty} \inf \int_\Omega \delta_1(x) |\nabla u_l - \nabla u_k|^{p(x)} dx = 0,$$

so, $u_k \xrightarrow[k\to\infty]{X} u$ and the space X is complete in its norm. □

Lemma 3.7. Let assumption 1-8 be satisfied. Then, a continuous functional J is weakly continuously differentiable and the Palais-Smale condition.

Proof . We assume that a sequence $\{u_k\} \subset X$ such that $\lim\limits_{k\to\infty} J(u_k) = \varpi$ and $\lim\limits_{k\to\infty} \|DJ(u_k)\| = 0$. We show that the sequence $\{u_k\}$ is bounded in X. If $\{u_k\}$is unbounded then there is a subsequence $\{u_{k(l)}\}$ such that $\|u_{k(l)}\|_X \geq l$ for all $l \in N$, so,

$$J(u_{k(l)}) \geq \|u_{k(l)}\|_X \left(\tilde{c} \|u_{k(l)}\|_X^{p_S - 1} - \frac{1}{\theta} \|DJ(u_{k(l)})\| \right) + c_6(\Omega).$$

Taking the limit as l tends to infinity, we conclude that $J\left(u_{k(l)}\right)$ tends to infinity, which contradicts to $\lim\limits_{k\to\infty} J\left(u_k\right) = \varpi$. Thus, sequence $\{u_k\}$ is bounded in X. Since $\{u_k\}$ is bounded in X, sequence $\{u_k\}$ converges weakly to some $u \in W_{1,0}^{p(\cdot)}\left(\Omega\right)$. We write

$$I\left(u\right) \leq \lim_{k\to\infty} I\left(u_k\right) = \lim_{k\to\infty}\left(T\left(u_k\right) + J\left(u_k\right) - K\left(u_k\right)\right) = T\left(u\right) + \varpi.$$

We have $\left\|DJ\left(u_k\right)\right\| \overset{k\to\infty}{\longrightarrow} 0$ and $DJ\left(u_k\right)\left(u-u_k\right) \overset{k\to\infty}{\longrightarrow} 0$. Next, we have

$$\lim_{k\to\infty} DT\left(u_k\right)\left(u-u_k\right) = \lim_{k\to\infty}\int_{\Omega} f\left(x,\ u_k\right)\left(u-u_k\right)dx = 0.$$

Thus, we deduce

$$\lim_{k\to\infty} DK\left(u_k\right)\left(u-u_k\right) = \lim_{k\to\infty}\int_{\Omega} d\left(x\right)u_k\left|u_k\right|^{p(x)-2}\left(u-u_k\right)dx = 0$$

since

$$\int_{\Omega} d\left(x\right)u_k\left|u_k\right|^{p(x)-2}\left(u-u_k\right)dx \leq k_{Hol}\left\|d\right\|_{L^\infty}\left\|u_k\right\|_{L^{p(\cdot)}(\Omega)}\left\|u-u_k\right\|_{L^{p(\cdot)}(\Omega)}.$$

Also, we get

$$\lim_{k\to\infty} DI\left(u_k\right)\left(u-u_k\right) = \lim_{k\to\infty}\left(DJ\left(u_k\right)\left(u-u_k\right) + DT\left(u_k\right)\left(u-u_k\right) - DK\left(u_k\right)\left(u-u_k\right)\right) = 0,$$

so

$$I\left(u\right) - \lim_{k\to\infty} I\left(u_k\right) = \lim_{k\to\infty}\left(I\left(u\right) - I\left(u_k\right)\right) \geq \lim_{k\to\infty} DI\left(u_k\right) = 0.$$

Therefore, we conclude

$$\lim_{k\to\infty} I\left(u_k\right) = I\left(u\right).$$

We show that $\lim\limits_{k\to\infty} u_k = u$ in X. Assume contrary then there is a number $\varepsilon > 0$ and a subsequence $\{u_k\}$ such that $\left\|u_k - u\right\|_X \geq \varepsilon$ for all $k \in N$. Taking $\tau = \frac{1}{2}$ in the uniform convexity condition (2), we obtain:

$$\frac{1}{2}I\left(u_k\right) + \frac{1}{2}I\left(u\right) - I\left(\frac{1}{2}u_k + \frac{1}{2}u\right) \geq \nu_1\rho_1\left(u_k - u\right) \geq \nu_1\varepsilon^{p_S}$$

for $0 < \varepsilon < 1$. Since we have established that $\lim_{k\to\infty} I(u_k) = I(u)$, taking the limit superior as $k \to \infty$ yields:

$$\begin{aligned}
\nu_1\varepsilon^{p_S} &\leq \limsup_{k\to\infty}\left[\frac{1}{2}I(u_k) + \frac{1}{2}I(u) - I\left(\frac{1}{2}u_k + \frac{1}{2}u\right)\right] \\
&= \frac{1}{2}I(u) + \frac{1}{2}I(u) - \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right) \\
&= I(u) - \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right).
\end{aligned}$$

Thus, we obtain:

$$I(u) - \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right) \geq \nu_1\varepsilon^{p_S}.$$

However, since $\frac{1}{2}u_k + \frac{1}{2}u$ converges weakly to u in X and I is weakly lower semicontinuous (as established earlier), we have:

$$I(u) \leq \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right).$$

Combining these inequalities gives:

$$0 = I(u) - I(u) \leq I(u) - \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right) \leq 0,$$

which contradicts $\nu_1 \varepsilon^{p_S} > 0$. Therefore, our assumption that $\|u_k - u\|_X \geq \varepsilon$ for all k must be false, and we conclude that $\lim_{k\to\infty} u_k = u$ in X. Since $\lim_{k\to\infty} I(u_k) = I(u)$, we obtain

$$I(u) - \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right) = \ = \limsup_{k\to\infty} \left(I(u) - I\left(\frac{1}{2}u_k + \frac{1}{2}u\right)\right) \geq \nu_1 \varepsilon^{p_S}.$$

However, we have

$$I\left(u\right) \leq \liminf_{k\to\infty} I\left(\frac{1}{2}u_k + \frac{1}{2}u\right)$$

since $\lim\limits_{k\to\infty} \left(\frac{1}{2}u_k + \frac{1}{2}u\right) \overset{weakly}{=} u$. Thus, we get a contradiction $0 \geq \nu_1 \varepsilon^{p_S}$, which proves that $\lim\limits_{k\to\infty} u_k = u$ in X. The functional J satisfies the Palais-Smale condition on X. □

Lemma 3.8. Let assumption 1)-8) be satisfied. Then, we have:

1) $J\left(0\right) = 0$, and there is a pair of positive number r and z such that

$$\alpha \leq \inf\left\{J\left(u\right)\ :\ u \in X,\ \|u\|_X = r\right\};$$

2) there is $\tilde{w} \in X$ such that $\|\tilde{w}\|_X > r,\ J\left(\tilde{w}\right) \leq J\left(0\right)$;

3) the $\Gamma = \left\{g \in C\left(\left[0,\ 1\right],\ X\right)\ :\ g\left(0\right) = 0,\ g\left(1\right) = \tilde{w}\right\}$ is not empty set.

Proof . 1) Straightforwardly, we have $I\left(0\right) = K\left(0\right) = T\left(0\right) = 0$ so $J\left(0\right) = 0$.

The second statement follows from

$$J\left(u\right) \geq \tilde{c}\rho_1\left(u\right) - \tilde{c}_1 \int_\Omega \delta_1\left(x\right) \left|\nabla u\right|^{\gamma(x)} dx \geq \tilde{c} r^{p_S} - \tilde{c}_1 r^{\gamma_m} \geq r^{p_S}\left(\tilde{c} - \tilde{c}_1 r^{\gamma_m - p_S}\right) = \alpha,$$

where $\|u\|_X = r$ and we assume that $r \leq 1$.

2) We assume that $s > 1$ and select $\tilde{v} \in C_C^\infty\left(\Omega\right),\ \tilde{v} \geq 0$, and $\Theta = \left\{x \in \Omega\ :\ \tilde{v}\left(x\right) \geq t_0\right\}$ with positive measure. We denote $\Theta_s = \left\{x \in \Omega\ :\ s\tilde{v}\left(x\right) \geq t_0\right\}$ and obtain

$$\begin{aligned}\int_{\Theta_s} F\left(x,\ s\tilde{v}\right) dx \geq &\int_{\Theta_s} \psi\left(x\right)\left(s\tilde{v}\left(x\right)\right)^\theta dx \\ =& s^\theta \int_{\Theta_s} \psi\left(x\right)\left(\tilde{v}\left(x\right)\right)^\theta dx \\ \geq& s^\theta \int_{\Theta} \psi\left(x\right)\left(\tilde{v}\left(x\right)\right)^\theta dx \\ =& s^\theta \int_\Theta \psi \tilde{v}^\theta dx > 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}J\left(s\tilde{v}\right) =& I\left(s\tilde{v}\right) + K\left(s\tilde{v}\right) - \int_{\Theta_s} F\left(x,\ s\tilde{v}\right) dx - \int_{\Omega\setminus\Theta_s} F\left(x,\ s\tilde{v}\right) dx \\ \leq& s^{p_S} I\left(\tilde{v}\right) + s^{p_S} K\left(\tilde{v}\right) - s^\theta \int_{\Theta_s} \psi\left(x\right)\left(\tilde{v}\left(x\right)\right)^\theta dx + c\, meas\left(\Omega\right).\end{aligned}$$

Since $\theta > p_S$ we conclude that $J\left(s\tilde{v}\right) \overset{s\to\infty}{\longrightarrow} -\infty$. Thus, there is a number s' such that $\|s'\tilde{v}\|_X > r$ and $J\left(s'\tilde{v}\right) \leq 0$. Therefore, for $\tilde{w} = s'\tilde{v} \in X$ such that $\|\tilde{w}\|_X > r$ and $J\left(\tilde{w}\right) \leq J\left(0\right)$.

3) We assume that $g \in C\left(\left[0,\ 1\right],\ X\right)$and define $g\left(s\right) = s\tilde{w}$ for all $s \in \left[0,\ 1\right]$. Thus, we conclude $g \in \Gamma$. The lemma is proven. □

Thus, the existence of a weak solution to the problem (1.1) has been proven, completely. □

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